MINIMUM PERMANENTS OF DOUBLY STOCHASTIC MATRICES WITH $k$ DIAGONAL $p \times p$ BLOCK SUBMATRICES

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ABSTRACT. For positive integers $k$ and $p \geq 3$, let

$$D = \begin{bmatrix}
0 & 0 & J_{2,kp} \\
0 & 0 & J_{p} \\
J_{kp,2} & J_{p} & O \\
& O & \ddots \\
& & & J_{p}
\end{bmatrix}$$

where $J_{p}$ is the $p \times p$ matrix whose entries are all 1. Then, we determine the minimum permanents and minimizing matrices over (1) the face of $\Omega(D)$ and (2) the face of $\Omega(D^*)$, where

$$D^* = D + \begin{bmatrix}
J_{2} & O \\
O & O
\end{bmatrix}.$$

1. Introduction

Let $\Omega_n$ denote the set of all $n$-square doubly stochastic matrices. This set is known to be a convex polytope of dimension $n^2 - 2n + 1$ in the Euclidean $n^2$-space. For an $n \times n$ matrix $A = [a_{ij}]$, the permanent of $A$, $\per A$, is defined by

$$\per A = \sum_{\sigma \in S_n} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)},$$

where $S_n$ stands for the symmetric group on the set $\{1, 2, \ldots, n\}$. For an $n \times n$ $(0, 1)$-matrix $D$, let $\Omega(D) = \{X \in \Omega_n \mid X \leq D\}$, where $X \leq D$ means that every entry of $X$ is less than or equal to the corresponding entry of $D$. Then, $\Omega(D)$ forms a face of $\Omega_n$ and every face of $\Omega_n$ is

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defined in this fashion [2]. Let $\mu(D)$ denote the minimum permanent over $\Omega(D)$. A matrix $A \in \Omega(D)$ is called a minimizing matrix over $\Omega(D)$, if $\text{per}A = \mu(D)$. The set of all minimizing matrices over $\Omega(D)$ is denoted by $\text{Min}(D)$.

Since the vertices of the polytope $\Omega(D)$ are the permutation matrices $P$ such that $P \leq D$, the barycenter of $\Omega(D)$ is the matrix

$$B_D = \frac{1}{\text{per}D} \sum_{P \leq D} P,$$

where the summation runs over all permutation matrices $P \leq D$ and $\Omega(D)$ is called barycentric if $B_D \in \text{Min}(D)$ [1]. For an $n \times n$ matrix $A$ and for $\alpha, \beta \subseteq \{1, 2, \ldots, n\}$, let $A[\alpha|\beta]$ denote the $|\alpha| \times |\beta|$ submatrix of $A$ lying in the rows $\alpha$ and columns $\beta$, and let $A(\alpha|\beta)$ denote the $(n - |\alpha|) \times (n - |\beta|)$ submatrix of $A$ lying in the rows complementary to $\alpha$ and columns complementary to $\beta$, where $|\gamma|$ denote the number of elements in the set $\gamma \subseteq \{1, 2, \ldots, n\}$. Let $I_n$ denote the identity matrix of order $n$ and $J_{m,n}$ denote the $m \times n$ matrix of 1’s. The matrix $J_{n,n}$ is denoted by $J_n$, for brevity and each column of $J_n$ is denoted by $e_n$. Let $E_{i,j}$ denote the $n$-square matrix of zeros and ones with exactly one nonzero entry at the $(i, j)$-position. For $i = 1, 2, \ldots, k$ and positive integers $n_i$ such that $\sum_{i=1}^k n_i = n$, let $A_i$ is the $n_i \times n_i$ matrix and let

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix}$$

is called block diagonal. Notationally, such matrix is often indicated as $A = A_1 \oplus \cdots \oplus A_k$ or, more briefly, $\bigoplus_{i=1}^k A_i$; this is called the direct sum of the matrices $A_i$.

Let for $n \geq 3$

$$U_{2,n} = \begin{bmatrix} O_2 & J_{2,n} \\ J_{n,2} & I_n \end{bmatrix}, \quad V_{2,n} = \begin{bmatrix} J_2 & J_{2,n} \\ J_{n,2} & I_n \end{bmatrix}.$$

Seok-Zun Song [5] proved that

$$\min \{\text{per}A \mid A \in \Omega(U_{2,n})\} = \frac{2(n-1)(n-2)^{n-2}}{n^{n+1}} \quad (n \geq 3)$$
and the minimum is achieved uniquely at

\[
\begin{bmatrix}
O_2 & \frac{1}{n} J_{2,n} \\
\frac{1}{n} J_{n,2} & \frac{n-2}{n} J_n
\end{bmatrix}.
\]

He also proved that

\[
\min \{ \text{per} A \mid A \in \Omega(U_{2,n}) \} = \min \{ \text{per} A \mid A \in \Omega(V_{2,n}) \}.
\]

For \( i = 1, 2, \ldots, k \) and positive integers \( n_i \) such that \( \sum_{i=1}^{k} n_i + 1 = n \), let \( U_{1,n} \) denote the \( n \times n \) \((0,1)\)-matrix defined by

\[
U_{1,n}[1,2,\ldots,n|1] = U_{1,n}[1|1,2,\ldots,n]^t = \begin{bmatrix} O \\ e_{n-1} \end{bmatrix}
\]

and

\[
U_{1,n}(1|1) = \bigoplus_{i=1}^{k} J_{n_i}.
\]

Seok-Su Do and S. H. Hwang [3] proved that

\[
\min \{ \text{per} A \mid A \in \Omega(U_{1,n}) \} = \alpha \prod_{i=1}^{k} \frac{n_i!}{(\alpha + n_i)^{n_i}}
\]

and the minimum is achieved uniquely at

\[
\begin{bmatrix}
\frac{\alpha}{\alpha+n_1} J_{n_1,1} & \frac{\alpha}{\alpha+n_1} J_{1,n_1} & \frac{\alpha}{\alpha+n_2} J_{1,n_2} & \cdots & \frac{\alpha}{\alpha+n_k} J_{1,n_k} \\
\frac{\alpha}{\alpha+n_2} J_{n_2,1} & \frac{\alpha}{\alpha+n_1} J_{1,n_2} & \frac{n_2}{\alpha+n_2} J_{n_2} & O & \cdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\frac{\alpha}{\alpha+n_k} J_{n_k,1} & \cdots & O & \frac{n_k}{\alpha+n_k} J_{n_k}
\end{bmatrix},
\]

where \( \alpha \) be the unique positive solution \( \varphi_{\alpha}(x) = \sum_{i=1}^{k} \frac{n_i}{x+n_i} - n + 2 = 0 \).

Also, they proved the next result as follows: let \( C = U_{1,n} + E_{1,1} \). For \( n \geq 3 \), the permanent function attains its minimum matrix over \( \Omega(C) \) uniquely the above matrix. In this paper, we prove the following results:
For a positive integer $k$ and $p \geq 3$, let

$$D = \begin{bmatrix}
0 & 0 & J_{2, kp} \\
0 & 0 & J_p \\
J_{kp, 2} & O & \ddots \\
\end{bmatrix}$$

Then we determine the minimizing matrix and the minimum permanent over;

(1) the face $\Omega(D)$ of the polytope of doubly stochastic matrices,

(2) the face $\Omega(D^*)$ of the polytope of doubly stochastic matrices, where

$$D^* = D + \begin{bmatrix}
J_2 & O \\
O & O \\
\end{bmatrix}.$$ 

2. Preliminaries

In this section, we introduce the well-known definitions and useful lemmas. An $n \times n$ matrix is called partly decomposable if it contains an $s \times (n-s)$ zero submatrix. A square matrix which is not partly decomposable is called fully indecomposable.

**Lemma 1.** [4] Let $D = [d_{ij}]$ be an $n \times n$ fully indecomposable $(0, 1)$-matrix and let $A = [a_{ij}] \in \text{Min}(D)$. Then $A$ is fully indecomposable and for $i, j$ with $d_{ij} = 1$, it holds that $\text{per} A(i|j) \geq \text{per} A$ where the inequality is an equality if $a_{ij} > 0$.

**Lemma 2.** [4] Let $D = [d_1, d_2, \ldots, d_n]$ be an $n \times n$ $(0, 1)$-matrix and let $A = [a_1, a_2, \ldots, a_n] \in \text{Min}(D)$. If $d_{j_1} = \cdots = d_{j_k}$, then the matrix obtained from $A$ by replacing each of $a_{j_1}, \ldots, a_{j_k}$ by $(a_{j_1} + \cdots + a_{j_k})/k$ also belongs to $\text{Min}(D)$.

**Lemma 3.** For $p \geq 3$ and $t \geq 1$, let $f(t) = t^2 - \frac{p+1}{p-1} t + 1$. Then, the value of function $f(t)$ is nonnegative.

**Proof.** Clearly, $f(t)$ is differentiable and continuous on $t \geq 1$. Since $\frac{p+1}{2(p-1)} < 1$, $f(t)$ is increasing function on $t \in [1, \infty)$. Then, $f(1) = 1 - \frac{p+1}{p-1} + 1 = \frac{p-3}{p-1} \geq 0$. Hence, the function $f(t)$ is always nonnegative. □
3. Results

For positive integer \(k\) and \(p \geq 3\), let \(U\) be the \((kp + 2) \times (kp + 2)\) matrix defined by

\[
U = \begin{bmatrix}
0 & 0 & \frac{1}{pk}J_{2,kp} \\
0 & 0 & \vdots \\
\frac{1}{pk}J_{kp,2} & cJ_p & O \\
\end{bmatrix}
\]

where \(c = \frac{pk - 2}{p^2k}\).

**Theorem 4.** For \(k \geq 1\) and \(p \geq 3\), the minimum permanent over the face \(\Omega(D)\) is \(\frac{(2kp - p - 1)(p)^k(pk - 2)^{pk - 2}}{p^{2pk - 1}k^{pk + 1}}\) and the minimum value is achieved uniquely at \(U\).

**Proof.** Let \(X \in \Omega(D)\) be a minimizing matrix and let

\[
A = (I_2 \oplus \bigoplus_{i=1}^{k} J_p)X(I_2 \oplus \bigoplus_{i=1}^{k} J_p).
\]

Then,

\[
A = \begin{bmatrix}
0 & 0 & a_1J_{2,p} & a_kJ_{2,p} \\
0 & 0 & \vdots & \vdots \\
a_1J_{2,p} & b_1J_p & b_2J_p & O \\
a_kJ_{2,p} & \vdots & \vdots & b_kJ_p
\end{bmatrix}
\]

for some real numbers \(a_i, b_i\) and by Lemma 2, \(A\) is also a minimizing matrix over \(\Omega(D)\).

Without loss of generality, we may assume that \(a_1 \geq a_2 \geq \cdots \geq a_k\) and let \(z_i = \frac{a_i}{b_i^2}\) for \(i = 1, 2, \ldots, k\). Since \(D\) is fully indecomposable, by Lemma 1, \(a_i \neq 0\) and \(b_i \neq 0\) for all \(i = 1, 2, \ldots, k\). Let \(B = A(1,2|1,2)\). Then

\[
\text{per}B = \prod_{i=1}^{k} p!b_i^p.
\]

For \(i = 1, 2, \ldots, k\), let \(T_0 = \{1, 2\}\) and \(T_i = \{l \in \mathbb{Z} | (i - 1)p + 3 \leq l \leq ip + 2\}\). Then, it is clear that if \(p, q\) are integers in the same \(T_i\), then

\[
\text{per}A(1|p) = \text{per}A(1|q).
\]
For \( j = 4, p + 3, 2p + 3, \ldots, p(k - 1) + 3 \), the matrix \( A(1,2|3,j) \) is equal to the one obtained from \( B \) by replacing the first two columns with

\[
A[1,2|1,2,\ldots,n]^t = A[1,2,\ldots,n|1,2] = \begin{bmatrix}
a_1 J_{p,2} \\
\vdots \\
a_k J_{p,2}
\end{bmatrix}.
\]

Since we notice that \( A(1,2|3,j) \) is partly decomposable for \( j = 4, p + 3, \ldots, p(k - 1) + 3 \), we have

\[
\text{per} A(1|3) = [(p - 1)a_1 \text{per} A(1,2|3,4) + pa_2 \text{per} A(1,2|3,p + 3) \\
+ pa_3 \text{per} A(1,2|3,2p + 3) + \cdots + pa_k \text{per} A(1,2|3,p(k - 1) + 3)] \text{per} B
\]
\[
= [(p - 1)a_1[(pl)^k a_1^2 b_1^{p-2} b_2^p \cdots b_k^p] + pa_2[(pl)^k 2a_1 a_2 b_1^{p-1} b_2^{p-1} \cdots b_k^p] \\
+ pa_3[(pl)^k 2a_1 a_3 b_1^{p-1} b_2^{p-1} b_3^{p-1} \cdots b_k^p] + \cdots \\
+ pa_k[(pl)^k 2a_1 a_k b_1^{p-1} b_2^{p-1} \cdots b_k^{p-1}]] \text{per} B.
\]

Since \( \text{per} B = \prod_{i=1}^k p!b_i^p \),

\[
\frac{\text{per} A(1,2|3,4)}{\text{per} B} = \frac{(pl)^k a_1^2 b_1^{p-2} b_2^p \cdots b_k^p}{\prod_{i=1}^k p!b_i^p} = \frac{a_1^2}{b_1^2},
\]

and for \( j = p + 3, 2p + 3, 3p + 3, \ldots, p(k - 1) + 3 \)

\[
\frac{\text{per} A(1,2|3,j)}{\text{per} B} = \frac{(pl)^k 2a_1 a_j b_1^{p-1} b_j^{p-1} \prod_{i \neq j} b_i^p}{\prod_{i=1}^k p!b_i^p} = \frac{2a_1 a_j}{b_1 b_j}.
\]

By (1) and (2), we can write that

\[
\text{per} A(1|3) = [(p - 1)a_1 \left(\frac{a_1}{b_1}\right)^2 + pa_2 \left(\frac{2a_1 a_2}{b_1 b_2}\right) + \cdots + pa_k \left(\frac{2a_1 a_k}{b_1 b_k}\right)] \text{per} B
\]
\[
= [(p - 1)a_1 z_1^2 + 2pa_2 z_1 z_2 + \cdots + 2pa_k z_1 z_k] \text{per} B.
\]

By similar method,

\[
\text{per} A(1|p(k - 1) + 3) = [2pa_1 z_1 z_k + 2pa_2 z_2 z_k + \cdots + 2pa_{k-1} z_{k-1} z_k + (p - 1)a_k z_k^2] \text{per} B.
\]
By Lemma 1, we see that

\[
0 = \text{per}A(1|3) - \text{per}A(1|p(k - 1) + 3)
\]
\[
= [(z_1 - z_k)((p - 1)a_1z_1 + 2pa_2z_2 + \cdots
+ 2p_{a_{k-1}}z_{k-1} + (p - 1)a_kz_k] - (p + 1)z_1z_k(a_1 - a_k)]\text{per}B
\]
\[
= [(z_1 - z_k)((p - 1)a_1z_1 + 2p \sum_{i=2}^{k-1} a_i z_i + (p - 1)a_kz_k]
\]
\[-(p + 1)z_1z_k(a_1 - a_k)]\text{per}B.
\]

Since \(z_1 - z_k = \frac{1}{pb_1b_k}(a_1 - a_k)\) and by Lemma 3, we can obtain that

\[
0 = \text{per}A(1|3) - \text{per}A(1|p(k - 1) + 3)
\]
\[
= \left(\frac{a_1 - a_k}{pb_1b_k}\right) [(p - 1)a_1z_1 + 2p \sum_{i=2}^{k-1} a_i z_i + (p - 1)a_kz_k]
\]
\[-(p + 1)z_1z_k(a_1 - a_k)]\text{per}B
\]
\[
= (a_1 - a_k)[(p - 1)z_1^2 - (p + 1)z_1z_k + \frac{(p - 1)}{pb_1}z_k^2]
\]
\[
+ \frac{2}{b_1b_k} \sum_{i=2}^{k-1} a_i z_i]\text{per}B
\]
\[
> (a_1 - a_k)[(p - 1)z_1^2 - (p + 1)z_1z_k + (p - 1)z_k^2]
\]
\[
+ \frac{2}{b_1b_k} \sum_{i=2}^{k-1} a_i z_i]\text{per}B
\]
\[
= (a_1 - a_k)[(p - 1)z_k^2\left(\frac{z_1}{z_k}\right)^2 - \frac{p + 1}{p - 1}\left(\frac{z_1}{z_k}\right) + 1] + \frac{2}{b_1b_k} \sum_{i=2}^{k-1} a_i z_i]\text{per}B
\]
\[
> 0,
\]

which is a contradiction. Since \(A\) ia doubly stochastic matrix, \(a_i = \frac{1}{p_k}\) and \(b_i = \frac{p_{k-2}}{p_k}\) for all \(i = 1, 2, \ldots, k\). Hence, we have \(\text{per}A = \frac{(2kp - p - 1)(p)k(pk-2)^{p-2}}{p^{pk}}\). So far we have proved that for any minimizing matrix \(X\) over \(\Omega(D)\),

\[(I_2 \oplus J_p \oplus \cdots \oplus J_p)X(I_2 \oplus J_p \oplus \cdots \oplus J_p) = U.\]
It remains to show the uniqueness of the minimizing matrix over $\Omega(D)$. Suppose that $X = [x_{ij}] \in \Omega(D)$ is a minimizing matrix such that $X \neq U = [u_{ij}]$. By (3), $X[T_0|T_i] = U[T_0|T_i]$ and $X[T_i|T_0] = U[T_i|T_0]$ for $i = 1, 2, \ldots, k$. So, there is an $i > 1$ such that $X[T_i^*|T_1^*] \neq U[T_i^*|T_1^*]$, where $T_i^* = T_0 \cup T_i$. Without loss of generality, we may assume that $i = 1$.

Case (1). $X[T_0|T_1] \neq U[T_0|T_1]$ or $X[T_1|T_0] \neq U[T_1|T_0]$.

We may assume that $X[T_0|T_1] \neq U[T_0|T_1]$ by taking transposition if necessary, and also that $x_{13} > u_{13}$ since $\sum_{j=3}^{p} x_{ij} = \sum_{j=3}^{p} u_{ij}$ by (3). Let $C = [c_{ij}] := (I_2 \oplus J_p \oplus \cdots \oplus J_p)X(I_3 \oplus J_{p-1} \oplus \cdots \oplus J_p)$. Then, by Lemma 2, $C$ is also a minimizing matrix over $\Omega(D)$, $c_{ij} = u_{ij}$ for all $(i, j) \in T_1^* \times T_1^*$ and

$$C[T_1^*|T_1^*] = \begin{bmatrix}
  x_{13} & b & \cdots & b \\
  O_2 & b' & c & \cdots & c \\
  u & v & \cdots & v \\
  u & v & \cdots & v \\
  uJ_{p,2} & : & : & : & : \\
  u & v & \cdots & v
\end{bmatrix}$$

where $u = u_{13}$. Since $C[T_1^*|T_1^*](I_2 \oplus J_p) = U[T_1^*|T_1^*]$ and $x_{13} > u_{13}$, we see that $c > u_{13}$ and $b < u_{13}$. Now,

$$0 = \text{per}C(1|3) - \text{per}C(2|3) = (p - 1)(c - b)\text{per}C(1, 2|3, 4) > 0,$$

a contradiction.

Case (2). $X[T_0|T_1] = U[T_0|T_1]$, $X[T_1|T_0] = U[T_1|T_0]$ but $X[T_1|T_1] \neq U[T_1|T_1]$.

In this case, we may assume that $x_{33} > u_{33}$. Let $H = [h_{ij}] := (I_3 \oplus J_{p-1} \oplus \cdots \oplus J_p)X(I_3 \oplus J_{p-1} \oplus \cdots \oplus J_p)$. Then, $H$ is also a minimizing matrix and $h_{ij} = u_{ij}$ for all $(i, j) \notin T_1^* \times T_1^*$ and

$$H[T_1^*|T_1^*] = \begin{bmatrix}
  uJ_{2,p} & \\
  O_2 & u_{J_{2,p}} \\
  x_{33} & r & \cdots & r \\
  r & t & \cdots & t \\
  uJ_{p,2} & : & : & : \\
  r & t & \cdots & t
\end{bmatrix}$$

where $u = u_{13}$. 
Since $x_{33} > u_{33}$, we see that $r < u_{33}$ and $t > u_{33}$ from (3) again. Thus, as before, we can show that $0 = \text{per}H(3|3) - \text{per}H(4|3) = (p - 1)(t - r)\text{per}H(3, 4|3, 4) > 0$, a contradiction. 

We recall that $D^* = D + \begin{bmatrix} J_2 & O \\ O & O \end{bmatrix}$.

From now on, we talk to the minimum permanent and the minimizing matrix over $\Omega(D^*)$.

Before we start the proof of some theorems and lemmas, let

$$A = \begin{bmatrix} zJ_{2,2} & a_1J_{2,p} & a_kJ_{2,p} \\ a_1J_{2,p} & b_1J_p & b_2J_p \\ a_kJ_{2,p} & b_2J_p & \cdots & b_kJ_p \end{bmatrix}$$

for some real numbers $a_i, b_i, z$ and $k \geq 4$.

**Lemma 5.** Let $z_i = \frac{a_i}{b_i}$. Then all $z_i$ are the same value for all $i = 1, 2, \ldots, k$.

**Proof.** Let $X$ be a minimizing matrix over $\Omega(D^*)$. By Lemma 2, $A^*$ is also a minimizing matrix over $\Omega(D^*)$. We may assume that $a_1 \geq a_2 \geq \cdots \geq a_k$ and let $z_i = \frac{a_i}{b_i}$ for all $i = 1, 2, \ldots, k$. Then, for the contrary, we assume that $a_1 > a_k$.

$$\text{per}A^*(1|3)$$

$$= [2z\text{per}A^*(1, 2|1, 3) + (p - 1)a_1\text{per}A^*(1, 2|3, 4) + pa_2\text{per}A^*(1, 2|3, p + 3) + pa_3\text{per}A^*(1, 2|3, 2p + 3) + \cdots + pa_k\text{per}A^*(1, 2|3, p(k - 1) + 3)]\text{per}B$$

$$= [2z[(p!)^ka_1b_1^{p-1}b_2^p \cdots b_k^p] + (p - 1)a_1[(p!)^ka_1^2b_1^{2p-2}b_2^p \cdots b_k^p] + \cdots + pa_k[(p!)^ka_1a_kb_1^{p-1}b_2^p \cdots b_k^{p-1}]}\text{per}B$$

Since $\text{per}B = \prod_{i=1}^k p^ib_i^p$,

$$\frac{\text{per}A^*(1, 2|3, 4)}{\text{per}B} = \frac{(p!)^ka_1^2b_1^{2p-2}b_2^p \cdots b_k^p}{\prod_{i=1}^k p^ib_i^p} = \frac{a_1}{b_1}$$

and

$$\frac{\text{per}A^*(1, 2|1, 3)}{\text{per}B} = \frac{(p!)^ka_1b_1^{p-1}b_2^p \cdots b_k^p}{\prod_{i=1}^k p^ib_i^p} = \frac{a_1}{b_1}.$$
Also, for $j = p + 3, 2p + 3, 3p + 3, \ldots, p(k - 1) + 3$,

\begin{equation}
\frac{\text{per} A^*(1, 2|3, j)}{\text{per} B} = \frac{(p!)^k 2 a_1 a_j b_1^{p-1} b_j^{p-1} \prod_{i \neq j} b_i^p}{\prod_{i=1}^k p! b_i^p} = \frac{2a_1 a_j}{b_1 b_j}.
\end{equation}

By (4), (5) and (6), we obtain that

\begin{align*}
\text{per} A^*(1|3) &= [2z(a_1/b_1) + (p - 1)a_1(a_1/b_1)^2 + pa_2(2a_1 a_2/b_1 b_2) + \cdots + pa_k(2a_1 a_k/b_1 b_k)]\text{per} B \\
&= [2zz_1 + (p - 1)a_1 z_1^2 + 2pa_2 z_1 z_2 + \cdots + 2pa_k z_1 z_k]\text{per} B.
\end{align*}

By similar method,

\begin{align*}
\text{per} A^*(1|p(k - 1) + 3) &= [2zz_k + 2pa_1 z_1 z_k + \cdots + 2pa_{k-1} z_{k-1} z_k + (p - 1)a_k z_k^2]\text{per} B.
\end{align*}

By Lemma 1, we see that

\begin{align*}
0 &= \text{per} A^*(1|3) - \text{per} A^*(1|p(k - 1) + 3) \\
&= [(z_1 - z_k)[2z + (p - 1)a_1 z_1 + 2pa_2 z_2 + \cdots + 2pa_{k-1} z_{k-1} + (p - 1)(a_1 - a_k)z_1 z_k]\text{per} B \\
&= [(z_1 - z_k)[2z + (p - 1)a_1 z_1 + 2p \sum_{i=2}^{k-1} a_i z_i + (p - 1)a_k z_k] \\
&\quad - (p + 1)z_1 z_k(a_1 - a_k)]\text{per} B.
\end{align*}

Since $z_1 - z_k = \frac{1}{pb_1 b_k}(a_1 - a_k)$ and by Lemma 3, we can obtain that

\begin{align*}
0 &= \text{per} A^*(1|3) - \text{per} A^*(1|p(k - 1) + 3) \\
&= \left[\frac{(a_1 - a_k)}{pb_1 b_k}\right][2z + (p - 1)a_1 z_1 + 2p \sum_{i=2}^{k-1} a_i z_i + (p - 1)a_k z_k] \\
&\quad - (p + 1)z_1 z_k(a_1 - a_k)]\text{per} B \\
&= (a_1 - a_k)\left[\frac{1}{pb_1 b_k}[2z + (p - 1)a_1 z_1 + 2p \sum_{i=2}^{k-1} a_i z_i + (p - 1)a_k z_k] \\
&\quad - (p + 1)z_1 z_k]\text{per} B.
\end{align*}
\[
(a_1 - a_k)\left[\frac{2z}{pb_1 b_k} + \frac{(p-1)}{pb_k} z_1^2 - (p+1)z_1 z_k + \frac{(p-1)}{pb_1} z_k^2 \right]
\]
\[+ \frac{2}{b_1 b_k} \sum_{i=2}^{k-1} a_i z_i \text{per} B
\]
\[
> (a_1 - a_k)\left[\frac{2z}{pb_1 b_k} + \frac{(p-1)}{pb_k} z_1^2 - (p+1)z_1 z_k + \frac{(p-1)}{pb_1} z_k^2 \right]
\]
\[+ \frac{2}{b_1 b_k} \sum_{i=2}^{k-1} a_i z_i \text{per} B
\]
\[
= (a_1 - a_k)\left[\frac{2z}{pb_1 b_k} + (p-1) z_k^2 \left[\left(\frac{z_1}{z_k}\right)^2 - \frac{p+1}{p-1} \left(\frac{z_1}{z_k}\right) + 1\right]\right]
\]
\[+ \frac{2}{b_1 b_k} \sum_{i=2}^{k-1} a_i z_i \text{per} B
\]
\[
> 0,
\]
which is a contradiction from \(a_1 > a_k\). Therefore \(a_1 = a_k\) and hence, \(a_i\) are the same value for \(i = 1, 2, \ldots, k\). Therefore, we obtain the result. \(\square\)

For the main theorem, we need the following Lemma:

**LEMMA 6.** For all \(i\), \(\frac{a_i}{b_i} \leq \frac{1}{2}\).

**Proof.** By Lemma 5, we have all \(\frac{a_i}{b_i}\) are the same value. So we may put \(\frac{a_i}{b_i} = z\) and assume that \(z > \frac{1}{2}\). Now, since \(A^*\) is row stochastic matrix, we obtain

\[
1 = 2z + \sum_{i=1}^{k} pa_i.
\]

By hypothesis, we have

\[
1 = 2z + \sum_{i=1}^{k} pa_i = 2z + p \sum_{i=1}^{k} a_i
\]
\[
> 2z + \frac{p}{2} \sum_{i=1}^{k} b_i.
\]
\[\text{(7)}\]
Since the $A^*$ is column stochastic matrix, we can change the equation (7)

$$1 > 2z + \frac{p}{2} \sum_{i=1}^{k} \frac{1 - 2a_i}{p}$$

$$= 2z + \frac{1}{2} \sum_{i=1}^{k} (1 - 2a_i)$$

$$= 2z + \frac{1}{2} [k - 2 \sum_{i=1}^{k} a_i].$$

Then, from the first and the last line in the equation (8), we have

$$2z < 1 - \frac{1}{2} [k - 2 \sum_{i=1}^{k} a_i]$$

$$< 1 - \frac{k}{2} + 1 = \frac{4-k}{2} < 0,$$

a contradiction. Therefore $\frac{g_i}{b_i} \leq \frac{1}{2}$ for all $i = 1, 2, \ldots, k.$

**Theorem 7.** For positive integers $p \geq 3$ and $k$, we have $\text{Min}(D^*) = \text{Min}(D)$.

**Proof.** Let $X$ be a minimizing matrix over $\Omega(D^*)$. By Lemma 2, $A^*$ is also a minimizing matrix over $\Omega(D^*)$. Then,

\[
\begin{align*}
\text{per}A^*(1|1) &= [z\text{per}A^*(1,2|1,2) + pa_1\text{per}A^*(1,2|3,4) + pa_2\text{per}A^*(1,2|3,p+3) \\
&\quad + pa_3\text{per}A^*(1,2|3,2p+3) + \cdots \\
&\quad + pa_k\text{per}A^*(1,2|3,p(k-1)+3)]\text{per}B \\
&= [z((p!)^{k-1}b_1^{p-1}b_2^p \cdots b_k^p] + pa_1[(p!)^{k-1}b_1^{p-1}b_2^p \cdots b_k^p] + pa_2[(p!)^{k-1}b_1^{p-1}b_2^p \cdots b_k^p] \\
&\quad + pa_3[(p!)^{k-1}b_1^{p-1}b_2^p \cdots b_k^p] + \cdots + pa_k[(p!)^{k-1}b_1^{p-1}b_2^p \cdots b_k^p]\text{per}B \\
&= [z + pa_1z_1 + \cdots + pa_kz_k]\text{per}B.
\end{align*}
\]

By the proof of Lemma 5, we get

\[
\text{per}A^*(1|3) = [2zz_1 + (p-1)a_1z_1^2 + 2pa_2z_1z_2 + \cdots + 2pa_kz_1z_k]\text{per}B.
\]
If $A^*[1, 2|1, 2] \neq O$, then Lemma 1 show that $\text{per} A^*(1|1) = \text{per} A^*(1|3) = \text{per} A^*$. But then

$$0 = \text{per} A^*(1|3) - \text{per} A^*(1|1)$$

$$= [z(2z_1 - 1) + a_1 z_1 ((p - 1)z_1 - p) + (2z_1 - 1) \sum_{i=2}^{k} pa_i z_i] \text{per} B.$$  \hspace{1cm} (9)

However, Lemma 6 yield that the right hand side of (9) is negative, which is a contradiction. Hence, we have $A^*[1, 2|1, 2] = O$, that is $\text{Min}(D^*) = \text{Min}(D)$.  \hspace{1cm} $\Box$

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References


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