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INVERTIBLE INTERPOLATION PROBLEMS IN ALG$\mathcal{L}$

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ABSTRACT. In this article, we investigate invertible interpolation problems in Alg$\mathcal{L}$: Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$ and let X and Y be operators acting on $\mathcal{H}$. When does there exist an invertible operator A in Alg$\mathcal{L}$ such that $AX = Y$?

1. Introduction

In this paper we are concerned with an interpolation problem in Alg$\mathcal{L}$. Given operators X and Y acting on a Hilbert space, when is there an invertible operator A in Alg$\mathcal{L}$ (usually satisfying some other conditions) such that $AX = Y$? The "other conditions" that have been of interest to us involve restricting A to lie in the algebra associated with a subspace lattice. Lance [7] initiated the discussion by considering a nest $\mathcal{N}$ and asking what conditions on x and y will guarantee the existence of an operator A in Alg$\mathcal{N}$ such that $Ax = y$. Hopenwasser [3] extended Lance's result to the case where the nest $\mathcal{N}$ is replaced by an arbitrary commutative subspace lattice $\mathcal{L}$; the conditions in both cases read the same. Munch [8] considered the problem of finding a Hilbert-Schmit operator A in Alg$\mathcal{N}$ that maps x to y, whereupon Hopenwasser [4] again extended to Alg$\mathcal{L}$. Anoussis, Katsoulis, Moore, and Trent [1] studied the problem of finding A so that $Ax = y$ and A is required to lie in certain ideals contained in Alg$\mathcal{L}$ (for a nest $\mathcal{L}$).

Roughly speaking, when an operator maps one thing to another, we think of the operator as the interpolating operator and the equation representing the mapping as the interpolation equation. The equations $Ax = y$ and $AX = Y$ are indistinguishable if spoken aloud, but we mean

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the change to capital letters to indicate that we intend to look at fixed operators X and Y, and ask under what conditions there will exist an operator A satisfying the equation $AX = Y$. Let x and y be vectors in a Hilbert space. Then $< x, y >$ means the inner product of vectors x and y. Note that the "vector interpolation" problem is a special case of the "operator interpolation" problem. Indeed, if we denote by $x \otimes u$ the rank-one operator defined by the equation $x \otimes u(w) =< w, u > x$, and if we set $X = x \otimes u$, and $Y = y \otimes u$, then the equations $AX = Y$ and $Ax = y$ represent the same restriction on A.

The simplest case of the operator interpolation problem relaxes all restrictions on A, requiring it simply to be a bounded operator. In this case, the existence of A is nicely characterized by the well-known factorization theorem of Douglas :

THEOREM D [2]. *Let Y and X be bounded operators on a Hilbert space $\mathcal{H}$. The following statements are equivalent:*

(1) *$rangeY^* \subseteq rangeX^*$;*

(2) *$Y^*Y \leq \lambda^2 X^*X$ for some $\lambda \geq 0$;*

(3) *there exists a bounded operator A on $\mathcal{H}$ so that $AX = Y$.*

Moreover, if (1), (2), and (3) are valid, then there exists a unique operator A so that

(a) *$\|A\|^2 = \inf\{\mu : Y^*Y \leq \mu X^*X\}$;*

(b) *$ker[Y^*] = ker[A^*]$; and*

(c) *$range[A^*] \subseteq range[X]^-$.*

We establish some notations and conventions. A (commutative) subspace lattice $\mathcal{L}$ is a strongly closed lattice of (commutative) projections acting on a Hilbert space $\mathcal{H}$. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. If $\mathcal{L}$ is a subspace lattice on a Hilbert space $\mathcal{H}$, then Alg$\mathcal{L}$ is the algebra of all bounded linear operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$.

2. Results

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a subspace lattice of orthogonal projections acting on $\mathcal{H}$ containing 0 and I. Let M be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{M}$ means the closure of M. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers. In this paper, we use the convention $\frac{0}{0} = 0$, when necessary.

THEOREM 1. *Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$. Let X and Y be operators acting on $\mathcal{H}$. Assume that rangeX and rangeY are dense in $\mathcal{H}$. Then the following are equivalent.*

(1) *There is an operator A in Alg$\mathcal{L}$ such that $AX = Y$, A is invertible and every E in $\mathcal{L}$ reduces A.*

(2) $\sup\left\{\frac{\|\sum_{i=1}^n E_iYf_i\|}{\|\sum_{i=1}^n E_iXf_i\|} : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H}\right\} < \infty$ *and*

$$\sup\left\{\frac{\|\sum_{i=1}^n E_iXf_i\|}{\|\sum_{i=1}^n E_iYf_i\|} : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H}\right\} < \infty.$$

Proof. If we assume that the conditions (2) holds, then there are operators A and B in Alg$\mathcal{L}$ such that $AX = Y$, $X = BY$ and every E in $\mathcal{L}$ reduces A and B by Theorem 1[5]. Since range$X$ and range$Y$ are dense in $\mathcal{H}$, $BA = I$ and $AB = I$. Hence A is invertible.

Conversely, by Theorem 1 [5],

$$\sup\left\{\frac{\|\sum_{i=1}^n E_iYf_i\|}{\|\sum_{i=1}^n E_iXf_i\|} : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H}\right\} < \infty.$$

Since $AE = EA$, $EA^{-1} = A^{-1}E$ for every E in $\mathcal{L}$. Hence A^{-1} is an operator in Alg$\mathcal{L}$. Since $AX = Y$, $X = A^{-1}Y$. So $A^{-1}(\sum_{i=1}^n E_iYf_i) = \sum_{i=1}^n E_iXf_i$, $n \in \mathbb{N}, E_i \in \mathcal{L}$ and $f_i \in \mathcal{H}$. Thus

$$\|\sum_{i=1}^n E_iXf_i\| \leq \|A^{-1}\|\|\sum_{i=1}^n E_iYf_i\|.$$

If $\|\sum_{i=1}^n E_iYf_i\| \neq 0$, then $\frac{\|\sum_{i=1}^n E_iXf_i)\|}{\|\sum_{i=1}^n E_iYf_i\|} \leq \|A^{-1}\| < \infty$. Hence $\sup\left\{\frac{\|\sum_{i=1}^n E_iXf_i)\|}{\|\sum_{i=1}^n E_iYf_i\|} : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H}\right\} < \infty.$ □

If we modify a little bit the proof of Theorem 1, we can get the following theorems. So we will omit the proof of the following theorem except that $AE = EA$ for every E in $\mathcal{L}$.

Let

$$\mathcal{M}_0 = \left\{\sum_{i=1}^n E_iXf_i : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H}\right\} \text{ and}$$

$$\mathcal{M}_1 = \left\{\sum_{i=1}^n E_iYf_i : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H}\right\}.$$

THEOREM 2. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let X and Y be operators acting on $\mathcal{H}$ such that $\mathcal{M}_0$ and $\mathcal{M}_1$ are dense in $\mathcal{H}$. Then the following statements are equivalent.*

(1) *There is an operator A in Alg$\mathcal{L}$ such that $AX = Y$, A is invertible and every E in $\mathcal{L}$ reduces A.*

(2) $\sup\left\{\frac{\|\sum_{i=1}^n E_iYf_i\|}{\|\sum_{i=1}^n E_iXf_i\|} : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H}\right\} < \infty$ *and*

$$\sup\left\{\frac{\|\sum_{i=1}^n E_iXf_i\|}{\|\sum_{i=1}^n E_iYf_i\|} : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H}\right\} < \infty.$$

Proof. (2) $\Rightarrow$ (1). Let $E, E_i \in \mathcal{L}$ and $f_i \in \mathcal{H}$. Then

$$\begin{aligned} AE(\sum_{i=1}^n E_iXf_i) &= A(\sum_{i=1}^n EE_iXf_i) \\ &= \sum_{i=1}^n EE_iYf_i \text{ and} \end{aligned}$$

$$\begin{aligned} EA(\sum_{i=1}^n E_iXf_i) &= E(\sum_{i=1}^n E_iYf_i) \\ &= \sum_{i=1}^n EE_iYf_i. \end{aligned}$$

Since $\mathcal{M}_0$ is dense in $\mathcal{H}$, every E in $\mathcal{L}$ reduces A. □

THEOREM 3. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Let $\{X_1, X_2, \cdots, X_n\}$ and $\{Y_1, Y_2, \cdots, Y_n\}$ be operators acting on $\mathcal{H}$. If there is an operator A in Alg$\mathcal{L}$ such that $AX_j = Y_j$ $(j = 1, 2, \cdots, n)$, A is invertible and every E in $\mathcal{L}$ reduces A, then*

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_if_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_if_{k,i}\|} : m_i \in \mathbb{N}, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} < \infty$$

and

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_if_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_if_{k,i}\|} : m_i \in \mathbb{N}, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} < \infty.$$

Proof. Since A is an operator in Alg$\mathcal{L}$, $Y_j = AX_j (j = 1, 2, \cdots, n)$ and $AE = EA$ for every E in $\mathcal{L}$,

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}\|} : m_i \in \mathbb{N}, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} < \infty$$

by Theorem 3[5]. Since $Y_j = AX_j$ and A is invertible, $X_j = A^{-1}Y_j (j = 1, 2, \cdots, n)$. Since $AE = EA$, $A^{-1}E = EA^{-1}$ for every E in $\mathcal{L}$, A^{-1} is an operator in Alg$\mathcal{L}$. Hence

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}\|} : m_i \in \mathbb{N}, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} < \infty.$$

□

THEOREM 4. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Let $\{X_1, X_2, \cdots, X_n\}$ and $\{Y_1, Y_2, \cdots, Y_n\}$ be operators acting on $\mathcal{H}$. Assume that the rangeX_k and the rangeY_k are dense in $\mathcal{H}$ for some k. If*

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}\|} : m_i \in \mathbb{N}, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} < \infty$$

and

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}\|} : m_i \in \mathbb{N}, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} < \infty,$$

then there is an operator A in Alg$\mathcal{L}$ such that $AX_j = Y_j (j = 1, 2, \cdots, n)$, A is invertible and every E in $\mathcal{L}$ reduces A.

Proof. By Theorem 4[5], there are operators A and B in Alg$\mathcal{L}$ such that $Y_j = AX_j$ and $X_j = BY_j$ $(j = 1, 2, \cdots, n)$ and every E in $\mathcal{L}$ reduces A and B. Since the rangeX_k and the rangeY_k are dense in $\mathcal{H}$, $Y_k = AX_k = ABY_k$ and $X_k = BY_k = BAX_k$. So $AB = I = BA$. Hence A is invertible. □

Let

$$\mathcal{N}_0 = \left\{\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i} : m_i \in \mathbb{N}, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} \text{ and}$$

$$\mathcal{N}_1 = \left\{\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i} : m_i \in \mathbb{N}, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\}.$$

THEOREM 5. *Let $\mathcal{L}$ be a commutative subspace lattice on a Hilbert space $\mathcal{H}$. Let $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$ be operators acting on $\mathcal{H}$. Assume that $\mathcal{N}_0$ and $\mathcal{N}_1$ are dense in $\mathcal{H}$. If*

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}\|} : l\leq n, m_i\in\mathbb{N}, E_{k,i}\in\mathcal{L} \text{ and } f_{k,i}\in\mathcal{H}\right\}<\infty$$

and

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}\|} : l\leq n, m_i\in\mathbb{N}, E_{k,i}\in\mathcal{L} \text{ and } f_{k,i}\in\mathcal{H}\right\}<\infty,$$

then there is an operator A in Alg$\mathcal{L}$ such that $Y_j = AX_j (j = 1, 2, \cdots, n)$, A is invertible and every E in $\mathcal{L}$ reduces A.

Proof. By Theorem 5[5], there are operators A and B in Alg$\mathcal{L}$ such that $Y_j = AX_j$, $X_j = BY_j$ $(j = 1, 2, \cdots, n)$ and every E in $\mathcal{L}$ reduces A and B. Since $\mathcal{N}_0$ and $\mathcal{N}_1$ are dense in $\mathcal{H}$, $AB = I = BA$. Hence A is invertible. □

With the similar proof of Theorem 5, we can get the following theorem. So we omit its proof.

THEOREM 6. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let $\{X_n\}$ and $\{Y_n\}$ be two infinite sequences of operators acting on $\mathcal{H}$. Assume that*

$$\mathcal{K}_0 = \left\{\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i} : m_i, l \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\}$$

and

$$\mathcal{K}_1 = \left\{\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i} : m_i, l \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\}$$

are dense in $\mathcal{H}$. If

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}\|} : m_i, l\in\mathbb{N}, E_{k,i}\in\mathcal{L} \text{ and } f_{k,i}\in\mathcal{H}\right\}<\infty$$

and

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}\|} : m_i, l\in\mathbb{N}, E_{k,i}\in\mathcal{L} \text{ and } f_{k,i}\in\mathcal{H}\right\}<\infty,$$

then there is an operator A in Alg$\mathcal{L}$ such that $AX_n = Y_n$, A is invertible and AE=EA for every E in $\mathcal{L}$ and all $n = 1, 2, \cdots$.

We omit the proof of the following theorem because it can be easily proved.

THEOREM 7. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Let $\{X_n\}$ and $\{Y_n\}$ be two infinite sequences of operators acting on $\mathcal{H}$. If there is an operator A in Alg$\mathcal{L}$ such that $AX_n = Y_n$ $(n = 1, 2, \cdots)$, A is invertible and every E in $\mathcal{L}$ reduces A, then*

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_if_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_if_{k,i}\|} : m_i, l \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} < \infty$$

and

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_if_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_if_{k,i}\|} : m_i, l \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} < \infty.$$

If we modify a little bit the proofs of Theorems 4 and 7, we can get the following theorem. So we omit its proof.

THEOREM 8. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Let $\{X_n\}$ and $\{Y_n\}$ be two infinite sequences of operators acting on $\mathcal{H}$. Assume that the rangeX_1 and the rangeY_1 are dense in $\mathcal{H}$. Then the following are equivalent.*

(1) *There is an operator A in Alg$\mathcal{L}$ such that $AX_n = Y_n$ for all $n = 1, 2, \cdots$, A is invertible and every E in $\mathcal{L}$ reduces A.*

(2)

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_if_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_if_{k,i}\|} : m_i, l \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} < \infty$$

and

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_if_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_if_{k,i}\|} : m_i, l \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} < \infty.$$

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