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UNIQUENESS OF POSITIVE STEADY STATES FOR WEAK COMPETITION MODELS WITH SELF-CROSS DIFFUSIONS

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Abstract. In this paper, we investigate the uniqueness of positive solutions to weak competition models with self-cross diffusion rates under homogeneous Dirichlet boundary conditions. The methods employed are upper-lower solution technique and the variational characterization of eigenvalues.

1. Introduction

Of concern is the uniqueness of positive solutions to 2×2 elliptic interacting system:

$$(1.1) \qquad \begin{cases} -(\alpha_1 + \beta_1 u + \gamma_1 v)\Delta u = u(a_1 - c_1 u - b_1 v) \\ -(\alpha_2 + \gamma_2 u + \beta_2 v)\Delta v = v(a_2 - b_2 u - c_2 v) \ \text{ in } \Omega, \\ (u, v) = (0, 0) \ \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded region with smooth boundary, α_i, a_i, b_i, c_i are positive constants and β_i, γ_i are nonnegative constants.

The system (1.1) is the steady state of the diffusive Lotka-Volterra competition model:

$$(1.2) \qquad \begin{cases} u_t - (\alpha_1 + \beta_1 u + \gamma_1 v)\Delta u = u(a_1 - c_1 u - b_1 v) \\ v_t - (\alpha_2 + \gamma_2 u + \beta_2 v)\Delta v = v(a_2 - b_2 u - c_2 v) \ \text{ in } \Omega \times (0, \infty), \\ (u, v) = (0, 0) \ \text{ on } \partial\Omega \times (0, \infty), \end{cases}$$

which has linear self-cross diffusion rates with respect to u, v. Here u, v may represent the densities of two species interacting competitively each

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other. An extended form for the steady states of (1.2) was studied in [11], which is the self-cross diffusive model:

$$(1.3)\qquad \begin{cases} -\Delta(d_1+\beta_{11}u+\beta_{12}v)u=(a_1-b_{11}u-b_{12}v)u \\ -\Delta(d_2+\beta_{21}u+\beta_{22}v)v=(a_2-b_{21}u-b_{22}v)v \quad \text{in } \Omega, \\ (u,v)=(0,0) \quad \text{on } \partial\Omega. \end{cases}$$

The authors gave sufficient conditions for the existence of positive solutions using the theory of fixed point on positive cones. The uniqueness of positive solutions to (1.3) has not been known yet, however we are able to provide sufficient conditions for the uniqueness of a positive coexistence to (1.1) which is a simpler form than (1.3).

We say that the system (1.1) has a positive solution (u,v) if $u(x)>0$ and $v(x)>0$ for all $x\in\Omega$. The existence of a positive solution (u,v) to the system (1.1) is called a *positive coexistence.* The system (1.1) is said to be in a *weak competition* if it satisfies $\frac{c_1}{b_2}>\frac{a_1}{a_2}>\frac{b_1}{c_2}$.(See [8].)

Notice that since the diffusion rates $\alpha_1+\beta_1u+\gamma_1v$, $\alpha_2+\gamma_2u+\beta_2v$ are strictly positive, the system (1.1) is equivalent to the system:

$$(1.4)\qquad \begin{cases} -\Delta u=u\left(\dfrac{a_1-c_1u-b_1v}{\alpha_1+\beta_1u+\gamma_1v}\right) \\ -\Delta v=v\left(\dfrac{a_2-b_2u-c_2v}{\alpha_2+\gamma_2u+\beta_2v}\right) \quad \text{in } \Omega, \\ (u,v)=(0,0) \ \text{ on } \partial\Omega. \end{cases}$$

Thus we shall use the form (1.4) to investigate the uniqueness of a positive coexistence for the system (1.1).

In the case that β_i, γ_i are equal to 0 and $\alpha_i=1$ in the system (1.1), by normalizing u and v appropriately, (1.1) becomes the following diffusive Lotka-Volterra competition model:

$$(1.5)\qquad \begin{cases} -\Delta u=u(a_1-u-b_1v) \\ -\Delta v=v(a_2-b_2u-v) \ \text{ in } \Omega \\ (u,v)=(0,0) \ \text{ on } \partial\Omega, \end{cases}$$

which there has been a great deal of work by many authors. See [1, 2, 3, 4, 9, 10] and the references therein. In [2, 4, 9, 10], the authors investigated the uniqueness of positive solution of the system (1.5). We point out that our uniqueness theorem covers their results since the model (1.5) is a special form of the system (1.1).

In [6], the authors investigated the existence of positive solutions of the following elliptic equations:

$$(1.6)\qquad \begin{cases} -\Delta u = uM(x,u,v) \\ -\Delta v = vN(x,u,v), \ \text{ in } \Omega \\ a\frac{\partial u}{\partial n} + \beta u = 0 \\ b\frac{\partial v}{\partial n} + \sigma v = 0 \ \text{ on } \partial\Omega, \end{cases}$$

where the functions M, N satisfy some conditions, β, σ are positive and a, b are nonnegative constants.

The system (1.4) is the special case of the model (1.6) with $M(x,u,v) = \frac{a_1 - c_1 u - b_1 v}{\alpha_1 + \beta_1 u + \gamma_1 v}$, $N(x,u,v) = \frac{a_2 - b_2 u - c_2 v}{\alpha_2 + \gamma_2 u + \beta_2 v}$ and $a = b = 0$. So we may employ the same method, the fixed point index theory, to show the existence of positive solutions of (1.4). Hence we mainly discuss about the uniqueness of positive solutions of (1.4)(or (1.1)) in this article.

In Section 2, we give some known lemmas and definitions which shall be needed later and state the positive coexistence theorem for the system (1.1). In Section 3, we provide sufficient conditions for the uniqueness of positive solutions to the system (1.1). The methods employed are upper-lower solution technique and the variational characterization of eigenvalues.

2. Preliminaries and coexistence

In this section, we state some known lemmas, definitions and notations which are useful in the following section. The positive coexistence theorem is also provided.

Throughout this article, we denote $\lambda_1(A)$ the first eigenvalue of an operator A under homogeneous Dirichlet boundary conditions.

Consider the following scalar equation:

$$(2.1)\qquad \begin{cases} \Delta u + u\Big(\dfrac{a - cu}{\alpha + \beta u}\Big) = 0 \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega, \end{cases}$$

where $\alpha > 0$, $a > 0$, $c > 0$ and $\beta \geq 0$.

The following lemma can be obtained from the result in [6] with $M(x,u) = \frac{a - cu}{\alpha + \beta u}$ under homogeneous Dirichlet boundary conditions.

LEMMA 2.1. *Suppose that* $\frac{a}{\alpha} > \lambda_1(-\Delta)$. *The problem* (2.1) *has a unique positive solution* $u \in C^2(\bar{\Omega})$.

One can observe that, by the maximum principle, the unique positive solution of (2.1) is bounded by $\frac{a}{c}$.

NOTATION 2.2. *For $\frac{a}{\alpha} > \lambda_1(-\Delta)$, let $\theta(a,c,\alpha,\beta)$ be the unique positive solution of*

$$\begin{cases} \Delta\theta + \theta\Big(\dfrac{a-c\theta}{\alpha+\beta\theta}\Big) = 0 \ \text{ in } \Omega \\ \theta = 0 \ \text{ on } \partial\Omega. \end{cases}$$

Assume that $a > \widetilde{a}$, $c < \widetilde{c}$, $\alpha < \widetilde{\alpha}$ and $\beta < \widetilde{\beta}$. Then for $\frac{a}{\alpha} > \frac{\widetilde{a}}{\widetilde{\alpha}} > \lambda_1(-\Delta)$, let $K\Big[\frac{(a,c,\alpha,\beta)}{(\widetilde{a},\widetilde{c},\widetilde{\alpha},\widetilde{\beta})}\Big]$ denote the supremum of $\frac{\theta(a,c,\alpha,\beta)}{\theta(\widetilde{a},\widetilde{c},\widetilde{\alpha},\widetilde{\beta})}$ on $\bar{\Omega}$.

The following lemma can be obtained by the obvious change of notations from Lemma 4 in [7].

LEMMA 2.3. (i) *$(\frac{a-c\theta}{\alpha+\beta\theta}) \mapsto \theta(a,c,\alpha,\beta)$ is a continuous mapping of $C(\overline{\Omega}\times\mathbb{R}^+) \to C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$.*

(ii) *If $\frac{a-c\theta}{\alpha+\beta\theta} \geq \frac{\tilde{a}-\tilde{c}\theta}{\tilde{\alpha}+\tilde{\beta}\theta} \not\equiv \frac{a-c\theta}{\alpha+\beta\theta}$ for $x \in \Omega$, then either $\theta(a,c,\alpha,\beta) > \theta(\tilde{a},\tilde{c},\tilde{\alpha},\tilde{\beta})$ (in the case $\frac{a}{\alpha} > \frac{\tilde{a}}{\tilde{\alpha}} > \lambda_1(-\Delta)$) or $\theta(a,c,\alpha,\beta) \equiv \theta(\tilde{a},\tilde{c},\tilde{\alpha},\tilde{\beta}) \equiv 0$ (in the other cases).*

Observe that since $\theta(a,c,\alpha,\beta)$, $\theta(\widetilde{a},\widetilde{c},\widetilde{\alpha},\widetilde{\beta}) > 0$ on Ω and $\bar{\Omega}$ is compact, $K\Big[\frac{(a,c,\alpha,\beta)}{(\widetilde{a},\widetilde{c},\widetilde{\alpha},\widetilde{\beta})}\Big]$ is finite. Moreover, the fact $\theta(a,c,\alpha,\beta) > \theta(\widetilde{a},\widetilde{c},\widetilde{\alpha},\widetilde{\beta})$ for $a > \widetilde{a}$, $c < \widetilde{c}$, $\alpha < \widetilde{\alpha}$, $\beta < \widetilde{\beta}$ by Lemma 2.3 implies $K\Big[\frac{(a,c,\alpha,\beta)}{(\widetilde{a},\widetilde{c},\widetilde{\alpha},\widetilde{\beta})}\Big] > 1$.

DEFINITION 2.4. *(i)* (u,v) is called an upper solution for the system (1.1) if

$$\begin{cases} -(\alpha_1+\beta_1 u+\gamma_1 v)\Delta u \geq u(a_1 - c_1 u - b_1 v) \\ -(\alpha_2+\gamma_2 u+\beta_2 v)\Delta v \geq v(a_2 - b_2 u - c_2 v) \ \text{ in } \Omega, \\ u \geq 0, \ \ v \geq 0 \ \text{ on } \partial\Omega, \end{cases}$$

(ii) (u, v) is a lower solution for the system (1.1), if

$$\begin{cases} -(\alpha_1+\beta_1 u+\gamma_1 v)\Delta u \leq u(a_1 - c_1 u - b_1 v) \\ -(\alpha_2+\gamma_2 u+\beta_2 v)\Delta v \leq v(a_2 - b_2 u - c_2 v) \ \text{ in } \Omega, \\ u \leq 0, \ \ v \leq 0 \ \text{ on } \partial\Omega. \end{cases}$$

Next we provide the existence theorem of positive solutions to the system (1.1)(or (1.4)). One can show that if $(\widehat{u},\widehat{v})$ is a solution of (1.1), then $\widehat{u} < \frac{a_1}{c_1}$, $\widehat{v} < \frac{a_2}{c_2}$. So using a weak competition condition, $\frac{c_1}{b_2} > \frac{a_1}{a_2} > \frac{b_1}{c_2}$, it is easy to check that $\frac{a_1-c_1u-b_1v}{\alpha_1+\beta_1u+\gamma_1v}$ and $\frac{a_2-b_2u-c_2v}{\alpha_2+\gamma_2u+\beta_2v}$ are decreasing in both u and v where $u \in [0,\frac{a_1}{c_1})$, $v \in [0,\frac{a_2}{c_2})$. Therefore we may adopt the existence result in [6] with $M(x,u,v) = \frac{a_1-c_1u-b_1v}{\alpha_1+\beta_1u+\gamma_1v}$, $N(x,u,v) =$

$\frac{a_2-b_2u-c_2v}{\alpha_2+\gamma_2u+\beta_2v}$ and $a = b = 0$ since our model is the special case of the system (1.6) in [6].

We consider the existence of semi-trivial solutions of problem (1.1). The system has exactly two semi-trivial solutions when exactly one of the species is absent. Applying Lemma 2.1, notice that there exists a unique positive solution, denoted by u_0, of

$$\begin{cases} -\Delta u = u\Big(\dfrac{a_1 - c_1 u}{\alpha_1 + \beta_1 u}\Big) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

if $\lambda_1(\Delta + \frac{a_1}{\alpha_1}) > 0$, and v_0 solves the equation:

$$\begin{cases} -\Delta v = v\Big(\dfrac{a_2 - c_2 v}{\alpha_2 + \beta_2 v}\Big) \text{ in } \Omega \\ v = 0 \text{ on } \partial\Omega \end{cases}$$

if $\lambda_1(\Delta+\frac{a_2}{\alpha_2}) > 0$. In fact, $u_0 = \theta(a_1, c_1, \alpha_1, \beta_1)$ and $v_0 = \theta(a_2, c_2, \alpha_2, \beta_2)$.

The following is the positive coexistence theorem for the system (1.1).

THEOREM 2.5. *Assume* $\frac{c_1}{b_2} > \frac{a_1}{a_2} > \frac{b_1}{c_2}$. *Suppose* $\frac{a_1}{\alpha_1} > \lambda_1(-\Delta)$ *and* $\frac{a_2}{\alpha_2} > \lambda_1(-\Delta)$. *If the first eigenvalue of the operators* $\Delta + \frac{a_1-b_1v_0}{\alpha_1+\gamma_1v_0}$ *and* $\Delta + \frac{a_2-b_2v_0}{\alpha_2+\gamma_2v_0}$ *has the same sign, i.e., both positive, negative, or zero, then the system* (1.1) *has a positive solution.*

Let $a_i^* = a_i - b_i\Big(\dfrac{a_j}{c_j}\Big)$, $\alpha_i^* = \alpha_i + \gamma_i\Big(\dfrac{a_j}{c_j}\Big)$ for $i, j = 1, 2$ and $i \neq j$. Consider the following condition on the system (1.1):

$$\frac{a_i^*}{\alpha_i^*} > \lambda_1(-\Delta) \text{ for } i = 1, 2. \tag{2.2}$$

Since the condition (2.2) implies $\lambda_1(\Delta + \frac{a_i}{\alpha_i}) > 0$ for $i = 1, 2$, there exist the solutions $\theta(a_i, c_i, \alpha_i, \beta_i)$ and $\theta(a_i^*, c_i, \alpha_i^*, \beta_i)$ for $i = 1, 2$, by Lemma 2.1. Using $\theta(a_i, c_i, \alpha_i, \beta_i) < \frac{a_i}{c_i}$ for $i = 1, 2$, one can see that $(\theta(a_1, c_1, \alpha_1, \beta_1), \theta(a_2, c_2, \alpha_2, \beta_2))$, $(\theta(a_1^*, c_1, \alpha_1^*, \beta_1), \theta(a_2^*, c_2, \alpha_2^*, \beta_2))$ is an upper/lower solution pair. Furthermore observe that the condition (2.2) guarantees the existence of positive solutions to (1.1) since (2.2) makes the assumptions $\lambda_1(\Delta + \frac{a_1-b_1v_0}{\alpha_1+\gamma_1v_0}) > 0$, $\lambda_1(\Delta + \frac{a_2-b_2u_0}{\alpha_2+\gamma_2u_0}) > 0$ in Theorem 2.5 hold. Alternatively, one can show the existence of positive solutions using upper-lower solution method and condition (2.2). We omit such a simple argument.

PROPOSITION 2.6. *Assume that* $\frac{c_1}{b_2} > \frac{a_1}{a_2} > \frac{b_1}{c_2}$ *and* (2.2) *holds. Then any coexistence state* (u, v) *for* (1.1) *satisfies* $\theta(a_1^*, c_1, \alpha_1^*, \beta_1) < u < \theta(a_1, c_1, \alpha_1, \beta_1)$ *and* $\theta(a_2^*, c_2, \alpha_2^*, \beta_2) < v < \theta(a_2, c_2, \alpha_2, \beta_2)$.

Proof. If (u, v) is a coexistence state for the system (1.4), then it is easy to see that u is a lower solution to the problem:

$$\Delta u + u\Big(\frac{a_1 - c_1 u}{\alpha_1 + \beta_1 u}\Big) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{2.3}$$

and v is a lower solution to the problem:

$$\Delta v + v\Big(\frac{a_2 - c_2 v}{\alpha_2 + \beta_2 v}\Big) = 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega.$$

Since any constant larger than $\frac{a_1}{c_1}$ is an upper solution of (2.3), the uniqueness of a positive solution u to (2.3) implies $u < \theta(a_1, c_1, \alpha_1, \beta_1) < \frac{a_1}{c_1}$. Similar argument gives $v < \theta(a_2, c_2, \alpha_2, \beta_2) < \frac{a_2}{c_2}$.

On the other hand, since $u < \frac{a_1}{c_1}$ and $v < \frac{a_2}{c_2}$, u is an upper solution of the equation:

$$\Delta u + u\Big(\frac{a_1^* - c_1 u}{\alpha_1^* + \beta_1 u}\Big) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{2.4}$$

and v is an upper solution of the problem:

$$\Delta v + v\Big(\frac{a_2^* - c_2 v}{\alpha_2^* + \beta_2 v}\Big) = 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega.$$

If (2.2) holds, then the unique solution $\theta(a_1^*, c_1, \alpha_1^*, \beta_1)$ of (2.4) exists by Lemma 2.1. Since u is an upper solution of (2.4), $u > \theta(a_1^*, c_1, \alpha_1^*, \beta_1)$. Similarly, $v > \theta(a_2^*, c_2, \alpha_2^*, \beta_2)$. □

Take $a = \max\{a_i\}$, $\alpha = \min\{\alpha_i\}$, $c = \min\{c_i\}$, $\beta = \min\{\beta_i\}$, $\widetilde{a} = \min\{a_i^*\}$, $\widetilde{\alpha} = \max\{\alpha_i^*\}$, $\widetilde{c} = \max\{c_i\}$ and $\widetilde{\beta} = \max\{\beta_i\}$. We impose the following assumption

$$\frac{\widetilde{a}}{\widetilde{\alpha}} > \lambda_1(-\Delta). \tag{2.5}$$

Then the assumption (2.5) implies the condition (2.2) and so guarantees the existence of $\theta(a, c, \alpha, \beta)$, $\theta(\widetilde{a}, \widetilde{c}, \widetilde{\alpha}, \widetilde{\beta})$. Moreover $\theta(a, c, \alpha, \beta) \geq \theta(a_i, c_i, \alpha_i, \beta_i)$ and $\theta(\widetilde{a}, \widetilde{c}, \widetilde{\alpha}, \widetilde{\beta}) \leq \theta(a_i^*, c_i^*, \alpha_i, \beta_i)$ are satisfied for $i = 1, 2$. Thus by Proposition 2.6, $K\Big[\frac{(a,c,\alpha,\beta)}{(\widetilde{a},\widetilde{c},\widetilde{\alpha},\widetilde{\beta})}\Big] \geq \frac{v}{u}, \frac{u}{v}$ where (u, v) is a positive solution of (1.1).

3. Uniqueness of positive solutions

In this section, we give sufficient conditions for the uniqueness of positive solutions of (1.1).

THEOREM 3.1. *Suppose that a weak competition model* (1.1) *satisfies* (2.5). *The coexistence state is unique provided that for* $i, j = 1, 2$, $i \neq j$, *the inequalities*

$$\frac{c_i b_i}{M_i} > \frac{\beta_i a_i}{\alpha_i^2} + \frac{M_i}{2\alpha_i^2} + \frac{M_j}{2\alpha_j^2} K\left[\frac{(a, c, \alpha, \beta)}{(\tilde{a}, \tilde{c}, \tilde{\alpha}, \tilde{\beta})}\right] \tag{3.1}$$

hold where $M_k = b_k\alpha_k + \gamma_k a_k + b_k\beta_k \frac{a_k}{c_k}$.

Proof. Suppose that (u_1, v_1) and (u_2, v_2) are positive solutions of the system (1.1)(or (1.4)). Then those satisfy

$$\Delta u_1 + u_1\left(\frac{a_1 - c_1u_1 - b_1v_1}{\alpha_1 + \beta_1u_1 + \gamma_1v_1}\right) = 0 \text{ in } \Omega, \quad u_1 = 0 \text{ on } \partial\Omega. \tag{3.2}$$

and

$$\Delta u_2 + u_2\left(\frac{a_1 - c_1u_2 - b_1v_2}{\alpha_1 + \beta_1u_2 + \gamma_1v_2}\right) = 0 \text{ in } \Omega, \quad u_2 = 0 \text{ on } \partial\Omega. \tag{3.3}$$

Let $w_1 = u_1 - u_2$, $w_2 = v_1 - v_2$. Subtracting (3.3) from (3.2) and some substitutions provide

$$\begin{cases} \Delta w_1 + \dfrac{1}{\alpha_1 + \beta_1u_1 + \gamma_1v_1}\{(a_1 - c_1u_1 - b_1v_1)w_1 \\ \qquad\qquad +\beta_1(\Delta u_2)w_1 + \gamma_1(\Delta u_2)w_2 \\ \qquad\qquad\qquad -c_1u_2w_1 - b_1u_2w_2\} = 0 \text{ in } \Omega, \\ w_1 = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.4}$$

Since $(u_1, v_1) > 0$ is a solution of (1.1), we may observe that the principal eigenvalue of the problem

$$\begin{cases} \Delta\psi + \psi\left(\dfrac{a_1 - c_1u_1 - b_1v_1}{\alpha_1 + \beta_1u_1 + \gamma_1v_1}\right) = \sigma_1\psi \text{ in } \Omega, \\ \psi = 0 \text{ on } \partial\Omega \end{cases}$$

is $\sigma_1 = 0$, with any multiple of u_1 as the eigenfunction.

By the variational property of eigenvalues, for any $z \in W_0^{1,2}(\Omega)$,

$$\int_\Omega \left[|\nabla z|^2 - z^2\left(\frac{a_1 - c_1u_1 - b_1v_1}{\alpha_1 + \beta_1u_1 + \gamma_1v_1}\right)\right] \geq 0. \tag{3.5}$$

Multiplying (3.4) by w_1, integrating by parts and using (3.5) imply

$$\int_\Omega \frac{1}{\alpha_1+\beta_1 u_1+\gamma_1 v_1}\{c_1 u_2 w_1^2 + b_1 u_2 w_2 w_1 - \beta_1(\Delta u_2) w_1^2 - \gamma_1(\Delta u_2) w_2 w_1\} \le 0. \tag{3.6}$$

Applying the same argument that we did in the above for (3.2)-(3.6) to the two equations involving v_1 and v_2 corresponding to (3.2)-(3.3), we obtain

$$\int_\Omega \frac{1}{\alpha_2+\gamma_2 u_2+\beta_2 v_2}\{b_2 v_1 w_1 w_2 + c_2 v_1 w_2^2 - \beta_2(\Delta v_1) w_2^2 - \gamma_2(\Delta v_1) w_1 w_2\} \le 0. \tag{3.7}$$

Add (3.6) to (3.7) to get

$$\begin{aligned} 0 &\ge \int_\Omega \Big[\frac{c_1 u_2 w_1^2 + b_1 u_2 w_2 w_1 - \beta_1(\Delta u_2) w_1^2 - \gamma_1(\Delta u_2) w_2 w_1}{\alpha_1+\beta_1 u_1+\gamma_1 v_1} \\ &\qquad + \frac{b_2 v_1 w_1 w_2 + c_2 v_1 w_2^2 - \beta_2(\Delta v_1) w_2^2 - \gamma_2(\Delta v_1) w_1 w_2}{\alpha_2+\gamma_2 u_2+\beta_2 v_2}\Big] \\ &= \int_\Omega \Big[\frac{c_1 u_2 - \beta_1(\Delta u_2)}{\alpha_1+\beta_1 u_1+\gamma_1 v_1} w_1^2 + \Big(\frac{b_1 u_2 - \gamma_1(\Delta u_2)}{\alpha_1+\beta_1 u_1+\gamma_1 v_1} \\ &\qquad + \frac{b_2 v_1 - \gamma_2(\Delta v_1)}{\alpha_2+\gamma_2 u_2+\beta_2 v_2}\Big) w_1 w_2 + \frac{c_2 v_1 - \beta_2(\Delta v_1)}{\alpha_2+\gamma_2 u_2+\beta_2 v_2} w_2^2\Big]. \end{aligned} \tag{3.8}$$

Observe that now it suffices to find conditions which make the right side of (3.8) positive definite to conclude $w_1 = w_2 = 0$ which shows the uniqueness of positive solutions.

For simplicity for the coefficients of w_1^2, $w_1 w_2$ and w_2^2 in (3.8),we use the notations :

$$A = \frac{c_1 u_2 - \beta_1(\Delta u_2)}{\alpha_1+\beta_1 u_1+\gamma_1 v_1},\ B = \frac{b_1 u_2 - \gamma_1(\Delta u_2)}{\alpha_1+\beta_1 u_1+\gamma_1 v_1},$$
$$C = \frac{b_2 v_1 - \gamma_2(\Delta v_1)}{\alpha_2+\gamma_2 u_2+\beta_2 v_2},\ D = \frac{c_2 v_1 - \beta_2(\Delta v_1)}{\alpha_2+\gamma_2 u_2+\beta_2 v_2}.$$

Plugging $-\Delta u_2 = u_2\big(\frac{a_1 - c_1 u_2 - b_1 v_2}{\alpha_1+\beta_1 u_2+\gamma_1 v_2}\big)$ in A, the coefficient A of w_1^2 is equal to $\frac{1}{\alpha_1+\beta_1 u_1+\gamma_1 v_1}[c_1\ u_2 + \beta_1 u_2\big(\frac{a_1 - c_1 u_2 - b_1 v_2}{\alpha_1+\beta_1 u_2+\gamma_1 v_2}\big)]$. For $u_2 \in [0, \frac{a_1}{c_1})$ and

$v_2 \in [0, \frac{a_2}{c_2})$, the fact of $\left(\frac{a_1 - c_1 u_2 - b_1 v_2}{\alpha_1 + \beta_1 u_2 + \gamma_1 v_2}\right)_u < 0$ implies that

$$\begin{aligned} & c_1 u_2 + \beta_1 u_2 \Big(\frac{a_1 - c_1 u_2 - b_1 v_2}{\alpha_1 + \beta_1 u_2 + \gamma_1 v_2} \Big) \\ = {} & u_2 \Big(\frac{c_1(\alpha_1 + \beta_1 u_2 + \gamma_1 v_2) + \beta_1 (a_1 - c_1 u_2 - b_1 v_2)}{\alpha_1 + \beta_1 u_2 + \gamma_1 v_2} \Big) \\ = {} & - u_2 (\alpha_1 + \beta_1 u_2 + \gamma_1 v_2) \Big(\frac{a_1 - c_1 u_2 - b_1 v_2}{\alpha_1 + \beta_1 u_2 + \gamma_1 v_2} \Big)_u > 0. \end{aligned}$$

So A is positive. Similarly, it is not hard to see that the coefficient D of w_2^2 is also positive.

On the other hand, by plugging $-\Delta u_2 = u_2 \big(\frac{a_1 - c_1 u_2 - b_1 v_2}{\alpha_1 + \beta_1 u_2 + \gamma_1 v_2} \big)$ in B and using $u_2 < \frac{a_1}{c_1}$,

$$\begin{aligned} B &= \frac{1}{\alpha_1 + \beta_1 u_1 + \gamma_1 v_1} [b_1 u_2 + \gamma_1 u_2 (\frac{a_1 - c_1 u_2 - b_1 v_2}{\alpha_1 + \beta_1 u_2 + \gamma_1 v_2})] \\ &= \frac{1}{\alpha_1 + \beta_1 u_1 + \gamma_1 v_1} \Big[\frac{b_1(\alpha_1 + \beta_1 u_2 + \gamma_1 v_2) + \gamma_1 (a_1 - c_1 u_2 - b_1 v_2)}{\alpha_1 + \beta_1 u_2 + \gamma_1 v_2} \Big] u_2 \\ &= \frac{1}{\alpha_1 + \beta_1 u_1 + \gamma_1 v_1} \Big[\frac{b_1(\alpha_1 + \beta_1 u_2) + \gamma_1 (a_1 - c_1 u_2)}{\alpha_1 + \beta_1 u_2 + \gamma_1 v_2} \Big] u_2 > 0. \end{aligned}$$

So B is positive. Similarly, we can check the positivity of C. Thus the coefficient $B + C$ of $w_1 \cdot w_2$ is positive. Therefore the right hand side of (3.8) can be considered as a quadratic form in the variables w_i for $i = 1, 2$.

Using $v < \frac{a_2}{c_2}$ and the fact that $u < \frac{a_1}{c_1}$ implies $a_1 u - c_1 u^2 > 0$, we have

$$\begin{aligned} \frac{a_1}{\alpha_1} u > -\Delta u = u \Big(\frac{a_1 - c_1 u - b_1 v}{\alpha_1 + \beta_1 u + \gamma_1 v} \Big) &> \frac{-b_1 uv}{\alpha_1 + \beta_1 u + \gamma_1 v} \\ &\geq \frac{-b_1 uv}{\alpha_1 + \gamma_1 v} \geq \frac{-b_1 u}{\alpha_1 (c_2/a_2) + \gamma_1} > -\frac{a_1}{\alpha_1} u. \end{aligned}$$

The last inequality follows from the fact:

$$\begin{aligned} & \frac{a_1}{\alpha_1} - \frac{b_1}{\alpha_1 (c_2/a_2) + \gamma_1} = \frac{a_1[\alpha_1 (c_2/a_2) + \gamma_1] - b_1 \alpha_1}{\alpha_1 (\alpha_1 (c_2/a_2) + \gamma_1)} \\ = {} & \frac{\frac{a_1}{a_2} \alpha_1 c_2 + a_1 \gamma_1 - b_1 \alpha_1}{\alpha_1 (\alpha_1 (c_2/a_2) + \gamma_1)} > \frac{\frac{b_1}{c_2} \alpha_1 c_2 + a_1 \gamma_1 - b_1 \alpha_1}{\alpha_1 (\alpha_1 (c_2/a_2) + \gamma_1)} \\ = {} & \frac{b_1 \alpha_1 + a_1 \gamma_1 - b_1 \alpha_1}{\alpha_1 (\alpha_1 (c_2/a_2) + \gamma_1)} = \frac{a_1 \gamma_1}{\alpha_1 (\alpha_1 (c_2/a_2) + \gamma_1)} \geq 0. \end{aligned}$$

Thus a solution (u, v) of (1.1) satisfies

$$\frac{a_1}{\alpha_1}u > -\Delta u > -\frac{a_1}{\alpha_1}u \tag{3.9}$$

and similarly,

$$\frac{a_2}{\alpha_2}v > -\Delta v > -\frac{a_2}{\alpha_2}v. \tag{3.10}$$

Since $0 < u_1,\ u_2 < \frac{a_1}{c_1}$ and $0 < v_1,\ v_2 < \frac{a_2}{c_2}$, we observe that $\alpha_1 \le \alpha_1+\beta_1 u_1+\gamma_1 v_1 \le \alpha_1+\beta_1(a_1/c_1)+\gamma_1(a_2/c_2)$ and $\alpha_2 \le \alpha_2+\gamma_2 u_2+\beta_2 v_2 \le \alpha_2 + \gamma_2(a_1/c_1) + \beta_2(a_2/c_2)$.

The inequality (3.9) and (3.10) implies that

$$\begin{aligned} A &= \frac{c_1 u_2}{\alpha_1 + \beta_1 u_1 + \gamma_1 v_1} - \frac{\beta_1(\Delta u_2)}{\alpha_1 + \beta_1 u_1 + \gamma_1 v_1} \\ &> \Big(\frac{c_1}{\alpha_1 + \beta_1(a_1/c_1) + \gamma_1(a_2/c_2)} - \frac{\beta_1 a_1}{\alpha_1^2}\Big)u_2 \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} D &= \frac{c_2 v_1}{\alpha_2 + \gamma_2 u_2 + \beta_2 v_2} - \frac{\beta_2(\Delta v_1)}{\alpha_2 + \gamma_2 u_2 + \beta_2 v_2} \\ &> \Big(\frac{c_2}{\alpha_2 + \gamma_2(a_1/c_1) + \beta_2(a_2/c_2)} - \frac{\beta_2 a_2}{\alpha_2^2}\Big)v_1. \end{aligned} \tag{3.12}$$

For B and C, we already calculated the following:

$$B = \frac{1}{\alpha_1 + \beta_1 u_1 + \gamma_1 v_1}\Big[\frac{b_1\alpha_1 + \gamma_1 a_1 + (b_1\beta_1 - \gamma_1 c_1)u_2}{\alpha_1 + \beta_1 u_2 + \gamma_1 v_2}\Big]u_2 \tag{3.13}$$

and

$$C = \frac{1}{\alpha_2 + \gamma_2 u_2 + \beta_2 v_2}\Big[\frac{b_2\alpha_2 + \gamma_2 a_2 + (b_2\beta_2 - \gamma_2 c_2)v_1}{\alpha_2 + \gamma_2 u_1 + \beta_2 v_1}\Big]v_1. \tag{3.14}$$

Next observe that (3.11)-(3.14) imply that $\int_\Omega [Aw_1^2 + (B+C)w_1 w_2 + Dw_2^2]$ is bigger than

$$\begin{aligned} \int_\Omega \Big[&\Big(\frac{c_1}{\alpha_1 + \beta_1(a_1/c_1) + \gamma_1(a_2/c_2)} - \frac{\beta_1 a_1}{\alpha_1^2}\Big)u_2 w_1^2 \\ &+ \Big(\frac{b_1\alpha_1 + \gamma_1 a_1 + (b_1\beta_1 - \gamma_1 c_1)u_2}{(\alpha_1 + \beta_1 u_1 + \gamma_1 v_1)(\alpha_1 + \beta_1 u_2 + \gamma_1 v_2)}\Big)u_2 w_1 w_2 \\ &+ \Big(\frac{b_2\alpha_2 + \gamma_2 a_2 + (b_2\beta_2 - \gamma_2 c_2)v_1}{(\alpha_2 + \gamma_2 u_1 + \beta_2 v_1)(\alpha_2 + \gamma_2 u_2 + \beta_2 v_2)}\Big)v_1 w_1 w_2 \\ &+ \Big(\frac{c_2}{\alpha_2 + \gamma_2(a_1/c_1) + \beta_2(a_2/c_2)} - \frac{\beta_2 a_2}{\alpha_2^2}\Big)v_1 w_2^2\Big]. \end{aligned} \tag{3.15}$$

Using the fact that B, $C > 0$ and $-w_1 w_2 \le w_1^2/2 + w_2^2/2$, we can see that (3.15) is larger than

$$\begin{aligned}(3.16)\quad \int_\Omega \Big[&\Big(\frac{c_1}{\alpha_1 + \beta_1(a_1/c_1) + \gamma_1(a_2/c_2)} - \frac{\beta_1 a_1}{\alpha_1^2}\Big) u_2 w_1^2 \\ &- \Big(\frac{b_1\alpha_1 + \gamma_1 a_1 + (b_1\beta_1 - \gamma_1 c_1)u_2}{(\alpha_1 + \beta_1 u_1 + \gamma_1 v_1)(\alpha_1 + \beta_1 u_2 + \gamma_1 v_2)}\Big) u_2 \Big(\frac{w_1^2}{2} + \frac{w_2^2}{2}\Big) \\ &- \Big(\frac{b_2\alpha_2 + \gamma_2 a_2 + (b_2\beta_2 - \gamma_2 c_2)v_1}{(\alpha_2 + \gamma_2 u_1 + \beta_2 v_1)(\alpha_2 + \gamma_2 u_2 + \beta_2 v_2)}\Big) v_1 \Big(\frac{w_1^2}{2} + \frac{w_2^2}{2}\Big) \\ &+ \Big(\frac{c_2}{\alpha_2 + \gamma_2(a_1/c_1) + \beta_2(a_2/c_2)} - \frac{\beta_2 a_2}{\alpha_2^2}\Big) v_1 w_2^2 \Big].\end{aligned}$$

So if we can get the inequality (3.16) > 0, the positive definiteness of the right hand side of (3.8) provides the uniqueness of the coexistence state for (1.1). Combining all terms involving w_1^2 in the right hand side of (3.16), the coefficient of w_1^2 becomes

$$\begin{aligned}(3.17)\quad &\Big(\frac{c_1}{\alpha_1 + \beta_1(a_1/c_1) + \gamma_1(a_2/c_2)} - \frac{\beta_1 a_1}{\alpha_1^2}\Big) u_2 \\ &- \Big(\frac{b_1\alpha_1 + \gamma_1 a_1 + (b_1\beta_1 - \gamma_1 c_1)u_2}{(\alpha_1 + \beta_1 u_1 + \gamma_1 v_1)(\alpha_1 + \beta_1 u_2 + \gamma_1 v_2)}\Big) \frac{u_2}{2} \\ &- \Big(\frac{b_2\alpha_2 + \gamma_2 a_2 + (b_2\beta_2 - \gamma_2 c_2)v_1}{(\alpha_2 + \gamma_2 u_2 + \beta_2 v_2)(\alpha_2 + \gamma_2 u_1 + \beta_2 v_1)}\Big) \frac{v_1}{2}\end{aligned}$$

and the coefficient of w_2^2 is

$$\begin{aligned}(3.18)\quad &\Big(\frac{c_2}{\alpha_2 + \gamma_2(a_1/c_1) + \beta_2(a_2/c_2)} - \frac{\beta_2 a_2}{\alpha_2^2}\Big) v_1 \\ &- \Big(\frac{b_1\alpha_1 + \gamma_1 a_1 + (b_1\beta_1 - \gamma_1 c_1)u_2}{(\alpha_1 + \beta_1 u_1 + \gamma_1 v_1)(\alpha_1 + \beta_1 u_2 + \gamma_1 v_2)}\Big) \frac{u_2}{2} \\ &- \Big(\frac{b_2\alpha_2 + \gamma_2 a_2 + (b_2\beta_2 - \gamma_2 c_2)v_1}{(\alpha_2 + \gamma_2 u_2 + \beta_2 v_2)(\alpha_2 + \gamma_2 u_1 + \beta_2 v_1)}\Big) \frac{v_1}{2}\end{aligned}$$

Using (3.13), (3.14) and the fact that the positivity of B, C implies $b_1\alpha_1 + \gamma_1 a_1 + (b_1\beta_1 - \gamma_1 c_1)u_2 > 0$ and $b_2\alpha_2 + \gamma_2 a_2 + (b_2\beta_2 - \gamma_2 c_2)v_1 > 0$, we can see that (3.17) is bigger than

$$\begin{aligned}(3.19)\qquad &\Big(\frac{c_1}{\alpha_1+\beta_1(a_1/c_1)+\gamma_1(a_2/c_2)}-\frac{\beta_1 a_1}{\alpha_1^2}\Big)u_2\\ &-\Big(\frac{b_1\alpha_1+\gamma_1 a_1+(b_1\beta_1-\gamma_1 c_1)u_2}{\alpha_1^2}\Big)\frac{u_2}{2}\\ &-\Big(\frac{b_2\alpha_2+\gamma_2 a_2+(b_2\beta_2-\gamma_2 c_2)v_1}{\alpha_2^2}\Big)\frac{v_1}{2}.\end{aligned}$$

Furthermore $u_2 < \frac{a_1}{c_1}$ and $v_1 < \frac{a_2}{c_2}$ imply that the right hand side of (3.19) is bigger than

$$\begin{aligned}(3.20)\qquad &\Big(\frac{c_1}{\alpha_1+\beta_1(a_1/c_1)+\gamma_1(a_2/c_2)}-\frac{\beta_1 a_1}{\alpha_1^2}\Big)u_2\\ &-\Big(\frac{b_1\alpha_1+\gamma_1 a_1+b_1\beta_1\frac{a_1}{c_1}}{\alpha_1^2}\Big)\frac{u_2}{2}\\ &-\Big(\frac{b_2\alpha_2+\gamma_2 a_2+b_2\beta_2\frac{a_2}{c_2}}{\alpha_2^2}\Big)\frac{v_1}{2}.\end{aligned}$$

So if we find conditions which satisfy (3.20) > 0, then we will get the desired results.

Divide the inequality (3.20) > 0 by u_2 to get

$$\begin{aligned}(3.21)\qquad &\Big(\frac{c_1}{\alpha_1+\beta_1(a_1/c_1)+\gamma_1(a_2/c_2)}-\frac{\beta_1 a_1}{\alpha_1^2}\Big)\\ &-\Big(\frac{b_1\alpha_1+\gamma_1 a_1+b_1\beta_1\frac{a_1}{c_1}}{2\alpha_1^2}\Big)\\ &-\Big(\frac{b_2\alpha_2+\gamma_2 a_2+b_2\beta_2\frac{a_2}{c_2}}{2\alpha_2^2}\Big)\Big(\frac{v_1}{u_2}\Big)>0.\end{aligned}$$

By the definition of $K\Big[\frac{(a,c,\alpha,\beta)}{(\tilde{a},\tilde{c},\tilde{\alpha},\tilde{\beta})}\Big]$, we get $\frac{v_1}{u_2}\le K\Big[\frac{(a,c,\alpha,\beta)}{(\tilde{a},\tilde{c},\tilde{\alpha},\tilde{\beta})}\Big]$. Moreover, a weak competition condition implies $\frac{a_1}{b_1}>\frac{a_2}{c_2}$. Using these facts, we can see that (3.21) holds if

$$\begin{aligned}(3.22)\qquad &\Big(\frac{c_1}{\alpha_1+\beta_1(a_1/c_1)+\gamma_1(a_1/b_1)}-\frac{\beta_1 a_1}{\alpha_1^2}\Big)\\ &-\Big(\frac{b_1\alpha_1+\gamma_1 a_1+b_1\beta_1\frac{a_1}{c_1}}{2\alpha_1^2}\Big)\\ &-\Big(\frac{b_2\alpha_2+\gamma_2 a_2+b_2\beta_2\frac{a_2}{c_2}}{2\alpha_2^2}\Big)K\Big[\frac{(a,c,\alpha,\beta)}{(\tilde{a},\tilde{c},\tilde{\alpha},\tilde{\beta})}\Big]>0.\end{aligned}$$

Applying the same method as we did in (3.19)-(3.22) to (3.18), we obtain

$$\begin{aligned}&\Big(\frac{c_2}{\alpha_2+\gamma_2(a_2/b_2)+\beta_2(a_2/c_2)}-\frac{\beta_2 a_2}{\alpha_2^2}\Big)\\&\quad-\Big(\frac{b_2\alpha_2+\gamma_2 a_2+b_2\beta_2\frac{a_2}{c_2}}{2\alpha_2^2}\Big)\\&\quad-\Big(\frac{b_1\alpha_1+\gamma_1 a_1+b_1\beta_1\frac{a_1}{c_1}}{2\alpha_1^2}\Big)K\Big[\frac{(a,c,\alpha,\beta)}{(\widetilde{a},\widetilde{c},\widetilde{\alpha},\widetilde{\beta})}\Big]>0.\end{aligned}$$

This completes the proof. □

Before we end this article, we discuss a known result in the literature which is an immediate consequence from our uniqueness theorem.

Consider the following diffusive Lotka-Volterra model:

$$\text{(3.23)}\qquad \begin{cases}-\Delta u=u(a_1-c_1u-b_1v)\\ -\Delta v=v(a_2-b_2u-c_2v)\quad \text{in } \Omega,\\ (u,v)=(0,0)\quad \text{on } \partial\Omega,\end{cases}$$

where a_i, b_i, c_i, $i=1,2$, are positive constants with $a_1\neq a_2$.

In [4], authors gave the condition for the uniqueness of positive solutions to (3.23) as the following:

$$\text{(3.24)}\quad 4c_1c_2>\frac{c_2b_1^2}{c_1}K(a_1,a_2-b_2\frac{a_1}{c_1})+2b_1b_2+\frac{c_1b_2^2}{c_2}K(a_2,a_1-b_1\frac{a_2}{c_2}).$$

Here $K(a_1,a_2-b_2a_1/c_1)=K\Big[\frac{(a_1,1,1,0)}{(a_2^*,1,1,0)}\Big]$ and $K(a_2,a_1-b_1a_2/c_2)=K\Big[\frac{(a_2,1,1,0)}{(a_1^*,1,1,0)}\Big]$.

COROLLARY 3.2. *The corresponding conditions to* (3.1) *for the system* (3.23) *imply the condition* (3.24).

Proof. If we take $\alpha_i=1$, $\beta_i=0$ and $\gamma_i=0$ in (1.1) for $i=1,2$, then the conditions (3.1) become

$$\text{(3.25)}\qquad 2c_1>b_1+b_2K\Big[\frac{(a,c,1,0)}{(\widetilde{a},\widetilde{c},1,0)}\Big],\quad 2c_2>b_2+b_1K\Big[\frac{(a,c,1,0)}{(\widetilde{a},\widetilde{c},1,0)}\Big].$$

Then (3.25) implies that

(3.26)

$$\begin{aligned}4c_1c_2&>\Big(b_1+b_2K\Big[\frac{(a,c,1,0)}{(\widetilde{a},\widetilde{c},1,0)}\Big]\Big)\Big(b_2+b_1K\Big[\frac{(a,c,1,0)}{(\widetilde{a},\widetilde{c},1,0)}\Big]\Big)\\&=b_1^2K\Big[\frac{(a,c,1,0)}{(\widetilde{a},\widetilde{c},1,0)}\Big]+b_1b_2+b_2^2K\Big[\frac{(a,c,1,0)}{(\widetilde{a},\widetilde{c},1,0)}\Big]+b_1b_2K\Big[\frac{(a,c,1,0)}{(\widetilde{a},\widetilde{c},1,0)}\Big]^2.\end{aligned}$$

Note $K\left[\frac{(a,c,1,0)}{(\tilde{a},\tilde{c},1,0)}\right]^2 > 1$, since $K\left[\frac{(a,c,1,0)}{(\tilde{a},\tilde{c},1,0)}\right] > 1$. By the definition of a, c, $\tilde{a}$ and $\tilde{c}$, $K\left[\frac{(a,c,1,0)}{(\tilde{a},\tilde{c},1,0)}\right] \geq K\left[\frac{(a_1,c_1,1,0)}{(a_2^*,c_2,1,0)}\right]$ and $K\left[\frac{(a,c,1,0)}{(\tilde{a},\tilde{c},1,0)}\right] \geq K\left[\frac{(a_2,c_2,1,0)}{(a_1^*,c_1,1,0)}\right]$. Using the fact $\theta(a,c,1,0) = \frac{1}{c}\theta(a,1,1,0)$, we have that $K\left[\frac{(a_1,c_1,1,0)}{(a_2^*,c_2,1,0)}\right] = \frac{c_2}{c_1}K\left[\frac{(a_1,1,1,0)}{(a_2^*,1,1,0)}\right]$ and $K\left[\frac{(a_2,c_2,1,0)}{(a_1^*,c_1,1,0)}\right] = \frac{c_1}{c_2}K\left[\frac{(a_2,1,1,0)}{(a_1^*,1,1,0)}\right]$. Thus we get the condition (3.24) which is the uniqueness condition in [4]. □

REMARK 3.3. The result in [3] also can be obtained from Theorem 3.1. Namely, a positive solution of the competition model :

$$(3.27) \qquad \begin{cases} -\Delta u = u(a-u-cv) \\ -\Delta v = v(a-eu-v) \quad \text{in } \Omega, \\ (u,v) = (0,0) \quad \text{on } \partial\Omega \end{cases}$$

is unique if $0 < c < 1$, $0 < e < 1$. This uniqueness theorem follows with ease : A weak competition condition implies their assumptions $0 < c, e < 1$. So if we take $\beta_1 = \beta_2 = 0$, $a_1 = a_2 = b$, $\gamma_1 = \gamma_2 = 0$, $\alpha_1 = \alpha_2 = 1$, $c_1 = c_2 = 1$. $b_1 = c$, $b_2 = e$ in (1.1), then the assumption (3.1) in Theorem 3.1, $c+e < 2$, holds always for the model (3.27) without any extra assumptions.

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