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RELATIVE VOLUME COMPARISON WITH INTEGRAL RADIAL CURVATURE BOUNDS

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ABSTRACT. In this paper, we generalize the Bishop-Gromov volume comparison theorem by considering an integral bound for the part of the radial Ricci curvature which lies below a given smooth function. We also establish a compactness theorem from this result.

1. Introduction

In 1997, P. Petersen and G. Wei generalized the classical volume comparison theorem to a situation where one only has an integral bound for the part of Ricci curvature which lies below a given number ([5]).

Like many other results on the volume comparison, they compared the volumes of concentric metric balls with those on the complete simply connected space forms of constant sectional curvature.

In this paper, we generalize their result to a situation where our model spaces do not have metrics of constant sectional curvature but their curvature may change sign.

For the construction of our model space, we first choose a constant $0 < l \leq \infty$ and a smooth function $K : [0, l) \to \mathbb{R}$ which are associated with the model space M^* and $x^* \in M^*$ as follows.

A complete Riemannian n- manifold M^* with the base point $x^* \in M^*$ is said to have the *radial sectional curvature* $K : [0, l) \to \mathbb{R}$ *at* x^* if the following is satisfied:

- The tangential cut locus $C_{x^*} \subset M^*_{x^*}$ at x^* is the sphere $\mathrm{S}^{n-1}(l)$ with radius l if $l < \infty$ and $C_{x^*} = \phi$ if $l = \infty$.

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- Along every geodesic $\gamma^* : [0, l) \to M^*$ emanating from $x^* \in M^*$, the sectional curvature satisfies

$$K_{M^*}(\dot{\gamma^*}(t), X) = K(t), \forall t \in (0, l], \ \forall X \in T_{\gamma^*(t)}M^*, \ X \perp \dot{\gamma^*}(t).$$

- If $l < \infty$, then $\gamma^*(l)$ is the first conjugate point to x^* along γ^*.

Several properties of the model space M^* can be found in [6]. Now suppose that we are given a complete Riemannian manifold M and $x \in M$. Let $c(\theta)$ be the *cut distance in the direction* $\theta \in \mathrm{S}_x M = \{\theta \in T_x M \mid \|\theta\| = 1\}$. We define $\omega(t, \theta)$ for $0 < t < c(\theta)$, $\theta \in \mathrm{S}_x M \subset T_x M$ by

$$\mathrm{dvol} = \omega(t, \theta) dt d\theta_{n-1},$$

where $d\theta_{n-1}$ is the standard volume element on the unit sphere $\mathrm{S}^{n-1} (= \mathrm{S}_x M)$. We define $\omega(t, \theta)$ to be zero for $t \geq c(\theta)$.

We then consider

$$k(p, K) = \int_{\mathrm{S}^{n-1}} \int_0^{c(\theta)} \max\{0, (n-1)K(t) - \mathrm{Ric}_-\}^p \omega(t, \theta) dt d\theta_{n-1},$$

where the function Ric_- is the lowest eigenvalue for the Ricci tensor of M.

Our main result is the following generalization of a Theorem in [5].

THEOREM 1.1. *Let (M^*, x^*) be a complete Riemannian n-manifold with the radial sectional curvature $K : [0, l) \to \mathbb{R}$ at the base point $x^* \in M^*$ and M be a complete Riemannian n-manifold with $x \in M$.*

Then given $p > \frac{n}{2}$ and $R < l$, there exists a constant $C(n, p, K, R)$ which is nondecreasing in R such that when $r < R$ we have

$$\left(\frac{\mathrm{vol}B(x, R)}{\mathrm{vol}B(x^*, R)}\right)^{\frac{1}{2p}} - \left(\frac{\mathrm{vol}B(x, r)}{\mathrm{vol}B(x^*, r)}\right)^{\frac{1}{2p}} \leq C(n, p, K, R) k(p, K)^{\frac{1}{2p}},$$

where $B(x, r)$ denotes the geodesic ball of radius r centered at x in M.

In case of the space form as a model space, the proof of the volume comparison theorem tends to depend heavily on the property of the mean curvature function on the comparison space with constant curvature for some analytic reasons.

However, the mean curvature function h_K in our case may be more complicated and more careful observation is needed to overcome the complexity of the shape of the mean curvature function in our case.

By using the same arguments as in the Theorem 1.7 of [5], we can also obtain a generalized precompactness theorem for manifolds with almost maximal volume.

THEOREM 1.2. *Under the same conditions as the above theorem and for given $c > 0$, one can find $\epsilon = \epsilon(n, p, K, r^2) > 0$ and $\delta = \delta(n, K, r^2) > 0$ such that the class of complete Riemannian n-manifolds with*

$$\mathrm{vol}B(x, r) \geq (1 - \delta)\mathrm{vol}B(x^*, r) \text{ for all } x \in M,$$

and

$$\int_{B(x,r)} \|\mathrm{Ric}\|^p \leq C, \;\; k(p, K) \leq \epsilon$$

is precompact in the C^α-topology ($\alpha < 2 - \frac{n}{p}$).

Since Theorem 1.2 follows immediately just by combining the above Theorem 1.1 with the arguments for the proof of Theorem 1.7 in [5], we omit the proof.

2. Proof of Theorem 1.1

Our proof follows basically from the arguments of section 2 and section 4 in [4].

First, let us introduce some notation.

The metric of the model space M^* with the radial sectional curvature $K : [0, l) \to \mathbb{R}$ at a base point x^* is expressed by using the geodesic polar coordinates around x^* as follows.

$$(ds^*)^2 = dt^2 + \omega_K^2 d\theta_{n-1}^2,$$

where $d\theta_{n-1}^2$ is the canonical metric on the standard unit sphere S^{n-1} and $\omega_K : [0, l) \to \mathbb{R}$ satisfies the Jacobi equation.

$$\omega_K'' + K\omega_K = 0,$$
$$\omega_K(0) = 0, \;\; \omega_K'(0) = 1.$$

If we let $h_K = \frac{\omega_K'}{\omega_K}$, then we have the following equation ([1]).

$$h_K' + \frac{h_K^2}{n-1} = -(n-1)K.$$

For a given complete Riemannian n-manifold (M, g), $\omega(t, \theta)$ is defined as in the previous section, that is, $\mathrm{dvol} = \omega(t, \theta) dt d\theta_{n-1}$ in the geodesic polar coordinates around $x \in M$.

In the similar way as in the model space, we let $h(t, \theta) = \frac{\omega\prime(t,\theta)}{\omega(t,\theta)}$, where $\omega\prime(t, \theta)$ is the derivation of $\omega(t, \theta)$ with respect to the first variable t and we also have the inequality

$$h' + \frac{h^2}{n-1} \leq -\mathrm{Ric}_-.$$

In fact, we know that h is the mean curvature of the distance sphere around x.

In order to generalize the mean curvature estimates in [4], we define

$$\psi(t,\theta) = (h(t,\theta) - h_K(t))_+,$$
$$\rho(t,\theta) = ((n-1)K(t) - \mathrm{Ric}_-(t,\theta))_+,$$

where $u_+ = \max(0,u)$.

From the above equality for h_K and inequality for h, we know that ψ and ρ satisfy

$$\psi' + \frac{\psi^2}{n-1} + \frac{2h_K\psi}{n-1} \le \rho. \tag{2.1}$$

We are now in a position to state our main lemma which generalizes Theorem 2.1 in [4] to our situation.

LEMMA 2.1. *With notations as above, we have for all $n \ge 2$, $p > \frac{n}{2}$ an estimate of the form*

$$\int_0^r \psi^{2p}(t,\theta)\omega dt \le C_K(n,p,r) \int_0^r \rho^p(t,\theta)\omega dt,$$

where $C_K(n,p,r)$ is an explicit constant depending only on the variables indicated.

REMARK. In the case $l < \infty$, we always suppose $r < l$ implicitly in the above lemma.

Proof. We shall follow the ideas of the proof of Theorem 2.1 in [4].

As in [4], we use the inequality (2.1). If we multiply this inequality by $\psi^{2p-2}\omega$ and integrate from 0 to r, then we obtain the following inequality. (For details, see the proof of Theorem 2.1 in [4])

$$\begin{aligned}&\left(\frac{1}{n-1} - \frac{1}{2p-1}\right)\int_0^r \psi^{2p}\omega \\ &\le \int_0^r \rho\psi^{2p-2}\omega \\ &\quad - \min_{t\in[0,r]}\{h_K(t)\}\left(\frac{2}{n-1} - \frac{1}{2p-1}\right)\int_0^r \psi^{2p-1}\omega.\end{aligned}$$

If $\min_{t\in[0,r]}\{h_K(t)\} \ge 0$, then by using the Hölder inequality, one immediately gets a bound which we want:

$$\int_0^r \psi^{2p}\omega \le \left(\frac{1}{n-1} - \frac{1}{2p-1}\right)^{-p} \int_0^r \rho^p\omega.$$

From now on, assume that $h_K^{\min}(K) := \min_{t\in[0,r]}\{h_K(t)\} < 0$ and let

$$H^+ := \{t \in (0,r) : h_K(t) \geq 0\},$$
$$H^- := \{t \in (0,r) : h_K(t) < 0\}.$$

Note that $(0,r) = H^+ \bigcup H^-$.

We shall now estimate the value $(\psi^{2p-1}\omega)(t)$ for $t \in (0,r) = H^+ \bigcup H^-$.

We first consider the case $t \in H^+$.

Put $a = \sup\{s < t : h_K(s) < 0\}$. (If $\{s < t : h_K(s) < 0\}$ is empty, then we put $a = 0$.) Then it is easy to check that $h_K(s) > 0$ on (a,t) and that $h_K(a) = 0$.

To estimate $(\psi^{2p-1}\omega)(a)$, we introduce two auxiliary functions $\tilde{h}_K := (h_K)_+$ and $\tilde{\psi} := (h - \tilde{h}_K)_+$.

It is easy to check that we still have

$$\tilde{\psi}' + \frac{\tilde{\psi}^2}{n-1} + \frac{2\tilde{h}_K\tilde{\psi}}{n-1} \leq \rho,$$
$$\tilde{\psi}(0) = 0.$$

Multiplying this inequality by $\tilde{\psi}^{2p-2}\omega$ and integrating from 0 to a, we obtain

$$\frac{1}{2p-1}(\tilde{\psi}^{2p-1}\omega)(a) + \left(\frac{1}{n-1} - \frac{1}{2p-1}\right)\int_0^a \tilde{\psi}^{2p}\omega$$
$$+ \left(\frac{2}{n-1} - \frac{1}{2p-1}\right)\int_0^a \tilde{\psi}^{2p-1}\tilde{h}_K\omega \leq \int_0^a \rho\tilde{\psi}^{2p-2}\omega.$$

Note that $\tilde{\psi}(a) = \psi(a)$ since $h_K(a) = 0$ and that $\tilde{\psi} \leq \psi$ since $h_K \leq \tilde{h}_K$.

So from the above inequality, we have

$$\frac{1}{2p-1}(\psi^{2p-1}\omega)(a) \leq \int_0^a \rho\tilde{\psi}^{2p-2}\omega$$
$$\leq \left(\int_0^a \rho^p\omega\right)^{\frac{1}{p}}\left(\int_0^a \psi^{2p}\omega\right)^{1-\frac{1}{p}}.$$

Now multiplying the inequality (2.1) by $\psi^{2p-2}\omega$ and integrating from a to t, we obtain

$$\frac{1}{2p-1}(\psi^{2p-1}\omega)(t) - \frac{1}{2p-1}(\psi^{2p-1}\omega)(a) + \left(\frac{1}{n-1} - \frac{1}{2p-1}\right)\int_a^t \psi^{2p}\omega$$
$$+ \left(\frac{2}{n-1} - \frac{1}{2p-1}\right)\int_a^t \psi^{2p-1}h_K\omega \leq \int_a^t \rho\psi^{2p-2}\omega.$$

Thus we have

$$\frac{1}{2p-1}(\psi^{2p-1}\omega)(t) \le \frac{1}{2p-1}(\psi^{2p-1}\omega)(a) + \int_a^t \rho\psi^{2p-2}\omega$$
$$\le 2(\int_0^r \rho^p\omega)^{\frac{1}{p}}(\int_0^r \psi^{2p}\omega)^{1-\frac{1}{p}}.$$

We next consider the case where $t \in H^-$.

Similarly, we let $c := \sup\{s < t : h_K(s) > 0\}$. Note that $h_K(c) = 0$. Then by the same arguments as before, we obtain

$$\frac{1}{2p-1}(\psi^{2p-1}\omega)(c) \le (\int_0^r \rho^p\omega)^{\frac{1}{p}}(\int_0^r \psi^{2p}\omega)^{1-\frac{1}{p}}.$$

Now note that if in the inequality (2.1) we drop the ψ^2 term and multiply through by ψ^{2p-2}, then we have

$$\psi'\psi^{2p-2} + \frac{2}{n-1}\psi^{2p-1}h_K^{\min}(r) \le \rho\psi^{2p-2}.$$

We multiply this by $(2p-1)$ and the integrating factor

$$\phi(t) = \exp(h_K^{\min}(r)\frac{2(2p-1)}{n-1}t)$$

and write this as

$$(\phi\psi^{2p-1})' \le (2p-1)\phi\rho\psi^{2p-2}$$
$$\le (2p-1)\rho\psi^{2p-2}.$$

If we multiply this inequality by ω and integrate from c to t, we get

$$(2.2) \qquad (\phi\psi^{2p-1}\omega)|_c^t - \int_c^t h\phi\psi^{2p-1}\omega \le (2p-1)\int_c^t \rho\psi^{2p-2}\omega,$$

which can be reduced to

$$(\phi\psi^{2p-1}\omega)|_c^t \le (2p-1)(\int_c^t h_+\psi^{2p-1}\omega + \int_c^t \rho\psi^{2p-2}\omega).$$

Recall that

$$\int_0^r \tilde{\psi}^{2p}\omega \le \left(\frac{1}{n-1} - \frac{1}{2p-1}\right)^{-p}\int_0^r \rho^p\omega.$$

This gives the following inequality.

$$\int_c^t h_+\psi^{2p-1}\omega \le (\int_0^r \tilde{\psi}^{2p}\omega)^{\frac{1}{2p}}(\int_0^r \psi^{2p}\omega)^{1-\frac{1}{2p}}$$
$$\le \left(\frac{1}{n-1} - \frac{1}{2p-1}\right)^{-\frac{1}{2}}(\int_0^r \rho^p\omega)^{\frac{1}{2p}}(\int_0^r \psi^{2p}\omega)^{1-\frac{1}{2p}}.$$

Consequently, we obtain from (2.3) that

$$\begin{aligned}
&(\phi\psi^{2p-1}\omega)(t)\\
&\le (\psi^{2p-1}\omega)(c)\\
&\quad + (2p-1)(\int_c^t h_+\psi^{2p-1}\omega + \int_c^t \rho\psi^{2p-2}\omega)\\
&\le (2p-1)(\int_0^r \rho^p\omega)^{\frac{1}{p}}(\int_0^r \psi^{2p}\omega)^{1-\frac{1}{p}}\\
&\quad + (2p-1)\left(\frac{1}{n-1}-\frac{1}{2p-1}\right)^{-\frac{1}{2}}(\int_0^r \rho^p\omega)^{\frac{1}{2p}}(\int_0^r \psi^{2p}\omega)^{1-\frac{1}{2p}}\\
&\quad + (2p-1)(\int_0^r \rho^p\omega)^{\frac{1}{p}}(\int_0^r \psi^{2p}\omega)^{1-\frac{1}{p}}.
\end{aligned}$$

We definitely get an estimate of $(\psi^{2p-1}\omega)(t)$ for $t\in(0,r)$ as follows.

$$\begin{aligned}
(\psi^{2p-1}\omega)(t) \le C_1(p,n,r,K)\{&(\int_0^r \rho^p\omega)^{\frac{1}{p}}(\int_0^r \psi^{2p}\omega)^{1-\frac{1}{p}}\\
&+(\int_0^r \rho^p\omega)^{\frac{1}{2p}}(\int_0^r \psi^{2p}\omega)^{1-\frac{1}{2p}}\}.
\end{aligned} \tag{2.3}$$

The proof of Theorem 2.1 in [4] can now be applied. (For details, See pp. 280–281 in [4])

So, with the inequality (2.4), we return to (2.2) and conclude the desired result.

$$\int_0^r \psi^{2p}\omega \le C(p,n,r,K)\int_0^r \rho^p\omega.$$

□

Now we can prove Theorem 1.1 by the similar arguments in the section 4 of [4] as follows.

Note first that as in the lemma 2.1 of [5], we see that the volume ratio satisfies

$$y'(r) \le C(n,K,r)(\int_{B(x,r)} \psi^{2p}\mathrm{dvol})^{\frac{1}{2p}}(\mathrm{vol}B(x^*,r))^{-\frac{1}{2p}}y(r)^{1-\frac{1}{2p}},$$

where

$$y(r) = \frac{\mathrm{vol}B(x,r)}{\mathrm{vol}B(x^*,r)},$$

$$C(n,K,r) = \max_{t\in[0,r]}\{\frac{\int_0^t C(t,s)ds}{\int_0^t \omega_K(s)ds}\},$$

and

$$C(w,u) = \frac{\omega_K(u)\omega_K(w)}{\min_{u\le s\le w}\{\omega_K(s)\}} \text{ for } u \le w.$$

Using lemma 2.1, we see that on $[0, R]$, the volume ratio $y(r)$ satisfies

$$y' \le Mf(r)y^{1-\frac{1}{2p}},\ M = C(n,p,K,R)\left(\int_{B(x,R)} \rho^p \mathrm{dvol}\right)^{\frac{1}{2p}}.$$

Integrating this inequality over $[r, R]$, we obtain the desired result. (For details, see the section 4 in [4].)

References

[1] R. Bishop and R. Crittenden, *Geometry of manifolds*, Academic Press, New York and London, 1964.
[2] Y. Machigashira, *Manifolds with pinched radial curvature*, Proc. Amer. Math. Soc. **118** (1993), no. 3, 979–985.
[3] Y. Machigashira and Katsuhiro Shiohama, *Riemannian manifolds with positive radial curvature*, Japan. J. Math. **19** (1993), no. 2, 419–430.
[4] P. Petersen and C. Sprouse, *Integral curvature bounds, distance estimates and applications*, J. Differential Geom. **50** (1998), no. 2, 269–298.
[5] P. Petersen and G. Wei, *Relative volume comparison with integral curvature bounds*, Geom. Funct. Anal. **7** (1997), no. 6, 1031–1045.
[6] Katsuhiro Shiohama, *Comparison Theorems for manifolds with radial curvature bounded below*, Josai Math. Monogr. **3** (2001), 81–91.

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