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A UNIFORM CONVERGENCE THEOREM FOR APPROXIMATE HENSTOCK-STIELTJES INTEGRAL

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Abstract. In this paper, we introduce, for each approximate distribution $\tilde{T}$ of $[a,b]$, the approximate Henstock-Stieltjes integral with value in Banach spaces. The Henstock integral is a special case of this where $\tilde{T} = \{(\tau, [a,b]) : \tau \in [a,b]\}$. This new concept generalizes Henstock integral and abstract Perron-Stieltjes integral. We establish a uniform convergence theorem for approximate Henstock-Stieltjes integral, which is an improvement of the uniform convergence theorem for Perron-Stieltjes integral by Schwabik [3].

1. Approximate Henstock-Stieltjes integral

Assume that X, Y and Z are Banach spaces and that there is a bilinear mapping $B : X \times Y \longrightarrow Z$. We use the short notation $xy = B(x,y)$ for the values of the bilinear form B for $x \in X, y \in Y$ and assume that

$$\|xy\|_Z \leq \|x\|_X \|y\|_Y.$$

Triples of Banach spaces X, Y, Z with these properties are called *bilinear triples* and they are denoted by $\mathcal{B} = (X, Y, Z)$(see [3]). For the case $\mathcal{B} = (\mathbb{R}, \mathbb{R}, \mathbb{R})$, we always assume $B(x,y) = xy$ (product).

Let $x : [a,b] \times [a,b] \longrightarrow X$. A tagged interval $(\tau, [c,d])$ consists of an interval $[c,d] \subset [a,b]$ and a point $\tau \in [c,d]$, and we write $x(\tau, d) - x(\tau, c)$ simply by $x(\tau, [c,d])$. The tagged interval $(\tau, [c,d])$ is *δ-fine* for a *gauge* δ (a positive function on [a,b]) if

$$[c,d] \subset (\tau - \delta(\tau), \tau + \delta(\tau)).$$

Let $P = \{(\tau_j, [c_j, d_j]) : 1 \leq j \leq m\}$ be a finite collection of non-overlapping tagged intervals in $[a,b]$. If $(\tau_j, [c_j, d_j])$ is δ-fine for each i and $\cup_{j=1}^{m} [c_j, d_j] = [a,b]$, then P is called a *δ-fine partition* of $[a,b]$.

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Assume that $\mathcal{B} = (X, Y, Z)$ is a bilinear triple and that functions $x : [a,b] \times [a,b] \longrightarrow X$ and $y : [a,b] \longrightarrow Y$ are given. We say that the Henstock-Stieltjes (shortly, H-) integral exists if there exists an element $I \in Z$ such that for every $\varepsilon > 0$, there is a gauge δ on $[a,b]$ such that for Riemann sums

$$S(dx, y, P) \equiv \sum_{j=1}^{m} x(\tau_j, J_j) y(\tau_j),$$

we have

$$\|S(dx, y, P) - I\|_Z < \varepsilon$$

provided $P = \{(\tau_j, J_j = [c_j, d_j]) : j = 1, ..., m\}$ is a δ-fine partition of $[a,b]$. We denote $I = (H) \int_a^b d[x(\tau,t)]y(\tau)$.

Note that for a Henstock integrable function $y : [a,b] \longrightarrow \mathbb{R}$ the Hensock integral $\int_a^b y(\tau)d\tau$ is equal to the Henstock-Stieltjes integral $(H) \int_a^b d[x(\tau,t)]y(\tau)$ if we take $x(\tau,t) = t$ (see [1]).

Let $\tilde{T}_\tau \subset [a,b]$ be a measurable set with $\tau \in \tilde{T}_\tau$ and $d_\tau \tilde{T}_\tau = 1$ (the density of $\tilde{T}_\tau$ at τ is 1), i.e. for the Lebesgue measure μ

$$d_\tau \tilde{T}_\tau = \lim_{h \to 0^+} \frac{\mu(\tilde{T}_\tau \cap (\tau - h, \tau + h))}{2h} = 1$$

if τ is not an end point of $\tilde{T}_\tau$, or

$$d_\tau \tilde{T}_\tau = \lim_{h \to 0^+} \frac{\mu(\tilde{T}_\tau \cap (\tau, \tau + h))}{h} = 1$$

$$(d_\tau \tilde{T}_\tau = \lim_{h \to 0^+} \frac{\mu(\tilde{T}_\tau \cap (\tau - h, \tau))}{h} = 1)$$

if τ is an end point of $\tilde{T}_\tau$. Such a collection $\tilde{T} = \{\tilde{T}_\tau : d_\tau \tilde{T}_\tau = 1,\ \tau \in [a,b]\}$ is called an *approximate distribution* on $[a,b]$. If a tagged interval $(\tau, [c,d])$ is δ-fine for a gauge δ and $\{c, \tau, d\} \subset \tilde{T}_\tau$, then we call $(\tau, [c,d])$ is $\delta(\tilde{T}_\tau)$-fine. A tagged partition $P = \{(\tau_j, [c_j, d_j]) : 1 \le j \le m\}$ of $[a,b]$ is said to be *$\delta(\tilde{T})$-fine* if $(\tau_j, [c_j, d_j])$ is $\delta(\tilde{T}_{\tau_j})$-fine for every $j = 1, 2, ..., m$. Now we define an approximate Henstock-Stieltjes integral.

Definition 1.1. Assume that $\mathcal{B} = (X, Y, Z)$ is a bilinear triple and that functions $x : [a,b] \times [a,b] \longrightarrow X$ and $y : [a,b] \longrightarrow Y$ are given. Let $\tilde{T}$ be an approximate distribution on the interval $[a,b]$. We say that *the approximate Henstock-Stieltjes (shortly, AH-) integral I with respect to $\tilde{T}$* exists if there is an element $I \in Z$ such that for every $\varepsilon > 0$ there is a gauge δ on $[a,b]$ such that

$$\|S(dx, y, P) - I\|_Z < \varepsilon$$

provided P is a $\delta(\tilde{T})$-fine partition of $[a,b]$. We denote $I = (\tilde{T} - AH)\int_a^b d[x(\tau,t)]y(\tau)$.

Let $\tilde{T}^0 = \{\tilde{T}_\tau = [a,b] \text{ for all } \tau \in [a,b]\}$. Then $\tilde{T}^0$ is an approximate distribution on $[a,b]$. Clearly, the AH-integral $(\tilde{T}^0 - AH)\int_a^b d[x(\tau,t)]y(\tau)$ is the same as the H-integral $(H)\int_a^b d[x(\tau,t)]y(\tau)$. In this note, we study some properties of AH-integrals, and establish a uniform convergence theorem for approximate Henstock-Stieltjes integral, which is an improvement of the uniform convergence theorem for Perron-Stieltjes integral by Schwabik [3].

There is a function which is not H-integrable (i.e., not AH-integrable with respect to $\tilde{T}^0$) but AH-integrable with respect to some other $\tilde{T}$.

EXAMPLE 1.2. Let us define a function $g : [0,1] \longrightarrow \mathbb{R}$ as follow:

$$g(s) = \begin{cases} \frac{1}{1-s}, & s = a_n \\ s, & s \in [0,1] - \{a_n : n \in \mathbb{N}\} \end{cases}$$

where $a_n = \frac{n}{n+1}$. Let us consider the bilinear triple $\mathcal{B} = (\mathbb{R}, \mathbb{R}, \mathbb{R})$. We define functions $x : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ and $y : [0,1] \longrightarrow \mathbb{R}$ as $x(\tau,t) = g(t)$ and

$$y(\tau) = \begin{cases} 0, & \tau = a_n \\ 1, & \tau \in [0,1] - \{a_n : n \in \mathbb{N}\} \end{cases}.$$

For any gauge δ on $[0,1]$ we can choose a $\delta-$fine partition

$$P = \{(\tau_j, [c_j, d_j]) : j = 1, ..., p\}$$

of $[0,1]$ such that

$$(\tau_p, [c_p, d_p]) = (1, [a_m, 1]), \quad (\tau_{p-1}, [c_{p-1}, d_{p-1}]) = (a_m, [c_{p-1}, a_m]),$$

$c_{p-1} = d_{p-2} \neq a_n$ for any $n = 1, 2, ..., m-1$, and

$$\{(a_1, [c_{j_1}, d_{j_1}]), ..., (a_{m-1}, [c_{j_{m-1}}, d_{j_{m-1}}])\} \subset P,$$
$$a_k \in (c_{j_k}, d_{j_k}) \quad k = 1, ..., m-1.$$

Then

$$S(dx, y, P) < 1 + (1 - \frac{1}{1-a_m}).$$

But

$$\lim_{m\to\infty} \left[1 - \frac{1}{1-a_m}\right] = -\infty.$$

This shows that the Henstock-Stieltjes integral $(H)\int_0^1 d[x(\tau,t)]y(\tau)$ does not exist. However we can show that the approximate Henstock-Stieltjes integral $(\tilde{T}-AH)\int_0^1 d[x(\tau,t)]y(\tau)$ exists with the following $\tilde{T}$:

Let $\tilde{T}=\{\{\tau\}\cup([0,1]-\{a_n\}_{n=1}^{\infty}):\tau\in[0,1]\}$. Then $\tilde{T}$ is an approximate distribution on $[0,1]$. Let $\varepsilon>0$ and define a gauge δ on $[a,b]$ by

$$\delta(\tau)=\begin{cases}1, & \tau\in[0,1]-\{a_n:n\in\mathbb{N}\}\\ \frac{\varepsilon}{2^{n+1}}, & \tau=a_n\end{cases}.$$

Let $P=\{(\tau_j,J_j=[c_j,d_j]):j=1,...p\}$ be an $\delta(\tilde{T})$-fine partition of $[0,1]$. And let

$$P_1=\{(\tau_j,J_j=[c_j,d_j])\in P:\tau_j=a_{n_j}\quad \text{for some}\quad n_j\}.$$

Then $\sum_{(\tau_j,J_j)\in P_1}x(\tau_j,J_j)y(\tau_j)=0$. If $\tau_j\neq a_n$, then $\tilde{T}_{\tau_j}(\in\tilde{T})$ contains no elements $a_n, n=1,2,....$ Hence

$$1-\sum_{n=1}^{\infty}\frac{\varepsilon}{2^n}\le\sum_{(\tau_j,J_j)\in P-P_1}x(\tau_j,J_j)y(\tau_j)\le 1.$$

Hence $|S(dx,y,P)-1|\le\varepsilon$, and this shows $(\tilde{T}-AH)\int_a^b d[x(\tau,t)]y(\tau)=1$.

THEOREM 1.3. *The approximate Henstock-Stieltjes integral is unique if it exits.*

Proof. Suppose $(\tilde{T}-AH)\int_a^b d[x(\tau,t)]y(\tau)=I_1$ and

$$(\tilde{T}-AH)\int_a^b d[x(\tau,t)]y(\tau)=I_2.$$

Let $\varepsilon>0$ be given. Then there is a gauge δ_k on $[a,b]$ such that

$$\|S(dx,y,P_k)-I_k\|_Z<\frac{\varepsilon}{2}$$

provided P_k is a $\delta_k(\tilde{T})$-fine partition of $[a,b]$, $k=1,2$. Put $\delta(t)=\min\{\delta_1(t),\delta_2(t)\}, t\in[a,b]$. Then any $\delta(\tilde{T})$-fine partition P of $[a,b]$ is a $\delta_k(\tilde{T})$-fine partition of $[a,b]$, $k=1,2$. Hence

$$\|I_1-I_2\|_Z\le\|S(dx,y,P)-I_1\|_Z+\|S(dx,y,P)-I_2\|_Z<\varepsilon.$$

This shows that the approximate Hensock-Stieltjes integral is unique if it exists. □

REMARK. Let $\tilde{T}^1$, $\tilde{T}^2$ be approximate distributions on $[a,b]$. Suppose $\tilde{T}^1$ is a refinement of $\tilde{T}^2$. If $(\tilde{T}^2-AH)\int_a^b d[x(\tau,t)]y(\tau)=I$ exists, then

$(\tilde{T}^1 - AH) \int_a^b d[x(\tau,t)]y(\tau)$ also exists and is equal to I. This can be seen easily because, if P is a $\delta(\tilde{T}^1)$-fine partition of $[a,b]$ then it is also a $\delta(\tilde{T}^2)$-fine partition of $[a,b]$.

From now on, all the integrals will be with respect to a fixed approximate distribution $\tilde{T}$ and we write

$$(\tilde{T}) \int_a^b d[x(\tau,t)]y(\tau)$$

for $(\tilde{T} - AH) \int_a^b d[x(\tau,t)]y(\tau)$.

Since

$$S(dx, c_1y_1 + c_2y_2, P) = c_1S(dx, y_1, P) + c_2S(dx, y_2, P)$$

and

$$S(d(c_1x_1 + c_2x_2), y, P) = c_1S(dx_1, y, P) + c_2S(dx_2, y, P)$$

for any $\delta(\tilde{T})$-fine partition P, we have the following theorem.

THEOREM 1.4. *Assume that $\mathcal{B} = (X, Y, Z)$ is a bilinear triple and that the functions $x : [a,b] \times [a,b] \longrightarrow X$ and $y_i : [a,b] \longrightarrow Y$ are such that the AH-integrals $(\tilde{T}) \int_a^b d[x(\tau,t)]y_i(\tau)$, $i = 1,2$ exist. Then for every $c_1, c_2 \in \mathbb{R}$ the integral $(\tilde{T}) \int_a^b d[x(\tau,t)](c_1y_1(\tau) + c_2y_2(\tau))$ exists and*

$$(\tilde{T}) \int_a^b d[x(\tau,t)](c_1y_1(\tau) + c_2y_2(\tau))$$
$$= c_1(\tilde{T}) \int_a^b d[x(\tau,t)]y_1(\tau) + c_2(\tilde{T}) \int_a^b d[x(\tau,t)]y_2(\tau).$$

If functions $x_i : [a,b] \times [a,b] \longrightarrow X$ and $y : [a,b] \longrightarrow Y$ are such that the AH- integrals $(\tilde{T}) \int_a^b d[x_i(\tau,t)]y(\tau), i = 1,2$ exist, then for every $c_1, c_2 \in \mathbb{R}$ the AH-integral $(\tilde{T}) \int_a^b d[c_1x_1(\tau,t) + c_2x_2(\tau,t)]y(\tau)$ exists and

$$(\tilde{T}) \int_a^b d[c_1x_1(\tau,t) + c_2x_2(\tau,t)]y(\tau)$$
$$= c_1(\tilde{T}) \int_a^b d[x_1(\tau,t)]y(\tau) + c_2(\tilde{T}) \int_a^b d[x_2(\tau,t)]y(\tau).$$

2. Uniform convergence theorem

Let $x : [a,b] \times [a,b] \longrightarrow X$ and $E \subset [a,b]$. Given an approximate distribution $\tilde{T}$ on $[a,b]$ and a gauge δ on $[a,b]$, let

$$ {}_{ap}V_a^b(x,\tilde{T},\delta,E) = \sup\left\{\sum_{\tau_j\in E} \|x(\tau_j,J_j)\|_X\right\} $$

for the supremum over all $\delta(\tilde{T})$-fine partition $P = \{(t_j,J_j) : j = 1,...,k\}$ of the interval $[a,b]$. The *approximate variation* of x with respect to $\tilde{T}$ in E is

$$ {}_{ap}V_a^b(x,\tilde{T},E) = \inf_{\delta>0}\left\{{}_{ap}V_a^b(x,\tilde{T},\delta,E)\right\}. $$

If ${}_{ap}V_a^b(x,\tilde{T},E) < \infty$, we say that x is *of approximately bounded variation* on E. In Example 1.2, we can show that ${}_{ap}V_0^1(x,\tilde{T},E = \{a_n : n \in \mathbb{N}\}) = 0$. However, it is easy to see that ${}_{ap}V_0^1(x,\tilde{T}^0,E = \{a_n : n \in \mathbb{N}\}) = \infty$.

We write

$$ {}_{ap}V_a^b(x,\tilde{T},\delta),\ {}_{ap}V_a^b(x,\tilde{T}) \ \text{ for } {}_{ap}V_a^b(x,\tilde{T},\delta,[a,b]),\ {}_{ap}V_a^b(x,\tilde{T},[a,b]) $$

respectively. (cf. [2], [3])

LEMMA 2.1. *Assume that $\mathcal{B} = (X,Y,Z)$ is a bilinear triple and that the functions $x : [a,b] \times [a,b] \longrightarrow X$ and $y : [a,b] \longrightarrow Y$ are given. And let $\tilde{T}$ be an approximate distribution on $[a,b]$ and $E \subset [a,b]$. If ${}_{ap}V_a^b(x,\tilde{T},E) = 0$ then for any $\varepsilon > 0$ there exists a $\delta(\tilde{T})$-fine partition $P = \{(\tau_j,J_j) : j = 1,...,k\}$ such that*

$$ \left\|\sum_{\tau_j\in E} x(\tau_j,J_j)y(\tau_j)\right\|_Z < \varepsilon. $$

Proof. Let $E_j = \{\tau \in E : j-1 \le \|y(\tau)\|_Y < j\}, j = 1,2,\cdots$. Then ${}_{ap}V_a^b(x,\tilde{T},E_j) = 0$ for any $j = 1,2,\cdots$ since ${}_{ap}V_a^b(x,\tilde{T},E) = 0$. For any given $\varepsilon > 0$, there exists a gauge δ_j on $[a,b]$ such that

$$ {}_{ap}V_a^b(x,\tilde{T},\delta_j,E_j) < \frac{\varepsilon}{j2^j}, (j = 1,2,...). $$

Define a gauge δ on $[a,b]$ by

$$ \delta(\tau) = \begin{cases} \delta_j(\tau), & \tau \in E_j, j = 1,2,... \\ \text{arbitrary}, & \text{elsewhere} \end{cases}. $$

For a $\delta(\tilde{T})$-fine partition $P = \{(t_j, J_j) : j = 1, ..., k\}$, we have

$$\left\| \sum_{\tau_j \in E} x(t_j, J_j) y(\tau_j) \right\|_Z \leq \sum_{\tau_j \in E} \|y(\tau_j)\|_Y \|x(t_j, J_j)\|_X$$

$$\leq \sum_{j=1}^{\infty} j[{}_{ap}V_a^b(x, \tilde{T}, \delta_j, E_j)] < \sum_{j=1}^{\infty} j \frac{\varepsilon}{j2^j} = \varepsilon.$$

□

LEMMA 2.2. *Assume that $\mathcal{B} = (X, Y, Z)$ is a bilinear triple and that AH-integral $(\tilde{T}) \int_a^b d[x(\tau, t)]y(\tau)$ exists. If x is of approximately bounded variation on $[a, b]$ and ${}_{ap}V_a^b(x, \tilde{T}, E) = 0$ for some $E \subset [a, b]$, then there exists a gauge δ such that ${}_{ap}V_a^b(x, \tilde{T}, \delta) < \infty$ and*

$$\left\| (\tilde{T}) \int_a^b d[x(\tau, t)]y(\tau) \right\|_Z \leq \sup_{\tau \in [a,b] - E} \|y(\tau)\|_Y \cdot {}_{ap}V_a^b(x, \tilde{T}, \delta)$$

Proof. Let $\varepsilon > 0$ be given. Since $\int_a^b d[x(\tau, t)]y(\tau)$ exists and x is of approximate bounded variation on $[a, b]$, there are gauges δ_1, δ_2 on $[a, b]$ such that

$$\left\| \sum_{j=1}^{k} x(\tau_j, J_j) y(\tau_j) - (\tilde{T}) \int_a^b d[x(\tau, t)]y(\tau) \right\|_Z < \varepsilon$$

for a $\delta_1(\tilde{T})$-fine partition P of $[a, b]$ and

$${}_{ap}V_a^b(x, \tilde{T}, \delta_2) < \infty.$$

And since ${}_{ap}V_a^b(x, \tilde{T}, E) = 0$, by Lemma 2.1, there is a gauge δ_3 on $[a, b]$ such that

$$\left\| \sum_{\tau_j \in E} x(\tau_j, J_j) y(\tau_j) \right\|_Z < \varepsilon$$

for a $\delta_3(\tilde{T})$-fine partition P of $[a, b]$. Put $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then any $\delta(\tilde{T})$-fine partition is a $\delta_k(\tilde{T})$-fine partition of $[a, b]$ $(k = 1, 2, 3)$. Hence ${}_{ap}V_a^b(x, \tilde{T}, \delta) \leq_{ap} V_a^b(x, \tilde{T}, \delta_2) < \infty$ and

$$\left\| \sum_{\tau_j \in E} x(\tau_j, J_j) y(\tau_j) \right\|_Z < \varepsilon$$

for a $\delta(\tilde{T})$-fine partition P of $[a,b]$. For this partition P, we have

$$\begin{aligned}
&\left\|(\tilde{T})\int_a^b d[x(\tau,t)]y(\tau)\right\|_Z \\
\leq\ &\left\|(\tilde{T})\int_a^b d[x(\tau,t)]y(\tau)-\sum_{j=1}^{k}x(\tau_j,J_j)y(\tau_j)\right\|_Z \\
&+\left\|\sum_{j=1}^{k}x(\tau_j,J_j)y(\tau_j)\right\|_Z \\
\leq\ &\varepsilon+\left\|\sum_{\tau_j\in E}x(\tau_j,J_j)y(\tau_j)\right\|_Z+\left\|\sum_{\tau_j\notin E}x(\tau_j,J_j)y(\tau_j)\right\|_Z \\
<\ &2\varepsilon+\sum_{\tau_j\notin E}\|y(\tau_j)\|_Y\,\|x(\tau_j,J_j)\|_X \\
<\ &2\varepsilon+\sup_{\tau\in[a,b]-E}\|y(\tau)\|_Y\sum_{\tau_j\notin E}\|x(\tau_j,J_j)\|_X \\
<\ &2\varepsilon+\sup_{\tau\in[a,b]-E}\|y(\tau)\|_Y\,{}_{ap}V_a^b(x,\tilde{T},\delta).
\end{aligned}$$

The statement is proved because $\varepsilon>0$ can be arbitrarily small. □

THEOREM 2.3 (Uniform Convergence Theorem). *Assume that* $\mathcal{B}=(X,Y,Z)$ *is a bilinear triple and that functions* $x:[a,b]\times[a,b]\longrightarrow X$ *and* $y,y_n:[a,b]\longrightarrow Y, n=1,2,....$*are given. If* x *is of approximately bounded variation on* $[a,b]$, *AH-integrals* $(\tilde{T})\int_a^b d[x(\tau,t)]y_n(\tau)$ *exist for* $n=1,2,...$, *and the sequence* $\{y_n\}$ *converges uniformly to* y *on* $[a,b]-E$, *where* ${}_{ap}V_a^b(x,\tilde{T},E)=0$, *then the AH-integral* $(\tilde{T})\int_a^b d[x(\tau,t)]y(\tau)$ *exists and*

$$(\tilde{T})\int_a^b d[x(\tau,t)]y(\tau)=\lim_{n\to\infty}(\tilde{T})\int_a^b d[x(\tau,t)]y_n(\tau).$$

Proof. Let $\varepsilon>0$ be given. Since the sequence y_n converges y uniformly on $[a,b]-E$ there is a positive integer N_1 such that for any $n>N_1$ and $\tau\in[a,b]-E$ we have

$$\|y_n(s)-y(s)\|_Y<\varepsilon.$$

By Lemma 2.2, there exists a gauge δ, such that ${}_{ap}V_a^b(x, \tilde{T}, \delta_1) < \infty$ and

$$\begin{aligned} &\left\| (\tilde{T}) \int_a^b d[x(\tau,t)]y_n(\tau) - (\tilde{T}) \int_a^b d[x(\tau,t)]y_m(\tau) \right\|_Z \\ \leq\ & \left\| (\tilde{T}) \int_a^b d[x(\tau,t)](y_n(\tau) - y_m(\tau)) \right\|_Z \\ \leq\ & \sup_{\tau\in[a,b]-E} \|y_n(\tau) - y_m(\tau)\|_Y \cdot {}_{ap}V_a^b(x, \tilde{T}, \delta_1) \leq 2\varepsilon_{ap}V_a^b(x, \tilde{T}, \delta_1) \end{aligned}$$

for $m, n > N_1$. Since Z is a Banach space this inequality implies that the limit

$$\lim_{n\to\infty} (\tilde{T}) \int_a^b d[x(\tau,t)]y_n(s) = I \in Z$$

exists. Let $N_2 \in \mathbb{N}$ be such that

$$\left\| (\tilde{T}) \int_a^b d[x(\tau,t)]y_m(\tau) - I \right\|_Z < \varepsilon$$

for $m > N_2$. Let now $m > N = \max(N_1, N_2)$ be fixed. Since the integral $(\tilde{T}) \int_a^b d[(\tau,t)]y_m(\tau)$ exists , there is a gauge $\delta_2(< \delta_1)$ on $[a,b]$ such that

$$\left\| \sum_{j=1}^k x(\tau_j, J_j)y_m(\tau_j) - (\tilde{T}) \int_a^b d[x(\tau,t)]y_m(\tau) \right\|_Z < \varepsilon$$

provided $P = \{(\tau_j, J_j) : j = 1, ..., k\}$ is a $\delta_2(\tilde{T})$- fine partition P of $[a,b]$. For such a $\delta_2(\tilde{T})$- fine partition P, we have

$$\begin{aligned} &\|S(dx, y, P) - I\|_Z \\ =\ & \left\| \sum_{j=1}^k x(\tau_j, J_j)y(\tau_j) - I \right\|_Z \\ \leq\ & \left\| \sum_{j=1}^k [x(\tau_j, J_j)y(\tau_j) - x(\tau_j, J_j)y_m(\tau_j)] \right\|_Z \\ &+ \left\| \sum_{j=1}^k x(\tau_j, J_j)y_m(\tau_j) - (\tilde{T}) \int_a^b d[x(\tau,t)]y_m(\tau) \right\|_Z \end{aligned}$$

$$
\begin{aligned}
&+\left\|(\tilde{T})\int_a^b d[x(\tau,t)]y_m(\tau)-I\right\|_Z \\
\leq\ & 2\varepsilon+\left\|\sum_{j=1}^{k} x(\tau_j,J_j)[y(\tau_j)-y_m(\tau_j)]\right\|_Z \\
\leq\ & 2\varepsilon+\left\|\sum_{\tau_j\in[a,b]-E} x(\tau_j,J_j)[y(\tau_j)-y_m(\tau_j)]\right\|_Z \\
&+\left\|\sum_{\tau_j\in E} x(\tau_j,J_j)[y(\tau_j)-y_m(\tau_j)]\right\|_Z \\
\leq\ & 2\varepsilon+\sup_{\tau\in[a,b]-E}\|y(\tau)-y_m(\tau)\|_Y\, {}_{ap}V_a^b(x,\tilde{T},\delta_2) \\
&+\left\|\sum_{\tau_j\in E} x(\tau_j,J_j)[y(\tau_j)-y_m(\tau_j)]\right\|_Z \\
\leq\ & 2\varepsilon+\varepsilon_{ap}V_a^b(x,\tilde{T},\delta_2)+\left\|\sum_{\tau_j\in E} x(\tau_j,J_j)[y(\tau_j)-y_m(\tau_j)]\right\|_Z .
\end{aligned}
$$

By Lemma 2.1 there is a gauge δ_3 such that whenever $P=\{(\tau_j,J_j): j=1,...,k\}$ is a $\delta_3(\tilde{T})$-fine partition of $[a,b]$ we have

$$\left\|\sum_{\tau_j\in E} x(\tau_j,J_j)[y(\tau_j)-y_m(\tau_j)]\right\|_z<\varepsilon.$$

Let $\delta=\min[\delta_2,\delta_3]$. Then, if we consider $\delta(\tilde{T})$-fine partitions P of $[a,b]$, we get

$$\|S(dx,y,P)-I\|_Z<2\varepsilon+\varepsilon_{ap}V_a^b(x,\tilde{T},\delta)+\varepsilon.$$

This means that the integral $(\tilde{T})\int_a^b d[x(\tau,t)]y(\tau)$ exists and

$$(\tilde{T})\int_a^b d[x(\tau,t)]y(\tau)=\lim_{n\to\infty}(\tilde{T})\int_a^b d[x(\tau,t)]y_n(\tau).$$

□

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