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Valuation of Options in Incomplete Markets*

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■ Abstract ■

The purpose of this paper is studying the valuation of option prices in incomplete markets. A market is said to be incomplete if the given traded assets are insufficient to hedge a contingent claim. This situation occurs, for example, when the underlying stock process follows jump-diffusion processes. Due to the jump part, it is impossible to construct a hedging portfolio with stocks and riskless assets. Contrary to the case of a complete market in which only one equivalent martingale measure exists, there are infinite numbers of equivalent martingale measures in an incomplete market. Our research here is focusing on risk minimizing hedging strategy and its associated minimal martingale measure under the jump-diffusion processes. Based on this risk minimizing hedging strategy, we characterize the dynamics of a risky asset and derive the valuation formula for an option price. The main contribution of this paper is to obtain an analytical formula for a European option price under the jump-diffusion processes using the minimal martingale measure.

Keyword : Option Price, Jump-Diffusion, Incomplete Market, Minimal Martingale Measure, Risk Minimizing Hedging Strategy

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1. Introduction

The problem of pricing of options from the price dynamics of stock is well understood in the context of complete markets. In this situation, a self-financing hedging portfolio is formed such that the terminal wealth of the portfolio has a random value equal to that of the option's payoff. However, if the underlying asset follows a jump-diffusion process, we cannot construct a dynamic portfolio to replicate the payoff of an option perfectly(i.e., perfect replication), and this security market is called incomplete. Contrary to a complete market in which only one equivalent martingale measure(or a risk-neutral measure) exists, in an incomplete market there are infinite numbers of equivalent martingale measures, and the risk exposure cannot always be eliminated completely by means of a judicious trading strategy. One can only hope to find the best strategy based on one's "reasonable" criterion.

One of the approaches to handle a valuation problem in incomplete market is based on the local risk minimizing strategy as discussed in several papers by Föllmer, Schweizer, and Sondermann(Föllmer and Sondermann(1986), Föllmer and Schweizer(1990), and Schweizer(1991)). The local risk minimizing strategy allows cash-inflows or outflows and looks for an admissible trading strategy that minimizes the magnitude of cash-flows. An equivalent martingale measure derived from this strategy is called the minimal martingale measure. Colwell and Elliott(1993) derive and explore the characteristics of the minimal martingale measure for the jump-diffusion model by applying the local risk minimizing strategy to a valuation problem in the presence of jumps.

Another available trading strategy is the variance minimizing hedging strategy used by Duffie and Jackson(1990), Duffie and Richardson(1991), Schäl(1994), Schweizer(1992, 1995, 1996), and Bertsimas, Kogan and Lo(2001). This strategy searches for an optimal self-financing strategy that minimizes the expected quadratic terminal risk with or without a prespecified initial investment. Bertsimas, Kogan, and Lo(2001), applying stochastic dynamic programming to the minimization of a mean variance function under Markov state-dynamics, derive recursive expression for the optimal replicating strategy. They show that the replicating cost that minimizes the mean variance function under an equivalent martingale measure corresponds to the equilibrium price of the option. El Karoui and Quenez(1995) assert that there is a price range for the actual market price of an option in an incomplete market and study its maximum and minimum price using stochastic control methods. One of their findings is that the maximum price is the selling price defined as the smallest price that allows the seller to hedge completely by a trading portfolio. A similar result is obtained for the minimum price.

Davis, Panas, and Zariphopoulou(1993), Davis and Zariphopoulou(1995), and Constantinides and Zariphopoulou(1999) suggest an option price as the maximum price at which a utility-maximizing investor would include the option in his or her portfolio. They characterize the fair prices for European and American options based on utility maximization in the presence of transaction costs. They transform this problem into that similar to Merton's problem(1992) of optimal consumption and portfolio choice for a single investor in an intertemporal economy. The investor's goal is to maximize his or her expected utility from terminal wealth and/or the expected utility of intermediate consumption and the goal is characterized by a value function. They then derive Hamilton-Jacobi-Bellman(HJB) equations for an option pricing problem under a suitably chosen utility function. As pointed out by Pham(1998), however, there is not in general a smooth solution of the HJB equation especially when the diffusion coefficients is degenerate. Therefore one is forced to use a notion of weak solutions such as viscosity solutions which has been first introduced to finance by Zariphopoulou(1999).

Using the risk minimizing trading strategy proposed by Föllmer, Schweizer and Sondermann(1986, 1990, and 1991) and under the minimal martingale measure driven by the strategy, we derive a pricing formula of a European option under jump-diffusion processes. In a simple case where we assume that the actual probability measure is a martingale measure(i.e., the actual world is risk-neutral), the option prices obtained from the risk-minimizing strategy coincide with those derived by Merton(1976). We find however that the ways of "balancing" market prices of two risk sources(diffusion and jump parts) between the two models are totally different. In other words, the Radon-Nikodym derivative process of the simple model is different from that of Merton's model, although the corresponding option pricing formulas are the same.

The outline of the rest of this paper is as follows. In section 2, we present the necessary frameworks, make assumptions, define the notations, introduce the minimal martingale measure, and discuss several related issues on option pricing problems in the presence of jumps under the minimal martingale measure. In section 3, we derive a pricing formula of a European option under the minimal martingale measure. Finally, section 4 concludes this paper.

2. Framework for Valuation in Incomplete Markets

In this section, we make some assumptions on a market, tradable assets, price processes of the assets, and the associated parameters on a probability space. Next, we begin in the next subsection with an introduction of Merton's model(1976). This is followed by a characterization of the equivalent martingale measures and discussion on the minimal martingale measure proposed by Föllmer, Schweizer, and Sondermann(Föllmer and Sondermann(1986), Föllmer and Schweizer(1990), and Schweizer(1991); hereafter we call the authors FSS). Finally, after discussing some properties of the measure, we derive the dynamics of the stock price in the presence of jumps under the minimal martingale measure.

2.1 Assumptions

We assume throughout this paper that(1) the capital markets are frictionless, and trading takes place continuously and without transaction costs,(2) there are two tradable assets in the market, a risky asset and a riskless asset,(3) the short-term interest rate is known and constant through time, and(4) the stock price follows a jump-diffusion process through time. The riskless asset B_t at time t is governed by the equations $dB_t = rB_t dt$, and $B_0 = 1$. We assume that the price of risky asset S_t is described by a stochastic differential equation of the form:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + d\left(\sum_{j=1}^{N_t} U_j\right). \qquad (1)$$

To be more rigorous, we consider a probability space $(\Omega, \mathrm{F}, \mathrm{P})$ with $\mathrm{F} = (F_t)_{t\in[0,T]}$, a filtration satisfying the usual conditions on which we define a standard Brownian motion $(W_t)_{t\geq 0}$, a Poisson process $(N_t)_{t\geq 0}$ with jump intensity λ (the average number of arrivals per unit time) and a sequence of $(U_j)_{j\geq 1}$ of independent, identically distributed random variables taking values in $(-1, \infty)$. We assume that the σ-algebras generated respectively by $(W_t)_{t\geq 0}, (N_t)_{t\geq 0}, (U_j)_{j\geq 1}$ are independent. For simplicity, we take the drift μ and the volatility σ to be a constant and assume that the asset pays no dividend. Then the dynamics of $(S_t)_{t\geq 0}$ is given by

$$S_t = S_0 \exp\{(\mu - \sigma^2/2)t + \sigma W_t\} \prod_{j=1}^{N_t}(1 + U_j), (2)$$

where $\prod_{j=1}^{0} = 1$

Now consider a trading strategy $\varphi = (\pi_t^0, \pi_t)$ where π_t^0 and π_t are the amount of riskless assets and stocks holding at time t, respectively. We assume that the trading strategy satisfies:

(i) π_t is F_t predictable,

(ii) π_t^0 is adapted,

(iii) $E\left\{\int_0^T \pi_t^2 \, d\tilde{S}_t + \left(\int_0^T |\pi_t \mu_t| dt\right)^2\right\} < \infty$,

(iv) the discounted value process $V_t(\varphi) \equiv \pi_t \tilde{S}_t + \pi_t^0$ has the right continuous sample paths and $V_t(\varphi) \in L^2(\Omega, \mathrm{P})$ for $0 \leq t \leq T$.

In(iii), μ_t denotes the expected rate of return on the stock, and $\tilde{S}_t$ is the discounted stock price at time t.

The discounted gain process from a trading strategy is defined to be $G_t(\varphi) \equiv \int_0^t \pi_u d_u \tilde{S}_u$, and the discounted cost process is $C_t(\varphi) \equiv V_t(\varphi) - G_t(\varphi)$. A trading strategy φ such that $C_t(\varphi) = C_0(\varphi)$ for all $0 \leq t \leq T$ is called self-financing. If, for some contingent claim $X \in L^2(\Omega, \mathrm{P})$, there is a self-financing trading strategy such that $V_t(\varphi) = X$, that is $X = C_0(\varphi) + \int_0^T \pi_u d\tilde{S}_u$ a.s., then φ is a riskless hedge portfolio for X and X is said to be attainable. If every contingent claim $X \in L^2(\Omega, \mathrm{P})$ is attainable, the market is called complete.

2.2 Merton's approach

Merton(1976) overcomes this valuation problem by assuming that the jump component of the stock's return represents non-systematic risk. According to the Capital Asset Pricing Model(CAPM), the expected return on all zero-beta securities must equal the riskless rate, and then he shows that the European option price $f(t)=F(S_t, T-t)$, which is a twice-continuously differentiable function of the stock and time, satisfies

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2}+(r-\lambda EU_1)S\frac{\partial F}{\partial S}-\frac{\partial F}{\partial r} \\ -rF+\lambda E\{F(S(1+U_1),\tau)-F(S,\tau)\}=0 \quad (3)$$

subject to the boundary conditions:

$$F(0,\tau)=0 \\ F(S,0)=\max(0,S-K) \quad (4)$$

where $\tau=T-t$, and K is the exercise price of the option. Note here that Equation(3) does not depend on μ. Merton also shows that the solution to Equation(3) for the European option price, when the current stock price is S, is given by:

$$F(S,\tau)= \\ \sum_{n=0}^{\infty}\frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}Ef^{BS} \quad (5) \\ \left(S\exp.(-\lambda\tau EU_1)\prod_{j=1}^{n}(1+U_j),\tau,K,\sigma^2,\gamma\right),$$

where $f^{BS}(S,\tau;K,\sigma^2,r)$ is the Black-Scholes option pricing formula for the no-jump case. If we assume that the size of the proportional jump has a log-normal distribution, then $\prod_{j=1}^{n}(1+U_j)$ will have a log-normal distribution with the variance of the logarithm of $\prod_{j=1}^{n}(1+U_j)$ equal to $n\delta^2$, where δ^2 denotes the variance of $\ln(1+U_j)$. In this special case, Merton shows that the value of European option is given by

$$F(S,\tau)=\sum_{n=0}^{\infty}\frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!} \\ f^{BS}(S,\tau;K,\nu_n,r_n) \quad (6)$$

where $\lambda'=\lambda E(1+U_j)$. The variable $f^{BS}(S,\tau;K,\nu_n,r_n)$ is the Black-Scholes option price, conditional on knowing that exactly n Poisson jumps will occur during the life of the option when the conditional variance rate (ν_n^2) is $\sigma^2+n\delta^2/\tau$ and the conditional risk-free rate $(r_n)$is $r-\lambda EU_1+\ln(1+EU_1)^n/\tau$.

In the next subsection, we show that the above formula for a European option is equivalent to the one obtained when the risk-minimizing trading strategy is used with the restriction that the actual probability measure is considered a martingale measure.

2.3 Minimal martingale measure

In this subsection, using the risk minimizing trading strategy proposed by FSS and under the minimal martingale measure driven by the strategy, we derive a pricing formula of a European option under jump-diffusion processes. To avoid any confusion, we let $\mathbb{P}^*$ denote the equivalent martingale measure and $\mathbb{Q}$ the minimal martingale measure respectively. Note that the minimal martingale measure is one of equivalent martingale measure.

We assume here the discounted stock price is a semi-martingale under $\mathbb{P}$. In this case, a local risk minimizing strategy(associated with the minimal martingale measure) is proposed by FSS. They look for an admissible trading strategy which minimizes, at t, the remaining risk

$$R_t(\varphi)\equiv E^{\tilde{P}}[(C_T(\varphi)-C_t(\varphi))^2 \mid F_t] \quad (7)$$

where C_t is a cost process at t defined earlier in the previous subsection.

Suppose the price process $\tilde{S}_t$ is a semi-martingale with the Doob-Meyer decomposition :

$$\tilde{S}_t = \tilde{S}_0 + M + A, \qquad (8)$$

where $M = (M_t)_{0 \le t \le T}$ is a square integrable martingale under $\mathbb{P}$, and $A = (A_t)_{0 \le t \le T}$ is a predictable process with paths of bounded variation such that $A_t = \int_0^t \alpha_s \, d\langle M \rangle_s$ for some predictable process, $\alpha = (\alpha_t)_{0 \le t \le T}$, where $\langle M \rangle$ is the quadratic variation of M.

FSS define the minimal martingale measure $\mathbb{Q}$ as an equivalent martingale measure such that

(1) $\tilde{S}_t$ is a martingale under $\mathbb{Q}$,

(2) $\mathbb{P} = \mathbb{Q}$ on F_0, and

(3) any square integrable $\mathbb{P}$-martingales orthogonal to M under $\mathbb{P}$ remains a martingale under $\mathbb{Q}$.

They also show that the minimal martingale measure $\mathbb{Q}$ is uniquely defined and exists if and only if

$$G_t = \exp\left(- \int_0^t \alpha_s dM_s - \frac{1}{2} \int_0^t \alpha_s^2 d\tilde{S}_s \right) \qquad (9)$$

is a square-integrable martingale under $\mathbb{P}$. In that case $\mathbb{Q}$ is given by :

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = G_T \qquad (10)$$

Colwell and Elliott(1993) show that the Radon-Nikodym density process G_t for Markovian models is given by

$$\begin{aligned} G_t(x_0) = 1 - \int_0^t G_s(x_0) g_s dW_s \\ - \int_0^t \int_{\mathbb{R}} G_s(x_0)[1 - h_s] \tilde{\nu}(ds, dy) \end{aligned} \qquad (11)$$

In general, g_t and h_t are suitably chosen so that $\tilde{S}_t$ is a $\mathbb{P}^*$-martingale; in that case $\mathbb{P}^*$ is called an equivalent martingale measure. Now $\tilde{S}_t$ is a $\mathbb{P}^*$-martingale if and only if

$$\mu + \lambda E U_1 - r = \sigma g_t + \lambda \int_{\mathbb{R}} y[1 - h_t] m(dy) \qquad (12)$$

where m is denoted by the law of the random variable U_j, g_t is interpreted as the market price of diffusion risk, and $1 - h_t$ as the market price of jump risk.

There are infinite numbers of ways to select g_t and h_t satisfying the above Equation(12) for finding an equivalent martingale measure. In other words, changing of measures is "adjusting" market price of risk embedded in the underlying assets. Note that the market price of risk is the expected excess return per unit risk over the riskfree rate. It measures the trade-offs between risk and return. Colwell & Elliott(1993) show that the risk minimizing strategy leads to the following selection of market price of risk and the corresponding Randon-Nikodym derivative process.

$$\begin{aligned} g &= \frac{(\mu + \lambda E U_1 - r)\sigma}{\sigma^2 + \lambda E U_1^2} \\ 1 - h &= \frac{(\mu + \lambda E U_1 - r) U_1}{\sigma^2 + \lambda E U_1^2} \\ G_t &= 1 - \int_0^t G_s \frac{\mu + \lambda E U_1 - r}{e^{-rs} S_s (\sigma^2 + \lambda E U_1^2)} dM_s \ . \end{aligned} \qquad (13)$$

Remark. The market price of risks in Merton's model, shown by Colwell and Elliott(1993) are given by

$$\begin{cases} g=(\mu+\lambda EU_1-r)/\sigma \\ 1-h=0 \end{cases} \quad (14)$$

As mentioned previously, the market price of jump risk in Merton's model is zero, and Merton(1976) only considers the diffusion risk inherent in risky asset's uncertainty. Comparing two sets of market prices of risks in Equation(13) and(14), we conclude that the measure under which Merton's option prices is derived and the minimal martingale measure are different. In the simple model we discuss in the last section of this Chapter, it is obvious that the market prices of diffusion and jump risks are all zero ($g=1-h=0$), since the left side of Equation(12) is zero. In other words, in the risk neutral world, the expected return of any risky asset is the risk-free rate.

Provided that there exists an equivalent martingale measure, we will consider the characteristics of the stock price dynamics under the new measure. Hence by Girsanov's theorem, $W_t^*=W_t+g\cdot t$ follows a standard Brownian motion on the space $(\Omega,\ \mathbb{F},\ \mathbb{Q})$. Zhang(1997) shows, under the transformation, that the martingale distribution of the jump size and martingale jump intensity are given by

(i) U_j are iid and $d\mathbb{Q}_{U_1}(x)=(1-\eta x)/(1-\eta EU_1)d\mathbb{P}_{U_1}(x)$

(ii) N_t^* is a Poisson process with intensity $\lambda^*=\lambda(1-\eta EU_1)$,

where

$$\eta\equiv\frac{\mu+\lambda EU_1-r}{\sigma^2+\lambda EU_1^2}\ . \quad (15)$$

The explicit solution G_t of the last Equation of (13), obtained by Zhang(1994), is given by

$$G_t=\exp(-\eta\sigma W_2\frac{1}{2}\eta^2\sigma^2 t)$$
$$\prod_{j=1}^{N_t}(1-\eta U_j)\exp(\eta\lambda tEU_1). \quad (16)$$

Positivity of G_t leads to the inequality $1-\eta U_1>0$. If $-1\le\eta\le0$, then $1-\eta U_1>0$ almost surely. If $\eta>0$, with the assumption that $1+U_j$ follows a log-normal distribution with mean m and variance δ^2, then $\mathbb{P}\{1-\eta U_j\le0\}=\mathbb{P}\left\{U_j\ge\frac{1}{\eta}\right\}=1-\Phi(\ln(1+1/\eta-m)/\delta)$. Thus if $\eta>0$, then the probability $\mathbb{P}\{1-\eta U_j\le0\}$ would have a positive value and so the equivalent probability measure might not be defined.

3. Analytical Formula for a European option price under minimal martingale measure

In this section, we derive an analytic formula for European option prices in the presence of jumps under the minimal martingale measure. This is new in the literature as we know. We also discuss a simple model in which the actual probability measure is a risk-neutral measure, and compare it with Merton's model.

Now, by changing measures with the transformation, $dW_t^*=dW_t+\eta\sigma dt$, and imposing the transformation on the Equation(1), we obtain

$$dS_t = S_t(\mu - \eta\sigma^2)dt + \sigma S_t dW_t^* + S_t d\left(\sum_{j=1}^{N_t^*} U_j\right) \quad (17)$$

Let $\mu^* = \mu - \eta\sigma^2$. The solution of S_t is given by:

$$S_t = S_0 \exp\{(\mu^* - \sigma^2/2)t + \sigma W_t^*\} \left(\prod_{j=1}^{N_t^*}(1+U_j)\right), \quad (18)$$

Let $\tilde{S}_t = e^{-rt}S_t$. Now,

$$\begin{aligned}
&E^{Q}(\tilde{S}_t \mid F_s) \\
&= \tilde{S}_s E^{Q}(\exp\{(\mu^* - r - \sigma^2/2)\}(t-s) + \sigma(W_t^* - W_s^*)\} \prod_{j=N_s^*+1}^{N_t} (1+U_j) \mid F_t) \\
&= \tilde{S}_s E^{Q}(\exp\{(\mu^* - r - \sigma^2/2)\} + \sigma(W_t^* - W_s^*)\} \prod_{j=1}^{N_t^* - N_s^*} (1+U_{N_s+j})) \\
&= \tilde{S}_s \exp\{(\mu^* - r)(t-s)\} E^{Q}\left(\prod_{j=N_s+1}^{N_t^*}(1+U_j)\right) \\
&= \tilde{S}_s \exp\{(\mu^* - r)(t-s)\} \exp[\lambda^*(t-s)E^{Q}U_1] \\
&= \tilde{S}_s \exp\{(\mu^* - r + \lambda^* E^{Q}U_1)(t-s)\}
\end{aligned}$$

where $\lambda^* = \lambda(1-\eta EU_1)$. From the above equation, we can conclude that$(\tilde{S}_s)$ is a martingale if and only if $\mu^* - r + \lambda^* E^{Q} U_1 = 0$. It is not hard to show that:

$$\begin{aligned}
&\mu^* - r + \lambda^* E^{Q}U_1 \\
&= (\mu - \eta\sigma^2) - r + \lambda(1-\eta EU_1)E^{Q}U_1 \\
&= \mu - r + \lambda EU_1 - \eta(\sigma^2 + \lambda EU_1^2) = 0,
\end{aligned}$$

since

$$\begin{aligned}
&E^{Q}U_1 \\
&= \int_{\mathbb{R}} x\, dQ_{U_1}(x) \\
&= \int_{\mathbb{R}} \frac{x(1-\eta x)}{1-\eta EU_1} dP_{U_1}(x) \\
&= \frac{1}{1-\eta EU_1} \int_{\mathbb{R}} x\, dP_{U_1}(x) - \eta \int_{\mathbb{R}} x^2 dP_{U_1}(x) \\
&= \frac{EU_1 - \eta EU_1^2}{1-\eta EU_1}. \quad (19)
\end{aligned}$$

Thus, we verify that $(\tilde{S}_s)$ is a martingale under Q. Now, the price process, Equation(17), can be rewritten by

$$dS_t = S_t(r - \lambda^* E^{Q}U_1)dt + \sigma S_t dW_t^* + S_t d\left(\sum_{j=1}^{N_t^*} U_j\right). \quad (20)$$

It is obvious that if we let $\eta = 0$, then $Q = P, \lambda^* = \lambda, W_t^* = W_t$, and $N_t^* = N_t$. This is the situation for which the actual measure is the risk-neutral measure. We call this a simple case and discuss this case in the end of this section.

Now we define a European option price. As Pham(1997) points out, each equivalent martingale measure(in this case, the minimal martingale measure Q) defines an admissible price of an option in the framework of Harrison and Pliska(1981, 1983). Suppose h is a real, twice differentiable payoff function, for $t > T$, the price of European option is given by

$$v(t) = E^{Q}[\exp\{-r(T-t)\} h(S_T) \mid F_t], \quad (21)$$

where $h(S_T) = (S_T - K)^+$ for a call, and

$h(S_T)=(K-S_T)^+$ for a put.

Now define $v(t)=F(t,x)$, then

$$F(t,x)$$
$$= E^Q\Big[e^{-r(T-t)}h\Big(x\exp\Big\{(\mu^*-\sigma^2/2)$$
$$(T-t)+\sigma W^*_{T-t}\Big\}\prod_{j=1}^{N^*_{T-t}}(1+U_j)\Big)\Big]$$
$$=E^Q\Big[e^{-r(T-t)}h\Big(x\exp\Big\{(r-\lambda^*E^QU_1-$$
$$\sigma^2/2)(T-t)\Big\}+\sigma W^*_{T-t}\prod_{j=1}^{N^*_{T-t}}(1+U_j)\Big)\Big]$$
$$=E^Q\Big[F_0(t,x\exp\Big\{-\lambda^*(T-t)$$
$$E^QU_1\Big\}\prod_{j=1}^{N^*_{T-t}}(1+U_j))\Big]$$
$$=\sum_{n=0}^{\infty}\exp\frac{\{-\lambda^*(T-t)(\lambda^*)^n(T-t)\}}{n!}$$
$$\times E^Q\Big[F_0\Big(t,x\exp\Big\{-\lambda^*(T-t)$$
$$E^QU_1\Big\}\prod_{j=1}^{n}(1+U_j)\Big)\Big] \quad (22)$$

where $F_0(t,x)$ is the Black-Scholes option pricing formula for the no-jump case defined by:

$$F_0(t,x)=E^Q[e^{-r(T-t)}h(x\exp\{(r-\sigma^2/2)$$
$$(T-t)+\sigma W^*_{T-t}\})]$$

Notice that if we set $\eta=0$, then the above equation is equivalent to the option pricing formula of Merton(1976).

Proposition. *If we assume that the random variable* $\ln(1+U_j)$ *follows a normal distribution with mean* m *and variance* δ^2*, then* $F(t,x)$ *can be computed analytically as follows.*

$$F(t,x)=\sum_{n=0}^{\infty}\frac{e^{-\lambda'(T-t)}(\lambda'(T-t))^n}{n!}\{e^{-r_n(T-t)}K\Phi(-d_2)-x\Phi(-d_1)\}, \quad (23)$$

where d_1, d_2 *are denoted by:*

$$-d_2=\frac{\ln(K/x)-(r_n-\nu_n^2/2)(T-t)}{\nu_n\sqrt{T-t}}$$
$$-d_1=-d_2+\nu_n\sqrt{T-t}, \quad (24)$$

where $\lambda\equiv\lambda^*(1+E^QU_1)$, $r_n\equiv r-\lambda^*E^QU_1+n\gamma/(T-t)$, and $\nu^2\equiv\sigma^2+n\delta^2/(T-t)$.

Note that $\gamma\equiv\ln E^Q(1+U_1)=\delta^2/2+m.$

proof : see the appendix.

The formula of option prices is very similar to Merton(1976). The only unknown parameter is η. Note that if we set $\eta=0$, the above Equation(23) is equivalent to what Merton(1976) derives.

It is clear as shown earlier in this section that the discounted stock price process is a martingale under the actual measure $\mathbb{P}$, if and only if $\mu=r-\lambda EU_1$. Thus we regard the actual measure with the condition $\mu=r-\lambda EU_1$ as the risk neutral measure in this market. Thus the process of stock price, Equation(1), can be rewritten by

$$\frac{dS_t}{S_t}=(r-\lambda EU_1)dt+\sigma dW_t+d\Big(\sum_{j=1}^{N_t}U_j\Big) \quad (25)$$

The stochastic differential equation(25) is a special case of Equation(20). Thus the European option price obtained in this simple case is the same as Merton's(1976). Although the market prices of risks are different as we see earlier in this section, the models render the same formula for a European option price.

4. Concluding Remarks

In this paper we study the valuation of European option prices under jump-diffusion processes, as an example of incomplete markets. Due to the jump part, the market is incomplete and so it is impossible to construct a hedging portfolio with stocks and riskless assets. Contrary to the case of a complete market in which only one equivalent martingale measure exists, there are infinite numbers of equivalent martingale measures in an incomplete market. Our research here is focusing on the well-known notion of risk minimizing strategy and its associated minimal martingale measure.

Based on the risk minimizing strategy introduced in the works of Föllmer, Schwezer, and Sondermann(1986, 1990, 1991), we characterize the dynamics of a risky asset and valuation formulas for option prices under jump-diffusion processes. In particular, we obtain an analytical formula for a European option price.

In addition to the risk-minimizing strategy applied in this paper to the jump- diffusion model, utilizing variance minimizing strategy or expected utility maximizing scheme is also an alternative way for the valuation of options in an incomplete market, and generalizing our procedure to these other strategies is a natural future research activity.

Appendix

Proof of Proposition:

Without loss of generality, we here consider European put price with $t=0$,

$x=S_0$, and exercise price, K. We first evaluate the expectation term in Equation(22).

$$E^Q(F_0(0, S_0\exp\{-\lambda^* TE^Q U_1\}\prod_{j=1}^{n}(1+U_1)))$$

$$=E^Q[e^{-rT}(K-S_T)^+ \mid n\ jumps]$$

(By recalling the process of B-S model)

$$=E^Q\left(e^{-rT}\left[K-S_0\exp\{-\lambda^* T E^Q U_1\}\prod_{j=1}^{n}(1+U_j)\exp(\sigma W_T+(r-\sigma^2/2)T)\right]^+\right)$$

$$=E^Q\left(e^{-rT}\left[K-S_0\exp\left(\sigma W_T+(r-\lambda^* E^Q U_1-\frac{1}{2}\sigma^2)T+\sum_{j=1}^{n}\ln(1+U_j)\right)\right]^+\right)$$

$$=E^Q\left(e^{-rT}\left[K-S_0\exp\left(\sigma W_T-(\lambda^* E^Q U_1+\frac{1}{2}\sigma^2)T+\sum_{j=1}^{n}\ln(1+U_j)\right)\right]^+\right)$$

$$=E^Q\left(\left[e^{-rT}K-S_0\exp\left\{\sqrt{\sigma^2 T+n\sigma^2}+nm-\lambda^* k^* T-\frac{1}{2}\sigma^2\right\}\right]^+\right),$$

where z follows a standard Gaussian law $N(0,1)$, $k^*=E^Q U_1$, and m, and δ^2 is mean and the variance of $\ln(1+U_1)$, respectively.

Note that

$$e^{-rT}K-S_0\exp z\sqrt{\sigma^2 T+n\delta^2}+nm-\lambda^* k^* T-\sigma^2 T/2\geq 0$$

$$\rightarrow z\leq \ln(K/S_0)-(r-\lambda^* k^*)T-nm+\sigma^2 T/2$$

$$\rightarrow z \le \ln(K/S_0) - \left(r - \lambda^* k^* + \frac{n\gamma}{T}\right)T + \frac{1}{2}\left(\sigma^2 + \frac{n\delta^2}{T}\right)T$$

$$\rightarrow z \le \ln(K/S_0) - (r_n - \nu_n^2/2)T$$

where

$$r_n = r - \lambda^* k^* + \frac{n\gamma}{T}$$
$$\nu_n^2 = \sigma^2 + \frac{n\delta^2}{T}$$

Now simple algebra with the previous result gives the European put price when the current stock price is S_0 at $t=0$ as follows: $F(0, S_0)$

$$= \sum_{n=0}^{\infty} \frac{e^{-\lambda^* T}(\lambda^* T)^n}{n!} E\Big(e^{-rT}K - S_0 \exp\{z\sqrt{\sigma^2 T + n\delta^2} + nm - \lambda^* k^* T - \sigma^2 T/2\} I_{\{z+d_2 \le 0\}}\Big)$$

$$= \sum_{n=0}^{\infty} \frac{e^{-\lambda^* T}(\lambda^* T)^n}{n!} \int_{-\infty}^{-d_2} (e^{-rT}K - S_0 \exp\{z\sqrt{\sigma^2 T + n\delta^2} + nm - \lambda^* k^* T - \sigma^2 T/2\}) \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz$$

$$= \sum_{n=0}^{\infty} \frac{e^{-\lambda^* T}(\lambda^* T)^n}{n!} \left\{ e^{-rT}K \int_{-\infty}^{-d_2} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz - S_0 \int_{-\infty}^{-d_2} \exp\{z\sqrt{\sigma^2 T + n\delta^2} + nm - \lambda^* k^* T - \sigma^2 T/2\} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz \right\}$$

$$= \sum_{n=0}^{\infty} \frac{e^{-\lambda^* T}(\lambda^* T)^n}{n!} \Big\{ \exp\{-n\gamma - \lambda^* k^* T - r_n T\} K \Phi(-d_2) - S_0 \int_{-\infty}^{-d_2} \exp\{z\sqrt{\sigma^2 T + n\delta^2} + nm - \lambda^* k^* T - \sigma^2 T/2\} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz \Big\} \quad \text{(A)}$$

Evaluation of the second integral by changing of variable $y = z - \nu_n \sqrt{T}$ gives

$$\int_{-\infty}^{-d_2} \exp\{z\sqrt{\sigma^2 T + n\delta^2} + nm - \lambda^* k^* T - \sigma^2 T/2\} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz$$

$$= \int_{-\infty}^{-d_1} \exp\{y\sqrt{\sigma^2 T + n\delta^2} + nm - \lambda^* k^* T - \sigma^2 T/2\} \frac{e^{-(y+\sqrt{\sigma^2 T + n\delta^2})^2/2}}{\sqrt{2\pi}} dy$$

$$= \exp\{n\delta^2/2 + nm - \lambda^* k^* T\} \int_{-\infty}^{-d_1} \frac{e^{-y/2}}{\sqrt{2\pi}} dy$$

$$= \exp\{n\gamma - \lambda^* k^* T\} \int_{-\infty}^{-d_1} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$$

$$= \exp\{n\gamma - \lambda^* k^* T\} \Phi(-d_1) \quad \text{(B)}$$

Combining Equation(A) with the Equation(B), We obtain:

$F(0, S_0)$

$$= \sum_{n=0}^{\infty} \frac{e^{-\lambda^* T}(\lambda^* T)^n}{n!} \{ \exp\{n\gamma - \lambda^* k^* T\} e^{-r_n T} K \Phi(-d_2) - S_0 \exp\{n\gamma - \lambda^* k^* T\} \Phi(-d_1)\}$$

$$= \sum_{n=0}^{\infty} \frac{e^{-\lambda' T}(\lambda' T)^n}{n!} \{e^{-r_n T} K \Phi(-d_2) - S_0 \Phi(-d_1)\}.$$

where $\lambda' \equiv \lambda^*(1 + E^Q U_1)$.

End of proof.

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