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MINIMUM PERMANENTS OF DOUBLY STOCHASTIC MATRICES WITH k DIAGONAL $p \times p$ BLOCK SUBMATRICES

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ABSTRACT. For positive integers k and $p \geq 3$, let

$$D=\left[\begin{array}{cc|cccc} 0 & 0 & & J_{2,kp} & & \\ 0 & 0 & & & & \\ \hline & & J_p & & & \\ & & & J_p & O & \\ J_{kp,2} & & O & & \ddots & \\ & & & & & J_p \end{array}\right]$$

where J_p is the $p \times p$ matrix whose entries are all 1. Then, we determine the minimum permanents and minimizing matrices over (1) the face of $\Omega(D)$ and (2) the face of $\Omega(D^*)$, where

$$D^* = D + \begin{bmatrix} J_2 & O \\ O & O \end{bmatrix}.$$

1. Introduction

Let Ω_n denote the set of all n-square doubly stochastic matrices. This set is known to be a convex polytope of dimension $n^2 - 2n + 1$ in the Euclidean n^2-space. For an $n \times n$ matrix $A = [a_{ij}]$, the permanent of A, per$A$, is defined by

$$\text{per}A = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)},$$

where S_n stands for the symmetric group on the set $\{1, 2, \ldots, n\}$. For an $n \times n$ $(0,1)$-matrix D, let $\Omega(D)$=$\{X \in \Omega_n | \ X \leq D\}$, where $X \leq D$ means that every entry of X is less than or equal to the corresponding entry of D. Then, $\Omega(D)$ forms a face of Ω_n and every face of Ω_n is

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defined in this fashion [2]. Let $\mu(D)$ denote the minimum permanent over $\Omega(D)$. A matrix $A \in \Omega(D)$ is called a *minimizing matrix* over $\Omega(D)$, if $\text{per}A = \mu(D)$. The set of all minimizing matrices over $\Omega(D)$ is denoted by $\text{Min}(D)$.

Since the vertices of the polytope $\Omega(D)$ are the permutation matrices P such that $P \leq D$, the *barycenter* of $\Omega(D)$ is the matrix

$$B_D = \frac{1}{\text{per}D} \sum_{P \leq D} P,$$

where the summation runs over all permutation matrices $P \leq D$ and $\Omega(D)$ is called *barycentric* if $B_D \in \text{Min}(D)$ [1]. For an $n \times n$ matrix A and for $\alpha, \beta \subseteq \{1, 2, \ldots, n\}$, let $A[\alpha|\beta]$ denote the $|\alpha| \times |\beta|$ submatrix of A lying in the rows α and columns β, and let $A(\alpha|\beta)$ denote the $(n - |\alpha|) \times (n - |\beta|)$ submatrix of A lying in the rows complementary to α and columns complementary to β, where $|\gamma|$ denote the number of elements in the set $\gamma \subseteq \{1, 2, \ldots, n\}$. Let I_n denote the identity matrix of order n and $J_{m,n}$ denote the $m \times n$ matrix of 1's. The matrix $J_{n,n}$ is denoted by J_n for brevity and each column of J_n is denoted by $\mathbf{e_n}$. Let $E_{i,j}$ denote the n-square matrix of zeros and ones with exactly one nonzero entry at the (i, j)-position. For $i = 1, 2, \ldots, k$ and positive integers n_i such that $\sum_{i=1}^{k} n_i = n$, let A_i is the $n_i \times n_i$ matrix and let

$$A = \begin{bmatrix} A_1 & & & & \\ & A_2 & \mathbf{0} & & \\ & & \ddots & & \\ & \mathbf{0} & & \ddots & \\ & & & & A_k \end{bmatrix}$$

is called *block diagonal*. Notationally, such matrix is often indicated as $A = A_1 \bigoplus \cdots \bigoplus A_k$ or, more briefly, $\bigoplus_{i=1}^{k} A_i$; this is called the *direct sum* of the matrices A_i.

Let for $n \geq 3$

$$U_{2,n} = \begin{bmatrix} O_2 & J_{2,n} \\ J_{n,2} & I_n \end{bmatrix}, \ V_{2,n} = \begin{bmatrix} J_2 & J_{2,n} \\ J_{n,2} & I_n \end{bmatrix}.$$

Seok-Zun Song [5] proved that

$$\min\{\text{per}A|\ A \in \Omega(U_{2,n})\} = \frac{2(n-1)(n-2)^{n-2}}{n^{n+1}} \quad (n \geq 3)$$

and the minimum is achieved uniquely at

$$\begin{bmatrix} O_2 & \frac{1}{n}J_{2,n} \\ \frac{1}{n}J_{n,2} & \frac{n-2}{n}I_n \end{bmatrix}.$$

He also proved that

$$\min\{\text{per}A|\ A \in \Omega(U_{2,n})\} = \min\{\text{per}A|\ A \in \Omega(V_{2,n})\}.$$

For $i = 1, 2, \ldots, k$ and positive integers n_i such that $\sum_{i=1}^{k} n_i + 1 = n$, let $U_{1,n}$ denote the $n \times n$ $(0,1)$-matrix defined by

$$U_{1,n}[1, 2, \ldots, n|1] = U_{1,n}[1|1, 2, \ldots, n]^t = \begin{bmatrix} O \\ \mathbf{e_{n-1}} \end{bmatrix}$$

and

$$U_{1,n}(1|1) = \bigoplus_{i=1}^{k} J_{n_i}.$$

Seok-Su Do and S. H. Hwang [3] proved that

$$\min\{\text{per}A|\ A \in \Omega(U_{1,n})\} = \alpha \prod_{i=1}^{k} \frac{n_i!}{(\alpha + n_i)^{n_i}}$$

and the minimum is achieved uniquely at

$$\begin{bmatrix} O & \frac{\alpha}{\alpha+n_1}J_{1,n_1} & \frac{\alpha}{\alpha+n_2}J_{1,n_2} & \cdots & \frac{\alpha}{\alpha+n_k}J_{1,n_k} \\ \frac{\alpha}{\alpha+n_1}J_{n_1,1} & \frac{n_1}{\alpha+n_1}J_{n_1} & & & \\ \frac{\alpha}{\alpha+n_2}J_{n_2,1} & & \frac{n_2}{\alpha+n_2}J_{n_2} & O & \\ \vdots & & & \ddots & \\ \vdots & & O & & \\ \frac{\alpha}{\alpha+n_k}J_{n_k,1} & & & & \frac{n_k}{\alpha+n_k}J_{n_k} \end{bmatrix},$$

where α be the unique positive solution $\varphi_\sigma(x) = \sum_{i=1}^{k} \frac{n_i}{x+n_i} - n + 2 = 0$.

Also, they proved the next result as follows: let $C = U_{1,n} + E_{1,1}$. For $n \geq 3$, the permanent function attains its minimum matrix over $\Omega(C)$ uniquely the above matrix. In this paper, we prove the following results:

For a positive integer k and $p \geq 3$, let

$$D = \left[\begin{array}{cc|cccc} 0 & 0 & & & & \\ 0 & 0 & & J_{2,kp} & & \\ \hline & & J_p & & & \\ & & & J_p & O & \\ & & & & \ddots & \\ J_{kp,2} & & O & & & J_p \end{array}\right]$$

Then we determine the minimizing matrix and the minimum permanent over;

(1) the face $\Omega(D)$ of the polytope of doubly stochastic matrices,

(2) the face $\Omega(D^*)$ of the polytope of doubly stochastic matrices,

where

$$D^* = D + \begin{bmatrix} J_2 & O \\ O & O \end{bmatrix}.$$

2. Preliminaries

In this section, we introduce the well-known definitions and useful lemmas. An $n \times n$ matrix is called *partly decomposable* if it contains an $s \times (n-s)$ zero submatrix. A square matrix which is not partly decomposable is called *fully indecomposable.*

LEMMA 1. [4] *Let $D = [d_{ij}]$ be an $n \times n$ fully indecomposable $(0,1)$-matrix and let $A = [a_{ij}] \in \mathrm{Min}(D)$. Then A is fully indecomposable and for i, j with $d_{ij} = 1$, it holds that $\mathrm{per}A(i|j) \geq \mathrm{per}A$ where the inequality is an equality if $a_{ij} > 0$.*

LEMMA 2. [4] *Let $D = [\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n]$ be an $n \times n$ $(0,1)$-matrix and let $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n] \in \mathrm{Min}(D)$. If $\mathbf{d}_{j_1} = \cdots = \mathbf{d}_{j_k}$, then the matrix obtained from A by replacing each of $\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_k}$ by $(\mathbf{a}_{j_1} + \cdots + \mathbf{a}_{j_k})/k$ also belongs to $\mathrm{Min}(D)$.*

LEMMA 3. *For $p \geq 3$ and $t \geq 1$, let $f(t) = t^2 - \frac{p+1}{p-1}t + 1$. Then, the value of function $f(t)$ is nonnegative.*

Proof. Clearly, $f(t)$ is differentiable and continuous on $t \geq 1$. Since $\frac{p+1}{2(p-1)} < 1$, $f(t)$ is increasing function on $t \in [1, \infty)$. Then, $f(1) = 1 - \frac{p+1}{p-1} + 1 = \frac{p-3}{p-1} \geq 0$. Hence, the function $f(t)$ is always nonnegative. $\square$

3. Results

For positive integer k and $p \geq 3$, let U be the $(kp+2) \times (kp+2)$ matrix defined by

$$U = \left[\begin{array}{cc|cccc} 0 & 0 & & \frac{1}{pk}J_{2,kp} & & \\ 0 & 0 & & & & \\ \hline & & cJ_p & & & \\ & & & cJ_p & O & \\ \frac{1}{pk}J_{kp,2} & & O & & \ddots & \\ & & & & & cJ_p \end{array}\right] \quad where\ c = \frac{pk-2}{p^2k}.$$

THEOREM 4. *For $k \geq 1$ and $p \geq 3$, the minimum permanent over the face $\Omega(D)$ is $\frac{(2kp-p-1)(p!)^k(pk-2)^{pk-2}}{p^{2pk-1}\,k^{pk+1}}$ and the minimum value is achieved uniquely at U.*

Proof. Let $X \in \Omega(D)$ be a minimizing matrix and let

$$A = (I_2 \oplus \bigoplus_{i=1}^{k} J_p)X(I_2 \oplus \bigoplus_{i=1}^{k} J_p).$$

Then,

$$A = \left[\begin{array}{cc|cccc} 0 & 0 & a_1J_{2,p} & & & a_kJ_{2,p} \\ 0 & 0 & & & & \\ \hline a_1J_{2,p} & & b_1J_p & & & \\ & & & b_2J_p & O & \\ & & O & & \ddots & \\ a_kJ_{2,p} & & & & & b_kJ_p \end{array}\right]$$

for some real numbers a_i, b_i and by Lemma 2, A is also a minimizing matrix over $\Omega(D)$.

Without loss of generality, we may assume that $a_1 \geq a_2 \geq \cdots \geq a_k$ and let $z_i = \frac{a_i}{b_i}$ for $i = 1, 2, \ldots, k$. Since D is fully indecomposable, by Lemma 1, $a_i \neq 0$ and $b_i \neq 0$ for all $i = 1, 2, \ldots, k$. Let $B = A(1,2|1,2)$. Then

$$\text{per}B = \prod_{i=1}^{k} p!b_i^p.$$

For $i = 1, 2, \ldots, k$, let $T_0 = \{1, 2\}$ and $T_i = \{l \in \mathbf{Z}|\ (i-1)p + 3 \leq l \leq ip + 2\}$. Then, it is clear that if p, q are integers in the same T_i, then

$$\text{per}A(1|p) = \text{per}A(1|q).$$

For $j = 4, p+3, 2p+3, \ldots, p(k-1)+3$, the matrix $A(1,2|3,j)$ is equal to the one obtained from B by replacing the first two columns with

$$A[1,2|1,2,\ldots,n]^t = A[1,2,\ldots,n|1,2] = \begin{bmatrix} a_1 J_{p,2} \\ \vdots \\ a_k J_{p,2} \end{bmatrix}.$$

Since we notice that $A(1,2|3,j)$ is partly decomposable for $j = 4,\ p+3, \ldots p(k-1)+3$, we have

$$\begin{aligned}
&\mathrm{per}A(1|3) \\
&= [(p-1)a_1 \mathrm{per}A(1,2|3,4) + pa_2 \mathrm{per}A(1,2|3,p+3) \\
&\quad + pa_3 \mathrm{per}A(1,2|3,2p+3) + \cdots + pa_k \mathrm{per}A(1,2|3,p(k-1)+3)]\mathrm{per}B \\
&= [(p-1)a_1[(p!)^k a_1^2 b_1^{p-2} b_2^p \cdots b_k^p] + pa_2[(p!)^k 2a_1a_2 b_1^{p-1} b_2^{p-1} \cdots b_k^p] \\
&\quad + pa_3[(p!)^k 2a_1a_3 b_1^{p-1} b_2^p b_3^{p-1} \cdots b_k^p] + \cdots \\
&\quad + pa_k[(p!)^k 2a_1a_k b_1^{p-1} b_2^p \cdots b_k^{p-1}]]\mathrm{per}B.
\end{aligned}$$

Since $\mathrm{per}B = \prod_{i=1}^k p! b_i^p$,

$$\frac{\mathrm{per}A(1,2|3,4)}{\mathrm{per}B} = \frac{(p!)^k a_1^2 b_1^{p-2} b_2^P \cdots b_k^p}{\prod_{i=1}^k p! b_i^p} = \frac{a_1^2}{b_1^2}, \tag{1}$$

and for $j = p+3, 2p+3, 3p+3, \ldots, p(k-1)+3$

$$\frac{\mathrm{per}A(1,2|3,j)}{\mathrm{per}B} = \frac{(p!)^k 2a_1a_j b_1^{p-1} b_j^{p-1} \prod_{i\neq j} b_i^p}{\prod_{i=1}^k p! b_i^p} = \frac{2a_1a_j}{b_1b_j}. \tag{2}$$

By (1) and (2), we can write that

$$\begin{aligned}
&\mathrm{per}A(1|3) \\
&= [(p-1)a_1\left(\frac{a_1}{b_1}\right)^2 + pa_2\left(\frac{2a_1a_2}{b_1b_2}\right) + \cdots + pa_k\left(\frac{2a_1a_k}{b_1b_k}\right)]\mathrm{per}B \\
&= [(p-1)a_1z_1^2 + 2pa_2z_1z_2 + \cdots + 2pa_kz_1z_k]\mathrm{per}B.
\end{aligned}$$

By similar method,

$$\begin{aligned}
&\mathrm{per}A(1|p(k-1)+3) \\
&= [2pa_1z_1z_k + 2pa_2z_2z_k + \cdots + 2pa_{k-1}z_{k-1}z_k + (p-1)a_kz_k^2]\mathrm{per}B.
\end{aligned}$$

By Lemma 1, we see that

$$\begin{aligned}
0 &= \mathrm{per}A(1|3) - \mathrm{per}A(1|p(k-1)+3) \\
&= [(z_1 - z_k)[(p-1)a_1 z_1 + 2pa_2 z_2 + \cdots \\
&\quad + 2pa_{k-1}z_{k-1} + (p-1)a_k z_k] - (p+1)z_1 z_k(a_1 - a_k)]\mathrm{per}B \\
&= [(z_1 - z_k)[(p-1)a_1 z_1 + 2p\sum_{i=2}^{k-1} a_i z_i + (p-1)a_k z_k] \\
&\quad -(p+1)z_1 z_k(a_1 - a_k)]\mathrm{per}B.
\end{aligned}$$

Since $z_1 - z_k = \frac{1}{pb_1 b_k}(a_1 - a_k)$ and by Lemma 3, we can obtain that

$$\begin{aligned}
0 &= \mathrm{per}A(1|3) - \mathrm{per}A(1|p(k-1)+3) \\
&= [\frac{(a_1 - a_k)}{pb_1 b_k}[(p-1)a_1 z_1 + 2p\sum_{i=2}^{k-1} a_i z_i + (p-1)a_k z_k] \\
&\quad -(p+1)z_1 z_k(a_1 - a_k)]\mathrm{per}B \\
&= (a_1 - a_k)[\frac{1}{pb_1 b_k}[(p-1)a_1 z_1 + 2p\sum_{i=2}^{k-1} a_i z_i + (p-1)a_k z_k] \\
&\quad -(p+1)z_1 z_k]\mathrm{per}B \\
&= (a_1 - a_k)[\frac{(p-1)}{pb_k} z_1^2 - (p+1)z_1 z_k + \frac{(p-1)}{pb_1} z_k^2 \\
&\quad + \frac{2}{b_1 b_k}\sum_{i=2}^{k-1} a_i z_i]\mathrm{per}B \\
&> (a_1 - a_k)[(p-1)z_1^2 - (p+1)z_1 z_k + (p-1)z_k^2 \\
&\quad + \frac{2}{b_1 b_k}\sum_{i=2}^{k-1} a_i z_i]\mathrm{per}B \\
&= (a_1 - a_k)[(p-1)z_k^2[(\frac{z_1}{z_k})^2 - \frac{p+1}{p-1}(\frac{z_1}{z_k}) + 1] + \frac{2}{b_1 b_k}\sum_{i=2}^{k-1} a_i z_i]\mathrm{per}B \\
&> 0,
\end{aligned}$$

which is a contradiction. Since A ia doubly stochastic matrix, $a_i = \frac{1}{pk}$ and $b_i = \frac{pk-2}{p^2 k}$ for all $i = 1, 2, \ldots, k$. Hence, we have $\mathrm{per}A = \frac{(2kp-p-1)(p!)^k (pk-2)^{pk-2}}{p^{2pk-1}\, k^{pk+1}}$. So far we have proved that for any minimizing matrix X over $\Omega(D)$,

$$(3) \qquad (I_2 \oplus J_p \oplus \cdots \oplus J_p)X(I_2 \oplus J_p \oplus \cdots \oplus J_p) = U.$$

It remains to show the uniqueness of the minimizing matrix over $\Omega(D)$. Suppose that $X = [x_{ij}] \in \Omega(D)$ is a minimizing matrix such that $X \neq U = [u_{ij}]$. By (3), $X[T_0|T_i] = U[T_0|T_i]$ and $X[T_i|T_0] = U[T_i|T_0]$ for $i = 1, 2, \ldots, k$. So, there is an $i > 1$ such that $X[T_i{}^*|T_i{}^*] \neq U[T_i{}^*|T_i{}^*]$, where $T_i{}^* = T_0 \cup T_i$. Without loss of generality, we may assume that $i = 1$.

Case (1). $X[T_0|T_1] \neq U[T_0|T_1]$ or $X[T_1|T_0] \neq U[T_1|T_0]$.

We may assume that $X[T_0|T_1] \neq U[T_0|T_1]$ by taking transposition if necessary, and also that $x_{13} > u_{13}$ since $\sum_{j=3}^{p} x_{ij} = \sum_{j=3}^{p} u_{ij}$ by (3). Let $C = [c_{ij}] := (I_2 \oplus J_p \oplus \cdots \oplus J_p)X(I_3 \oplus J_{p-1} \oplus \cdots \oplus J_p)$. Then, by Lemma 2, C is also a minimizing matrix over $\Omega(D)$, $c_{ij} = u_{ij}$ for all $(i,j) \in T_1{}^* \times T_1{}^*$ and

$$C[T_1{}^*|\, T_1{}^*] = \begin{bmatrix} \begin{matrix} \\ O_2 \\ \\ \end{matrix} & \begin{matrix} x_{13} & b & \cdots & b \\ b' & c & \cdots & c \end{matrix} \\ \begin{matrix} \\ \\ uJ_{p,2} \\ \\ \end{matrix} & \begin{matrix} u & v & \cdots & v \\ u & v & \cdots & v \\ \vdots & \vdots & \vdots & \vdots \\ u & v & \cdots & v \end{matrix} \end{bmatrix}$$

where $u = u_{13}$. Since $C[T_1{}^*|\, T_1{}^*](I_2 \oplus J_p) = U[T_1{}^*|\, T_1{}^*]$ and $x_{13} > u_{13}$, we see that $c > u_{13}$ and $b < u_{13}$. Now,

$$0 = \mathrm{per}C(1|3) - \mathrm{per}C(2|3) = (p-1)(c-b)\mathrm{per}C(1,2|3,4) > 0,$$

a contradiction.

Case (2). $X[T_0|T_1] = U[T_0|T_1]$, $X[T_1|T_0] = U[T_1|T_0]$ but $X[T_1|T_1] \neq U[T_1|T_1]$.

In this case, we may assume that $x_{33} > u_{33}$. Let $H = [h_{ij}] :=(I_3 \oplus J_{p-1} \oplus \cdots \oplus J_p)X(I_3 \oplus J_{p-1} \oplus \cdots \oplus J_p)$. Then, H is also a minimizing matrix and $h_{ij} = u_{ij}$ for all $(i,j) \notin T_1{}^* \times T_1{}^*$ and

$$H[T_1{}^*|\, T_1{}^*] = \begin{bmatrix} O_2 & uJ_{2,p} \\ uJ_{p,2} & \begin{matrix} x_{33} & r & \cdots & r \\ r & t & \cdots & t \\ \vdots & \vdots & \vdots & \vdots \\ r & t & \cdots & t \end{matrix} \end{bmatrix}$$

where $u = u_{13}$.

Since $x_{33} > u_{33}$, we see that $r < u_{33}$ and $t > u_{33}$ from (3) again. Thus, as before, we can show that $0 = \text{per}H(3|3) - \text{per}H(4|3) = (p-1)(t-r)\text{per}H(3,4|3,4) > 0$, a contradiction. $\square$

We recall that $D^* = D + \begin{bmatrix} J_2 & O \\ O & O \end{bmatrix}$.

From now on, we talk to the minimum permanent and the minimizing matrix over $\Omega(D^*)$.

Before we start the proof of some theorems and lemmas, let

$$A = \left[\begin{array}{c|ccccc} zJ_{2,2} & a_1J_{2,p} & & & & a_kJ_{2,p} \\ \hline a_1J_{2,p} & b_1J_p & & & & \\ & & b_2J_p & O & & \\ & & & & & \\ & O & & \ddots & & \\ a_kJ_{2,p} & & & & & b_kJ_p \end{array}\right]$$

for some real numbers a_i, b_i, z and $k \geq 4$.

LEMMA 5. *Let $z_i = \frac{a_i}{b_i}$. Then all z_i are the same value for all $i = 1, 2, \ldots, k$.*

Proof. Let X be a minimizing matrix over $\Omega(D^*)$. By Lemma 2, A^* is also a minimizing matrix over $\Omega(D^*)$. We may assume that $a_1 \geq a_2 \geq \cdots \geq a_k$ and let $z_i = \frac{a_i}{b_i}$ for all $i = 1, 2, \ldots, k$. Then, for the contrary, we assume that $a_1 > a_k$.

$$\begin{aligned}
& \text{per}A^*(1|3) \\
= \; & [2z\text{per}A^*(1,2|1,3) + (p-1)a_1\text{per}A^*(1,2|3,4) \\
& + pa_2\text{per}A^*(1,2|3,p+3) + pa_3\text{per}A^*(1,2|3,2p+3) + \cdots \\
& + pa_k\text{per}A^*(1,2|3,p(k-1)+3)]\text{per}B \\
= \; & [2z[(p!)^k a_1 {b_1}^{p-1} {b_2}^p \cdots {b_k}^p] + (p-1)a_1[(p!)^k a_1^2 b_1^{p-2} {b_2}^p \cdots {b_k}^p] + \cdots \\
& + pa_k[(p!)^k a_1 a_k b_1^{p-1} b_2^p \cdots b_k^{p-1}]]\text{per}B.
\end{aligned}$$

Since $\text{per}B = \prod_{i=1}^k p! b_i^p$,

$$\frac{\text{per}A^*(1,2|3,4)}{\text{per}B} = \frac{(p!)^k a_1^2 b_1^{p-2} b_2^P \cdots b_k^p}{\prod_{i=1}^k p! b_i^p} = \frac{a_1^2}{b_1^2} \tag{4}$$

and

$$\frac{\text{per}A^*(1,2|1,3)}{\text{per}B} = \frac{(p!)^k a_1 b_1^{p-1} b_2^P \cdots b_k^p}{\prod_{i=1}^k p! b_i^p} = \frac{a_1}{b_1}. \tag{5}$$

Also, for $j = p+3, 2p+3, 3p+3, \ldots, p(k-1)+3$,

$$(6)\qquad \frac{\mathrm{per}A^*(1,2|3,j)}{\mathrm{per}B} = \frac{(p!)^k 2a_1a_j b_1^{p-1} b_j^{p-1}\prod_{i\neq j} b_i^p}{\prod_{i=1}^k p! b_i^p} = \frac{2a_1a_j}{b_1b_j}.$$

By (4), (5) and (6), we obtain that

$$\begin{aligned}
&\mathrm{per}A^*(1|3)\\
=\ & [2z(\frac{a_1}{b_1}) + (p-1)a_1(\frac{a_1}{b_1})^2 + pa_2(\frac{2a_1a_2}{b_1b_2}) + \cdots + pa_k(\frac{2a_1a_k}{b_1b_k})]\mathrm{per}B\\
=\ & [2zz_1 + (p-1)a_1z_1^2 + 2pa_2z_1z_2 + \cdots + 2pa_kz_1z_k]\mathrm{per}B.
\end{aligned}$$

By similar method,

$$\begin{aligned}
&\mathrm{per}A^*(1|p(k-1)+3)\\
=\ & [2zz_k + 2pa_1z_1z_k + \cdots + 2pa_{k-1}z_{k-1}z_k + (p-1)a_kz_k^2]\mathrm{per}B.
\end{aligned}$$

By Lemma 1, we see that

$$\begin{aligned}
0 &= \mathrm{per}A^*(1|3) - \mathrm{per}A^*(1|p(k-1)+3)\\
&= [(z_1 - z_k)[2z + (p-1)a_1z_1 + 2pa_2z_2 + \cdots + 2pa_{k-1}z_{k-1}\\
&\quad +(p-1)a_kz_k] - (p+1)(a_1 - a_k)z_1z_k]\mathrm{per}B\\
&= [(z_1 - z_k)[2z + (p-1)a_1z_1 + 2p\sum_{i=2}^{k-1} a_iz_i + (p-1)a_kz_k]\\
&\quad -(p+1)z_1z_k(a_1 - a_k)]\mathrm{per}B.
\end{aligned}$$

Since $z_1 - z_k = \frac{1}{pb_1b_k}(a_1 - a_k)$ and by Lemma 3, we can obtain that

$$\begin{aligned}
0 &= \mathrm{per}A^*(1|3) - \mathrm{per}A^*(1|p(k-1)+3)\\
&= [\frac{(a_1 - a_k)}{pb_1b_k}[2z + (p-1)a_1z_1 + 2p\sum_{i=2}^{k-1} a_iz_i + (p-1)a_kz_k]\\
&\quad -(p+1)z_1z_k(a_1 - a_k)]\mathrm{per}B\\
&= (a_1 - a_k)[\frac{1}{pb_1b_k}[2z + (p-1)a_1z_1 + 2p\sum_{i=2}^{k-1} a_iz_i + (p-1)a_kz_k]\\
&\quad -(p+1)z_1z_k]\mathrm{per}B
\end{aligned}$$

$$\begin{aligned}
&= (a_1 - a_k)[\frac{2z}{pb_1b_k} + \frac{(p-1)}{pb_k}z_1^2 - (p+1)z_1z_k + \frac{(p-1)}{pb_1}z_k^2 \\
&\quad + \frac{2}{b_1b_k}\sum_{i=2}^{k-1} a_iz_i]\mathrm{per}B \\
&> (a_1 - a_k)[\frac{2z}{pb_1b_k} + (p-1)z_1^2 - (p+1)z_1z_k + (p-1)z_k^2 \\
&\quad + \frac{2}{b_1b_k}\sum_{i=2}^{k-1} a_iz_i]\mathrm{per}B \\
&= (a_1 - a_k)[\frac{2z}{pb_1b_k} + (p-1)z_k^2[(\frac{z_1}{z_k})^2 - \frac{p+1}{p-1}(\frac{z_1}{z_k}) + 1] \\
&\quad + \frac{2}{b_1b_k}\sum_{i=2}^{k-1} a_iz_i]\mathrm{per}B \\
&> 0,
\end{aligned}$$

which is a contradiction from $a_1 > a_k$. Therefore $a_1 = a_k$ and hence, a_i are the same value for $i = 1, 2, \ldots, k$. Therefore, we obtain the result. □

For the main theorem, we need the following Lemma:

LEMMA 6. *For all i, $\frac{a_i}{b_i} \leq \frac{1}{2}$.*

Proof. By Lemma 5, we have all $\frac{a_i}{b_i}$ are the same value. So we may put $\frac{a_i}{b_i} = z$ and assume that $z > \frac{1}{2}$. Now, since A^* is row stochastic matrix, we obtain

$$1 = 2z + \sum_{i=1}^{k} pa_i.$$

By hypothesis, we have

$$\begin{aligned}
1 &= 2z + \sum_{i=1}^{k} pa_i = 2z + p\sum_{i=1}^{k} a_i \\
&> 2z + \frac{p}{2}\sum_{i=1}^{k} b_i.
\end{aligned} \tag{7}$$

Since the A^* is column stochastic matrix, we can change the equation (7)

$$
\begin{aligned}
1 &> 2z + \frac{p}{2}\sum_{i=1}^{k}\frac{(1-2a_i)}{p} \\
&= 2z + \frac{1}{2}[\sum_{i=1}^{k}(1-2a_i)] \qquad (8)\\
&= 2z + \frac{1}{2}[k - 2\sum_{i=1}^{k} a_i].
\end{aligned}
$$

Then, from the first and the last line in the equation (8), we have

$$
\begin{aligned}
2z &< 1 - \frac{1}{2}[k - 2\sum_{i=1}^{k} a_i] \\
&< 1 - \frac{k}{2} + 1 = \frac{4-k}{2} < 0,
\end{aligned}
$$

a contradiction. Therefore $\frac{a_i}{b_i} \leq \frac{1}{2}$ for all $i = 1, 2, \ldots, k$. □

THEOREM 7. *For positive integers $p \geq 3$ and k, we have* $\mathrm{Min}(D^*) = \mathrm{Min}(D)$.

Proof. Let X be a minimizing matrix over $\Omega(D^*)$. By Lemma 2, A^* is also a minimizing matrix over $\Omega(D^*)$. Then,

$$
\begin{aligned}
& \mathrm{per}A^*(1|1) \\
= \ & [z\mathrm{per}A^*(1,2|1,2) + pa_1\mathrm{per}A^*(1,2|3,4) + pa_2\mathrm{per}A^*(1,2|3,p+3) \\
& + pa_3\mathrm{per}A^*(1,2|3,2p+3) + \cdots \\
& + pa_k\mathrm{per}A^*(1,2|3,p(k-1)+3)]\mathrm{per}B \\
= \ & [z[(p!)^k b_1^p b_2^p \cdots b_k^p] + pa_1[(p!)^k b_1^{p-1} b_2^p \cdots b_k^p] + pa_2[(p!)^k b_1^p b_2^{p-1} \cdots b_k^p] \\
& + pa_3[(p!)^k b_1^p b_2^p b_3^{p-1} \cdots b_k^p] + \cdots + pa_k[(p!)^k b_1^p b_2^p \cdots b_k^{p-1}]]\mathrm{per}B \\
= \ & [z + pa_1 z_1 + \cdots + pa_k z_k]\mathrm{per}B.
\end{aligned}
$$

By the proof of Lemma 5, we get

$$
\mathrm{per}A^*(1|3) = [2zz_1 + (p-1)a_1 z_1^2 + 2pa_2 z_1 z_2 + \cdots + 2pa_k z_1 z_k]\mathrm{per}B.
$$

If $A^*[1,2|1,2] \neq O$, then Lemma 1 show that per$A^*(1|1) =$ per$A^*(1|3) =$ perA^*. But then

$$(9) \quad \begin{aligned} 0 &= \text{per}A^*(1|3) - \text{per}A^*(1|1) \\ &= [z(2z_1 - 1) + a_1 z_1((p-1)z_1 - p) + (2z_1 - 1)\sum_{i=2}^{k} p a_i z_i]\text{per}B. \end{aligned}$$

However, Lemma 6 yield that the right hand side of (9) is negative, which is a contradiction. Hence, we have $A^*[1,2|1,2] = O$, that is Min$(D^*) =$ Min(D). □

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