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RADICALS OF A LEFT-SYMMETRIC ALGEBRA ON A NILPOTENT LIE GROUP

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Abstract. The purpose of this paper is to compare the radicals of a left symmetric algebra considered in [1] when the associated Lie algebra is nilpotent. In this case, we show that all the radicals considered there are equal. We also consider some other radicals and show they are also equal.

1. Introduction

A real (or complex) *affine space* is a set $\mathbb{E}^n$ provided with a simply transitive action of a vector space $\mathbb{R}^n$ (or $\mathbb{C}^n$). Let $\mathrm{Aff}(\mathbb{E}^n)$ denote the group of affine transformations of $\mathbb{E}^n$. On a manifold M^n, if there is a coordinate atlas modeled on an affine space such that the coordinate changes are locally affine, that is, the restrictions of affine transformations of $\mathbb{E}^n$, then such a structure will be called an *affine structure* on M. A manifold with an affine structure will be called an *affinely flat* manifold, or simply an *affine* manifold. If M, M' are affine manifolds and $f : M \to M'$ is a map, then f is said to be *affine* when f is locally affine in affine coordinates.

By a well known theorem of differential geometry, an affine manifold M can be considered as a manifold with a *flat affine* connection, that is, which admits a linear connection ∇ whose torsion and curvature tensor vanish.

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Many examples of affine manifolds are obtained as quotients of Lie groups with left invariant flat affine structures. Such structures have been studied by many authors. It is well known that there is a correspondence between the set of simply transitive affine actions of a simply connected Lie group G on an affine space and the set of left-symmetric algebra structures on the Lie algebra of G. (See below for this correspondence and the definitions.) Hence the problems on Lie groups with left invariant flat affine structures can be considered as purely algebraic problems. The general structure theory for a left-symmetric algebra seems to be difficult, but the structures of certain classes of left-symmetric algebras are known. Especially for the case when the associated Lie algebra A^- of a left-symmetric algebra A is nilpotent, Mizuhara showed that A/R is the direct sum of simple algebras (see [8], [10]), where R is the Koszul radical of A.

In this paper we study and compare the various radicals of a left-symmetric algebra A when A^- is nilpotent. From the purely algebraic viewpoint there are well known radicals, the *Albert radical* $\alpha(A)$ and *solvable radical Rad* A, which always exist in finite dimensional non-associative algebras and the *left nilpotent radical* N which exists in finite dimensional left-symmetric algebras. The *Koszul radical* R (see [4], [1]) and *complete radical* C are coming from the geometry of left-symmetric algebra. Other radicals to be investigated are $A^{\perp_\sigma}$, $A^{\perp_\eta}$ and $A^{\perp_\tau}$, the *radicals of a symmetric bilinear form* given by σ, η and τ respectively. (See Section 3 for definitions.) When the associated Lie algebra A^- is nilpotent, we show that all these radicals are equal.

MAIN THEOREM. *Let A be a left-symmetric algebra over $\mathbb{R}$ or over $\mathbb{C}$. If A^- is nilpotent, then $N = Rad\ A = \alpha(A) = C = R = A^{\perp_\sigma} = A^{\perp_\eta} = S = A^{\perp_\tau}$, where $S = \{a \in A \mid \rho_a$ is nilpotent $\}$.*

Let's recall some basic relations between a left-invariant flat affine structure on a Lie group and a left-symmetric algebra structure on its Lie algebra which are used throughout the paper. Let G be a simply connected Lie group with a left invariant flat affine connection ∇. Define a "product" on the Lie algebra $\mathfrak{g}$ of G by $xy = \nabla_x y$, for $x, y \in \mathfrak{g}$. Then the "torsion free" and "flat" conditions correspond to the following algebraic equations respectively :

$$xy - yx = [x, y], \tag{1}$$

$$(x, y, z) = (y, x, z), \tag{2}$$

where $(x, y, z) = (xy)z - x(yz)$ and $[x, y]$ denotes the Lie algebra product in $\mathfrak{g}$ for all $x, y, z \in \mathfrak{g}$. An algebra A satisfying (2) is called a *left-symmetric* algebra and hence $(\mathfrak{g}, \nabla)$ becomes a left symmetric algebra. Defining the bracket on A by (1), we obtain a Lie algebra $A^-(= \mathfrak{g})$ which is said to be *associated* with A. Conversely, if G is simply connected and xy denotes a product defined on $\mathfrak{g}$ satisfying (1) and (2), then the left invariant connection given by $\nabla_x y = xy$ defines a left invariant flat affine structure ∇ on G. Therefore, if G is simply connected, then the study of left-symmetric algebras defined on $\mathfrak{g}$ with $(\mathfrak{g}, \nabla)^- = \mathfrak{g}$ is equivalent to the study of left invariant flat affine structures on G.

If λ and ρ denote the left and right multiplication operators in A, that is, $\lambda_x(y) = xy = \rho_y(x)$ for $x, y \in A$, then the condition (2) is equivalent to each of the following two conditions :

$$\lambda_{[x,y]} = [\lambda_x, \lambda_y], \tag{3}$$

$$[\rho_x, \lambda_y] = \rho_x \rho_y - \rho_{yx} \tag{4}$$

for all $x, y \in A$.

2. Left invariant affine structures on Lie groups

An affine structure on a Lie group G is called *left invariant* if for each $g \in G$ the left translations l_g are affine maps. Let's review briefly the geometry of a left invariant affinely flat Lie group. We refer the reader to [6] for the details and generalizations. Let G be a simply connected Lie group of dimension n with Lie algebra $\mathfrak{g}$ and with a left invariant flat affine structure ∇ on G, so that $A = (\mathfrak{g}, \nabla)$ becomes a left-symmetric algebra over $\mathbb{R}$ with multiplication xy given by $\nabla_x y$. It is well-known that there exists a developing map $D : G \to \mathbb{E}^n$ which is an affine local diffeomorphism and we obtain a Lie group homomorphism $\phi = \phi_D : G \to \mathrm{Aff}(\mathbb{E}^n)$ such that $\phi(g) \circ D = D \circ l_g$ for $g \in G$. For a given point $x \in \mathbb{E}^n$, let Ev_x be the evaluation of $\mathrm{Aff}(\mathbb{E}^n)$ on $\mathbb{E}^n$ given by $Ev_x(f) = f(x)$, for $f \in \mathrm{Aff}(\mathbb{E}^n)$ and define $ev_x : G \to \mathbb{E}^n$ by the composition, $ev_x = Ev_x \circ \phi$. Then we have that $D = ev_{x_0}$, where $x_0 = D(e)$, e is the identity of G and D becomes a covering map onto its image $\Omega = D(G) \subset \mathbb{E}^n$.

Take the point $x_0 = 0$ as the origin in $\mathbb{E}^n$. Then the affine space $\mathbb{E}^n$ becomes a vector space $\mathbb{K}^n$ where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. The affine group

$\mathrm{Aff}(\mathbb{E}^n)$ can be written as a semi-direct product $\mathrm{GL}(\mathbb{K}^n) \ltimes \mathbb{K}^n$ and its Lie algebra can be correspondingly written as a sum $\mathfrak{aff}(\mathbb{E}^n) = \mathfrak{gl}(\mathbb{K}^n) + \mathbb{K}^n$. Hence ϕ has two components $\phi = (L, q) : G \to \mathrm{GL}(\mathbb{K}^n) \ltimes \mathbb{K}^n$ and $d\phi = (h, t) : \mathfrak{g} \to \mathfrak{gl}(\mathbb{K}^n) + \mathbb{K}^n$. Since $D = ev_0$ is a covering map, the differential $d(ev_0)|_e$ is a linear isomorphism from T_eG to $T_0\Omega$ and in fact, $d(ev_0)|_e = t$. Identifying $\mathfrak{g} = T_eG$ with $\mathbb{K}^n = T_0\mathbb{E}^n = \mathbb{E}^n$ via $d(ev_0)|_e$, we have $d\phi = (\lambda, Id)$, where $\lambda_x y = xy$ for x, y in $A = (\mathfrak{g}, \nabla)$ and we obtain the following commutative diagram.

$$\text{(2-1)}\qquad \begin{array}{ccccc} \mathfrak{g} & \xrightarrow{d\phi=(\lambda,Id)} & \mathfrak{aff}(\mathfrak{g}) = \mathfrak{gl}(\mathfrak{g}) + \mathfrak{g} & \xrightarrow{d(Ev_x)|_{Id}} & \mathfrak{g} \\ \Big\downarrow \exp & & \Big\downarrow \exp & & \\ G & \xrightarrow{\phi=(L,q)} & \mathrm{Aff}(\mathfrak{g}) = \mathrm{GL}(\mathfrak{g}) \ltimes \mathfrak{g} & \xrightarrow{Ev_x} & \mathfrak{g}. \end{array}$$

Using this notation, we can express the affine action as follows: For $g = \exp a \in G$ and $x \in \mathfrak{g}$,

$$ev_x(g) = g \cdot x = L_g(x) + q_g = e^{\lambda_a}(x) + \text{“}e^a - 1\text{”},$$

where “$e^a - 1$” $= a + \frac{1}{2!}\lambda_a(a) + \frac{1}{3!}{\lambda_a}^2(a) + \cdots$ and hence we can deduce that $d(ev_x)|_e = 1 + \rho_x$.

This function $1 + \rho_x$ plays significant roles in the geometry of a left-symmetric algebra $A = (\mathfrak{g}, \nabla)$. For example, we know that (G, ∇) is geodesically complete if and only if $D = ev_0 : G \to \mathbb{E}^n = \mathfrak{g}$ is onto, and this is equivalent to the fact that $d(ev_x)|_e = 1 + \rho_x$ is bijective for all x [6], or equivalently $p(x) := \det(1 + \rho_x) > 0$ for all x. In fact, this is equivalent to $p(x) \equiv 1$. (See [3], [14].) Notice that $p(x) \equiv 1$ if and only if ρ_x is nilpotent for all $x \in A$ from the equation $\det(1 + t\rho_x) = 1 + s_1(\rho_x)\, t + \cdots + s_n(\rho_x)\, t^n$, where $s_r(\rho_x)$ is r-th symmetric polynomial of the eigenvalues of ρ_x. Furthermore ρ_x is nilpotent for all $x \in A$ if and only if $\mathrm{tr}\rho_x = 0$ for all $x \in A$. This can be deduced immediately from the fact that

$$\text{(2-2)}\qquad \mathrm{tr}\ (\rho_x)^n = \mathrm{tr}\ \rho_{x^n},$$

where for $x \in A$, $x^{k+1} = {\rho_x}^k(x)$, $k = 1, 2, \cdots$ [1].

3. Radicals of a left-symmetric algebra

In [1], various radicals of a left-symmetric algebra are studied and we recall them briefly.

Let A be an arbitrary algebra over a field F. A is said to be *simple* if $A^2 \neq 0$ and it contains no proper ideals. If A is a direct sum of finitely many simple ideals of A, then A is said to be *semisimple*. The *Albert radical*, $\alpha(A)$ is defined to be the intersection of all maximal ideals M of A such that $A^2 \not\subseteq M$. If such maximal ideals do not exist, then we let $\alpha(A) = A$. The following theorem is shown in [2].

THEOREM 3.1. *Let A be a finite dimensional algebra over a field F. Then the Albert radical $\alpha(A)$ is the unique minimal ideal of A such that the quotient algebra $A/\alpha(A)$ is semisimple or 0.*

Recall that an ideal I of A is called *solvable* if $I^{(k)} = 0$ for some integer $k \geq 0$ where $I^{(k)}$ is defined inductively by $I^{(0)} = I$, $I^{(i+1)} = I^{(i)}I^{(i)}$. It is well known that for dim $A < \infty$ there exists a unique maximal solvable ideal *Rad* A of A, called the *solvable radical* of A [13].

Let I be an ideal of A. For $k > 0$, denote by ${}^{<k>}I$ the linear span of $\lambda_{a_1} \cdots \lambda_{a_{k-1}}(a_k)$ for all $a_1, \cdots, a_k \in I$. If ${}^{<k>}I = 0$ for some $k > 0$, then I is said to be *left nilpotent*. If I and J are left nilpotent ideals of a finite-dimensional left-symmetric algebra A over F, then so is $I + J$ [1]. Thus A contains a unique maximal left nilpotent ideal N containing all left nilpotent ideals of A. We call N the *left nilpotent radical* of A.

Let A be a left-symmetric algebra over a field $\mathbb{K}$, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. Given a Lie algebra homomorphism $s : A^- \to \mathbb{K}$, define a symmetric bilinear form by $\langle x, y \rangle = s(x \cdot y)$. Let's denote $B^\perp = \{x \in A \mid \langle x, b \rangle = 0$ for all $b \in B\}$ for a subspace B of A. If $B = A$, then $A^\perp$ is called the *radical* of $\langle , \rangle$ in A. By the left-symmetries (3) or (4), both $\mathrm{tr}\lambda$ and $\mathrm{tr}\rho$ are Lie algebra homomorphism of A^- into $\mathbb{K}$. Hence $\eta(x, y) = \mathrm{tr}\lambda_{xy}$ and $\sigma(x, y) = \mathrm{tr}\rho_{xy} = \mathrm{tr}\rho_x\rho_y$ belong to this type of forms. Also define $\tau(x, y) = \mathrm{tr}\lambda_x\lambda_y$ and denote their radicals by $A^{\perp_\eta}$, $A^{\perp_\sigma}$ and $A^{\perp_\tau}$ respectively. Clearly $A^{\perp_\eta} = A^{\perp_\sigma}$ if A^- is unimodular, that is, tr $ad_x = 0$ for all x in A.

The *Koszul radical* R of a left-symmetric algebra $A = (\mathfrak{g}, \nabla)$ is defined by

$$R = \{a \in A \mid a + \Omega = \Omega\},$$

where Ω is the developing image of the left-invariant affinely flat Lie group G with Lie algebra $\mathfrak{g} = A^-$. Using the characteristic polynomial $p(x) = \det(1 + \rho_x)$, Helmstetter [4] showed that over $\mathbb{C}$,

$$R = \{a \in A \mid p(x + a) = p(x) \text{ for all } x \in A\}$$

and R becomes the largest left ideal contained in the Lie ideal $K = \ker(\mathrm{tr}\rho)$. This also holds over $\mathbb{R}$. (See [1].)

Recall that a left symmetric algebra $A = (\mathfrak{g}, \nabla)$ is complete if and only if the Lie group G with Lie algebra $\mathfrak{g} = A^-$ acts transitively on $\mathbb{E}^n$. This is equivalent to the fact that $\mathrm{tr}\rho_x = 0$ for all $x \in A$. In [1] the following is proved.

PROPOSITION 3.2. *Let A be a left-symmetric algebra and N, R, $A^{\perp_\sigma}$ be left-nilpotent radical, Koszul radical and radical of σ respectively. Then we have that*

(i) *N, R and $A^{\perp_\sigma}$ are complete as left-symmetric algebras,*
(ii) *R is the unique maximal complete left ideal in A.*

Let A be a left-symmetric algebra. Then we will show that if two ideals I and J of A are complete as left-symmetric algebra, then $I + J$ is also a complete ideal of A. From the discussion in Section 2, the following can be shown easily.

PROPOSITION 3.3. *Let B be a left-symmetric subalgebra of A and H be the Lie subgroup of G with its Lie algebra $\mathfrak{h} = B^-$. Then*

(i) *the developing map ev_H of H at 0 is the restriction of the developing map ev_0 of G,*
(ii) *if B is complete as a left-symmetric algebra, then $B \subset \Omega = ev_0(G)$.*
(iii) *if B is an ideal of A, then the developing map $\tilde{ev} : G/H \to \Omega/B \subset A/B$ is naturally induced from ev_0.*

PROPOSITION 3.4. *Let $0 \to I \to A \to J \to 0$ be an exact sequence of left-symmetric algebras. Then A is complete if and only if I and J are complete.*

Proof. Let H be the normal subgroup of G whose Lie algebra is I^-. Then the Lie algebra of G/H is $A/I = J$. Let $\tilde{\Omega} = \Omega/I \subset A/I$. Assume that A is complete. Then (i) I is complete : Indeed for $a \in I$, $\rho_a = \begin{pmatrix} \overline{\rho}_a & * \\ 0 & 0 \end{pmatrix}$. Then since A is complete, $0 = \mathrm{tr}\rho_a = \mathrm{tr}\overline{\rho}_a$. Thus I is complete. (ii) J is complete : For $\overline{x} = x + I \in A/I$, since A is complete, there exists $g \in G$ such that $g \cdot 0 = x$. Then $\overline{g} \cdot \overline{0} = \overline{g \cdot 0} = \overline{x}$. This shows that G/H acts on A/I transitively.

Conversely, assume that I and J are complete. Then for $x \in A$, there is $g \in G$ such that $\overline{g} \cdot \overline{0} = \overline{x}$ since J is complete, and hence $g \cdot 0 \in x + I$. This implies that $g \cdot I = x + I$ since I is invariant under the action of linear part of g and hence $g^{-1} \cdot x \in I$. Since I is also complete, there exists $h \in H$ such that $h \cdot 0 = g^{-1} \cdot x$. Thus $gh \cdot 0 = x$. In fact, gHg^{-1} acts on $x + I \subset A$. □

COROLLARY 3.5. *Let I and J be complete ideals of A. Then $I + J$ is a complete ideal of A.*

Proof. Let I and J be complete ideals of A. Then we have a short exact sequence $0 \to I \to I + J \to (I + J)/I \to 0$. Since I, J and $(I + J)/I \cong J/(I \cap J)$ are complete, by Proposition 3.4, $I + J$ is also complete. □

REMARK. We can also show Corollary 3.5 directly using the Helmstetter characterization of R as follows. Let I and J be complete ideals of A. We know that I and J are in $R \subset K$ and also the ideal $I + J$ is also in $R \subset K = ker(tr\rho)$. Then for $x \in I + J$, $\rho_x = \begin{pmatrix} \overline{\rho}_x & * \\ 0 & 0 \end{pmatrix}$, where $\overline{\rho}_x : I + J \to I + J$. Since $I + J \subset K$, $0 = \mathrm{tr}\rho_x = \mathrm{tr}\overline{\rho}_x$. This shows that $I + J$ is complete.

It follows from the above corollary that there exists a unique maximal complete ideal, called the *complete radical*, C containing all the complete ideals of A. We know that $N \subset C \subset R$ since N is a complete ideal and R is the maximal complete left ideal of A.

We summarize the inclusion relations between these radicals considered so far.(Compare with [1].)

THEOREM 3.6. *Let A be a left-symmetric algebra and N, $Rad\ A$, $\alpha(A)$, C, R, $A^{\perp_\sigma}$ be the left-nilpotent radical, solvable radical, Albert radical, complete radical, Koszul radical and radical of σ respectively. Also let $S = \{a \in A \mid \rho_a$ is nilpotent $\}$ and $K = \ker \mathrm{tr}\rho$. Then we have that*

$$N \subset Rad\ A \subset \alpha(A),$$

$$N \subset C \subset R \subset A^{\perp_\sigma} \subset S \subset K.$$

The inclusions appeared in Theorem 3.6 in general can not be replaced by the equality in solvable Lie algebra as shown in [1].

Let A be a left-symmetric algebra. Then by the left-symmetry (3) of A, the mapping λ defined by $\lambda : A \to \mathfrak{gl}(A)$, $x \mapsto \lambda_x$, is a homomorphism of Lie algebra A^- into the Lie algebra $\mathfrak{gl}(A)$ of all linear endomorphisms of A. As is well known (see for example [12]), we have a weight space decomposition as follows.

THEOREM 3.7. *If A^- is a nilpotent Lie algebra over $\mathbb{C}$, then there exist distinct weights $\{\alpha_i\}_{i=1}^{l}$ of $\lambda : A^- \to \mathfrak{gl}(A)$. Denote by A_i the weight space corresponding to the weight α_i. Then*

(i) $A = A_1 \oplus \cdots \oplus A_l$,
(ii) *there exists a basis of A_i such that for any element $x \in A$,*

$$\lambda_x|_{A_i} = \begin{pmatrix} \alpha_i(x) & & * \\ & \ddots & \\ 0 & & \alpha_i(x) \end{pmatrix}.$$

Recall that the Koszul radical R is the maximal left ideal of A contained in $K = \ker \operatorname{tr} \rho$. In general, R is not an ideal of A. (See [4].) But the following structure theorem was proved by Mizuhara ([8], [9] and [10]).

THEOREM 3.8 (MIZUHARA). *Let A be a left-symmetric algebra whose associated Lie algebra A^- is nilpotent. Then*

(i) *R is an ideal of A containing $[A, A]$;*
(ii) *Over $\mathbb{C}$, $R = \cap \ker \alpha_i = \{x \mid \alpha_i(x) = 0$ for all $i\}$;*
(iii) *Over $\mathbb{C}$, there is a basis $(e_{i1}, \cdots, e_{in_i})$ of A_i such that for any x in A,*
$R = \{x = (x_{11}, \cdots, x_{1n_1}, x_{21}, \cdots, x_{ln_l}) \mid x_{1n_1} = \cdots = x_{ln_l} = 0\}$,
where $n_i = \dim A_i$ and $e_i := e_{in_i}$ is an idempotent;
(iv) *Over $\mathbb{C}$, there exists a commutative subalgebra $B = < e_1, \cdots, e_l > \cong \mathbb{C} \oplus \cdots \oplus \mathbb{C}$ of A such that A is a semi-direct sum of R and B, where ρ_{e_i} acts as 1 on A_i and 0 on A_j for $j \neq i$;*
(v) *Over $\mathbb{R}$, there exists a commutative subalgebra $B \cong \mathbb{C}_{\mathbb{R}} \oplus \cdots \oplus \mathbb{C}_{\mathbb{R}} \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R}$ of A such that A is a semi-direct sum of R and B.*

REMARK 3.9. (i) $A = \mathbb{R}$ or $\mathbb{C}$ is a left-symmetric algebra given by $\{ee = e\}$ so that A is the field itself. In this case $R = A^{\perp_\sigma} = 0$ and $\mathbb{R}$ or $\mathbb{C}$ is simple. (ii) $A = \mathbb{C}_{\mathbb{R}}$ is the underlying real left-symmetric algebra of the complex left-symmetric algebra $\mathbb{C}$ and it is given by the presentation $< e_1, e_2 \mid e_1e_1 = e_1,\ e_1e_2 = e_2,\ e_2e_1 = e_2,\ e_2e_2 = -e_1 >$. Then $R = A^{\perp_\sigma} = 0$ and $\mathbb{C}_{\mathbb{R}}$ is simple.

REMARK 3.10. In fact, for the real case, the complexification $A_{\mathbb{C}}$ of A gives a semi-direct sum $R_{\mathbb{C}} + B_{\mathbb{C}}$ and it can be shown easily that $R_{\mathbb{C}}$ is the complexification of the Koszul radical R of A and $B_{\mathbb{C}}$ is the complexification of $B = B_{\mathbb{C}} \cap A$, from which we can deduce that B is of the form given in Theorem 3.8(v).

4. Proof of the main theorem

Suppose that A is a left-symmetric algebra whose associated Lie algebra A^- is nilpotent. We show that the radicals in Theorem 3.6 are all identical.

CLAIM 1. $N = C = R$.

Proof. It is known that A is left-nilpotent if and only if A is complete and A^- is a nilpotent Lie algebra. (See [11], [4], [5] for proofs.) Then by Theorem 3.8, R is an ideal. Since R is complete and $R^-(< A^-)$ is nilpotent, R is left nilpotent and hence $R \subset N$ by the maximality of N. This shows that $N = C = R$. □

CLAIM 2. $R = A^{\perp_\sigma} = S$.
Denote $a^k = {\rho_a}^{k\text{-}1}(a)$ *for* $a \in A$.

(i) *If* $a \in S = \{x \mid \rho_x \text{ is nilpotent }\}$, *then* $a^k \in K$ *for all integers* $k \geq 1$.
(ii) *If* $a^k \in K$ *for all integers* $k \geq 1$, *then* $a \in R$.

Proof. (i) This follows from the fact that $0 = \mathrm{tr}(\rho_a)^k = \mathrm{tr}\ \rho_{a^k}$.

(ii) First for the complex case, we have the weight space decomposition of A, $A = A_1 \oplus \cdots \oplus A_l$ and from Theorem 3.8 $A = R + B$, where B is a commutative subalgebra generated by $e_1, \cdots, e_l$ and $e_i e_j = \delta_{ij} e_j$.

Let $a = r + s \in K$ with $a^k \in K$, where $r \in R$ and $s = \sum_{i=1}^l t_i e_i \in B$. Since R is an ideal, $a^k \in K$ can be written as $a^k = \tilde{r} + \sum_{i=1}^l (t_i)^k e_i$, where $\tilde{r} \in R \subset K$. Then $0 = \mathrm{tr}\rho_{a^k} = \mathrm{tr}\rho_{\tilde{r}} + \sum_{i=1}^l (t_i)^k \mathrm{tr}\rho_{e_i}$, where $\mathrm{tr}\rho_{\tilde{r}} = 0$ and $d_i := \mathrm{tr}\rho_{e_i} = \dim A_i \geq 1$. Now we have $\sum_{i=1}^l (t_i)^k d_i = 0$ for every integer $k \geq 1$ with $d_i \geq 1$, and hence we conclude that $t_i = 0$ for all $i = 1, \cdots, l$. This shows that $a = r \in R$. Hence, by Theorem 3.6 we have $R = A^{\perp_\sigma} = S$. For the real case, consider the complexification $A_{\mathbb{C}}$ of A and apply the above argument. It can be shown easily that $K(A_{\mathbb{C}}) = K(A) \otimes \mathbb{C}$ and we obtain the same result since $R(A_{\mathbb{C}}) = R(A) \otimes \mathbb{C}$ and $R(A) = R(A_{\mathbb{C}}) \cap A$. □

CLAIM 3. $R = A^{\perp_\tau}$.

(i) λ_a *is nilpotent if and only if* ρ_a *is nilpotent,*

(ii) $R = A^{\perp_\tau}$.

Proof. (i) By Theorem 3.7, if λ_a is nilpotent for a in A, then we have $\alpha_i(a) = 0$ for every i. Then by Theorem 3.8 and 3.6, $a \in \cap \ker \alpha_i = R \subset S$. Hence ρ_a is nilpotent. Conversely, suppose that ρ_a is nilpotent for $a \in A$. Then $a \in S$, and so by Claim 2, $a \in R$. Then we have $\alpha_i(a) = 0$ for all i, by Theorem 3.8. Hence λ_a is nilpotent. For the real case, again use the complexification as above to obtain the same conclusion.

(ii) Let $a \in R$. Then we have that $\alpha_i(a) = 0$ for all i. Thus by Theorem 3.7, the main diagonal entries of $\lambda_a\lambda_x$ are all zero for x in A and hence $\mathrm{tr}\lambda_a\lambda_x = 0$. This shows that $R \subset A^{\perp_\tau}$. Note that $\mathrm{tr}\lambda_a\lambda_x$ remains unchanged through the complexification.

For the converse, note that $A^{\perp_\tau}$ is a Lie ideal of A^- since $[A, A] \subset R \subset A^{\perp_\tau}$ and $\tau([x, y], z) = \tau(x, [y, z])$ for all $x, y, z \in A$. From Theorem 3.8, if we can show that $A^{\perp_\tau} \cap < e_1, \cdots, e_l >= 0$, then we have $R = A^{\perp_\tau}$ since $R \subset A^{\perp_\tau}$.

To prove $A^{\perp_\tau} \cap < e_1, \cdots, e_l >= 0$, consider λ_{e_i} which is given by an upper triangular matrix by Theorem 3.7 with the main diagonal entries 0 on A_j for $j \neq i$ and 1 on A_i. This follows from Theorem 3.8(iv) and the nilpotency of $ad_x = \lambda_x - \rho_x$. Let $a = \sum_{i=1}^l t_i e_i \in A^{\perp_\tau}$. Then $0 = \tau(a, a) = \mathrm{tr}\lambda_a\lambda_a = \sum_{i=1}^l \sum_{j=1}^l t_i t_j \mathrm{tr}\lambda_{e_i}\lambda_{e_j} = \sum_{i=1}^l (t_i)^2 \mathrm{tr}\lambda_{e_i}\lambda_{e_i}$ $= \sum_{i=1}^l (t_i)^2 \dim A_i$. Similarly for any integer $k \geq 2$, $a^k = \sum_{i=1}^l (t_i)^k e_i$ and hence $0 = \tau(a, a^k) = \mathrm{tr}\lambda_a\lambda_{a^k} = \sum_{i=1}^l (t_i)^{k+1} \dim A_i$. Therefore we have $t_i = 0$ for any i. This completes the proof. Also use the fact that $A^{\perp_\tau} \otimes \mathbb{C} = A_{\mathbb{C}}{}^{\perp_\tau}$ to conclude the same for the real case. □

CLAIM 4. $\alpha(A) \subset R$.

Proof. By Theorem 3.8, we have that $A/R \cong \mathbb{C} \oplus \cdots \oplus \mathbb{C}$ (over $\mathbb{C}$) or $A/R \cong \mathbb{C}_{\mathbb{R}} \oplus \cdots \oplus \mathbb{C}_{\mathbb{R}} \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R}$ (over $\mathbb{R}$) is semisimple. Then since $\alpha(A)$ is minimal with respect to the property of Theorem 3.1, $\alpha(A) \subset R$. □

This completes the proof of the main theorem. Recall that $A^{\perp_\eta} = A^{\perp_\sigma}$, which follows from the fact that A^- is unimodular.

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