WEAKLY KRULL AND RELATED PULLBACK DOMAINS

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ABSTRACT. Let $T$ be an integral domain, $M$ a nonzero maximal ideal of $T$, $K = \frac{T}{M}$, $\varphi : T \to K$ the canonical map, $D$ a proper subring of $K$, and $R = \varphi^{-1}(D)$ the pullback domain. Assume that for each $x \in T$, there is a $u \in T$ such that $u$ is a unit in $T$ and $ux \in R$. In this paper, we show that $R$ is a weakly Krull domain (resp., GWFD, AWFD, WFD) if and only if $htM = 1$, $D$ is a field, and $T$ is a weakly Krull domain (resp., GWFD, AWFD, WFD).

1. INTRODUCTION

Recall that an integral domain $D$ is called a weakly Krull domain if

$$D = \cap_{P \in X(D)} D_P$$

and this intersection has finite character, that $D$ is a generalized weakly factorial domain (GWFD) if each nonzero prime ideal of $D$ contains a primary element, that $D$ is an almost weakly factorial domain (AWFD) if for each nonzero nonunit element $x$ of $D$, there is a positive integer $n = n(x)$ such that $x^n$ can be written as a product of primary elements, and that $D$ is a weakly factorial domain (WFD) if each nonzero nonunit element of $D$ is a product of primary elements. Clearly, a WFD is an AWFD and an AWFD is a GWFD. It is also known that a GWFD is a weakly Krull domain Anderson, Chang & Park [4, Corollary 2.3].

Let $T$ be an integral domain, $M$ a nonzero maximal ideal of $T$, $K = \frac{T}{M}$, $\varphi : T \to K$ the canonical map, $D$ a proper subring of $K$, and $R = \varphi^{-1}(D)$ the
pullback domain.

\[
\begin{array}{ccc}
R = \varphi^{-1}(D) & \longrightarrow & D \\
\downarrow & & \downarrow \\
T & \longrightarrow & K = T/M
\end{array}
\]

(□)

We shall refer to \( R \) as a pullback domain of type (□) and as a pullback domain of type (□*) if for each \( 0 \neq x \in T \), there is a \( u \in U(T) \) such that \( ux \in R \). If \( R \) is a pullback domain of type (□), then \( M \) is a divisorial ideal (and hence a \( t \)-ideal) of \( R \), \( M \) has the same height in both \( R \) and \( T \), and for any prime ideal \( P(\not\in M) \) of \( R \), \( T \subseteq R_P \) and \( R_P = T_{PR_P \cap T} \) (cf. Fontana & Gabelli [9, p. 805]). One can show that if \( T = K + M \) (and hence \( R = D + M \)), then \( R \) is a pullback domain of type (□*).

In Anderson, Chang & Park [5, Section 2], we showed that if \( T = K + M \) (and hence \( R = D + M \)), then \( R \) is a weakly Krull domain (resp., GWFD, AWFD, WFD) if and only if \( \text{ht} \, M = 1 \), \( D \) is a field, and \( T \) is a weakly Krull domain (resp., GWFD, AWFD, WFD). The purpose of this paper is to generalize these results to a pullback domain of type (□) or (□*). That is, we show that if \( R \) is a pullback domain of type (□), then \( R \) is a weakly Krull domain if and only if \( \text{ht} \, M = 1 \), \( D \) is a field, and \( T \) is a weakly Krull domain; and if \( R \) is of type (□*), then \( R \) is a GWFD (resp., AWFD, WFD) if and only if \( \text{ht} \, M = 1 \), \( D \) is a field, and \( T \) is a GWFD (resp., AWFD, WFD).

Let \( D \) be an integral domain with quotient field \( K \) and \( I \) a nonzero fractional ideal of \( D \). Then

\[
I^{-1} = \{ x \in K | xI \subseteq D \}, I_v = (I^{-1})^{-1}, \quad \text{and} \quad I_t = \cup \{ J_v | (0) \neq J \subseteq I \text{ is finitely generated} \}.
\]

If \( I_v = I \) (resp., \( I_t = I \)), \( I = (x_1, \ldots, x_n)_v \) for some \((0) \neq (x_1, \ldots, x_n) \subseteq I\), then \( I \) is said to be a divisorial ideal (resp., \( t \)-ideal, finite type \( t \)-ideal). An ideal of \( D \) maximal among proper integral \( t \)-ideals is called a maximal \( t \)-ideal. A fractional ideal \( I \) is \( t \)-invertible if \((II^{-1})_t = D \). It is well known that a maximal \( t \)-ideal is a prime ideal, every proper integral \( t \)-ideal is contained in a maximal \( t \)-ideal, a \( t \)-invertible \( t \)-ideal is of finite type, and a \( t \)-invertible prime \( t \)-ideal is a maximal \( t \)-ideal. We say that \( D \) has \( t \)-dimension one, denoted by \( t \)-dim\(|(D) = 1\), if each maximal \( t \)-ideal of \( D \) has height-one.
Let $T(D)$ be the set of $t$-invertible fractional $t$-ideals of an integral domain $D$. Then $T(D)$ is an abelian group under the $t$-product $I \ast J = (IJ)_t$, and hence the quotient group $Cl(D) = T(D)/\text{Prin}(D)$, called the class group of $D$, is also an abelian group, where $\text{Prin}(D)$ is the subgroup of $T(D)$ of nonzero principal fractional ideals of $D$. If $D$ is a Krull domain, then $Cl(D)$ is the usual divisor class group, and if $D$ is a Prüfer domain, then $Cl(D)$ is the ideal class group of invertible ideals (or Picard group) of $D$.

All rings considered in this paper are commutative integral domains with identity and for an integral domain $D$, $U(D)$ denotes the set of unit elements of $D$ and $X^1(D)$ is the set of height-one prime ideals of $D$. A nonzero nonunit element $a$ of $D$ is said to be primary if $aD$ is a primary ideal. It is known that if $aD$ is primary, then $\sqrt{aD}$ is a maximal $t$-ideal. The reader is referred to Gilmer [12, §32 and §34] and Zafrullah [16] for the $t$-operation; to Anderson & Zafrullah [1], Anderson, Mott & Zafrullah [2], Anderson, Chang & Park [4, 5] for weakly Krull and related domains; to Brewer & Rutter [8], Fontana & Gabelli [9], Gabelli & Houstongh [11], Lucas [15] for pullback domains; to Anderson [3], Bouvier [6], Bouvier & Zafrullah [7], Fontana & Gabelli [9], Fossum [10] for the class group; and to Fossum [10], Gilmer [12], Kaplansky [14] for standard notations and definitions.

We first study when the pullback domain $R$ is weakly Krull. Recall that a weakly Krull domain has $t$-dimension one Anderson, Mott & Zafrullah [2, Lemma 2.1].

**Theorem 1** (cf. Anderson, Chang & Park [5, Theorem 2.3]). Let $R$ be a pullback domain of type $(\square)$. Then $R$ is a weakly Krull domain if and only if $htM = 1$, $D$ is a field, and $T$ is a weakly Krull domain.

**Proof.** $(\Rightarrow)$ Assume that $R$ is a weakly Krull domain. Then $t\text{-dim}(R) = 1$, and since $M$ is a $t$-ideal of $R$, $M$ is a height-one maximal $t$-ideal of $R$. If $a \in D \setminus \{0\}$, then $\varphi^{-1}(aD)$ is an invertible ideal of $R$ such that $M \subsetneq \varphi^{-1}(aD) \subseteq R$ (cf. Fontana & Gabelli [9, Corollary 1.7]). Hence $M$ being a maximal $t$-ideal of $R$ implies that $D$ is a field.

We next show that $T$ is weakly Krull. Let $Q(\neq M)$ be a maximal ideal of $T$, and let $P = Q \cap R$. Then $T_Q = R_P$, and since $R_P$ is weakly Krull Anderson, Chang & Park [5, Lemma 2.1(2)],

$$T_Q = \cap\{T_{Q'} | Q' \in X^1(T) \text{ and } Q' \subseteq Q\}$$
so

\[ T = \bigcap_{Q \in \text{Max}(T)} T_Q = \bigcap_{Q' \in X^1(T)} T_{Q'}. \]

Note that for each \( Q' \in X^1(T) \setminus \{M\}, \) \( T_{Q'} = R_{Q' \cap R} \) (and hence \( \text{ht}(Q' \cap R) = 1 \)) and \( T \) is an overring of \( R. \) Hence the intersection \( T = \bigcap_{Q' \in X^1(T)} T_{Q'} \) has finite character, and thus \( T \) is weakly Krull.

\( (\Leftarrow) \) Assume that \( \text{ht}M = 1, \) \( D \) is a field, and \( T \) is weakly Krull. Let \( M_1 \) be a maximal ideal of \( R \) such that \( M_1 \neq M. \) Then \( R_{M_1} = T_Q \) for some prime ideal \( Q \) of \( T. \) Note that \( T_Q \) is weakly Krull Anderson, Chang & Park [5, Lemma 2.1(2)]; so

\[ T_Q = R_{M_1} = \bigcap \{ R_P \mid P \in X^1(R) \text{ and } P \subseteq M_1 \}. \]

Since \( R = \bigcap \{ R_{M'} \mid M' \text{ is a maximal ideal of } R \} \) and \( \text{ht}M = 1, \) we have \( R = \bigcap_{P \in X^1(R)} R_P. \) Moreover, since \( R \subseteq T \) and for each \( P \in X^1(R) \setminus \{M\}, \) \( R_P = T_{Q'} \) for some \( Q' \in X^1(T), \) the intersection \( R = \bigcap_{P \in X^1(R)} R_P \) has finite character, and thus \( R \) is weakly Krull. \( \square \)

Our next corollary, which was observed in the proof of Theorem 1 above, will be very useful in the subsequent arguments.

**Corollary 2.** Let \( R \) be a pullback domain of type \( (\Box). \) If \( R \) is a weakly Krull domain, then \( X^1(R) = \{ Q \cap R \mid Q \in X^1(T) \} \) and for each \( Q \in X^1(T) \setminus \{M\}, \) \( R_{Q \cap R} = T_Q. \)

**Theorem 3** (cf. Anderson, Chang & Park [5, Theorem 2.4]). If \( R \) is a pullback domain of type \( (\Box^*), \) then \( R \) is a GWFD if and only if \( \text{ht}M = 1, \) \( D \) is a field, and \( T \) is a GWFD.

**Proof.** \( (\Rightarrow) \) Assume that \( R \) is a GWFD. Then since a GWFD is weakly Krull Anderson, Chang & Park [4, Corollary 2.3], by Theorem 1 above, \( \text{ht}M = 1, \) \( D \) is a field, and \( T \) is weakly Krull (and hence \( t\text{-dim}(T) = 1 \)).

Let \( Q \in X^1(T) \) and \( P = Q \cap R. \) Then \( hP = 1 \) (Corollary 2), and so \( P = \sqrt{aR} \) for some \( a \in R \) (cf. Anderson, Chang & Park [4, Theorem 2.2]). Thus \( Q = \sqrt{aT} \) since \( Q \) is the unique prime ideal of \( T \) lying over \( P \) and \( t\text{-dim}(T) = 1. \)

\( (\Leftarrow) \) Assume that \( \text{ht}M = 1, \) \( D \) is a field, and \( T \) is a GWFD. Then as a GWFD is weakly Krull, \( R \) is weakly Krull by Theorem 1. Let \( P \in X^1(R) \setminus \{M\} \) and \( Q \in X^1(T) \) such that \( R_P = T_Q \) (Corollary 2). Then there is an \( x \in R \) such that \( Q = \sqrt{xT} \) (cf. Anderson, Chang & Park [4, Theorem 2.2]) since \( T \) is a GWFD and \( R \) is of type \( (\Box^*). \) If \( P' \) is a minimal prime ideal of \( xR, \) then \( P' \) is a \( t\)-ideal of \( R, \) and hence \( \text{ht}P' = 1 \) (note that \( t\text{-dim}(R) = 1); \) so \( P' = Q' \cap R \) for some \( Q' \in X^1(T). \)
Hence $x \in Q'$, and so $Q = Q'$ and $P = P'$. Therefore, $P = \sqrt{XR}$, and thus $R$ is a GWFD Anderson, Chang & Park [4, Theorem 2.2]. □

The proof of Theorem 3 shows that the " ⇒ " implication in Theorem 3 holds for a pullback domain of type (□). Recall that an integral domain $D$ is an AWFD if and only if $D$ is a weakly Krull domain and $Cl(D)$ is torsion Anderson, Mott & Zafrullah [2, Theorem 3.4].

**Theorem 4** (cf. Anderson, Chang & Park [5, Theorem 2.5]). *If $R$ is a pullback domain of type (□*), then $R$ is an AWFD if and only if $htM = 1$, $D$ is a field, and $T$ is an AWFD.*

**Proof.** ($⇒$) Assume that $R$ is an AWFD. Then $R$ is weakly Krull Anderson, Mott & Zafrullah [2, Theorem 3.4]; so by Theorem 1 and Anderson, Mott & Zafrullah [2, Theorem 3.4], it suffices to show that if $J$ is a $t$-invertible $t$-ideal of $T$, then $(J^n)_t$ is principal for some integer $n \geq 1$. Since $M$ is a $t$-ideal of $T$ (note that $htM = 1$) and $J$ is $t$-invertible, $JJ^{-1} \not\subseteq M$. Thus there is a $u \in J^{-1}$ such that $uJ \not\subseteq M$. Replacing $J$ with $uJ$, we may assume that $J \not\subseteq M$. Since $J$ is $t$-invertible and $R$ is of type (□*), there are some $x_1, \ldots, x_n \in R$ such that $J = ((x_1, \ldots, x_n)T)_v = (IT)_t$, where $I = (x_1, \ldots, x_n)R$.

Clearly, $I \not\subseteq M$, and hence $IR_M = RM$. For $P \in X^1(R) \setminus \{M\}$, let $Q \in X^1(T)$ such that $Q \cap R = P$ and $RP = TQ$ (Corollary 2). Then since $JT_Q$ is principal Kang [13, Corollary 2.7], $(IR_P)_t = (IT)_t = ((IT)_tT_Q)_t = (IT)_tT_Q = JT_Q$ is principal Kang [13, Lemma 3.4] (note that $Q$ is a prime $t$-ideal of $T$ and $J$ is $t$-invertible). So $I$ is $t$-locally principal, and hence $I$ is $t$-invertible Kang [13, Corollary 2.7]. Thus as $Cl(R)$ is torsion, $(I^n)_t = aR$ for some $a \in R$ and integer $n \geq 1$ Anderson, Mott & Zafrullah [2, Theorem 3.4].

We claim that $(J^n)_t = aT$. Let $Q \in X^1(T) \setminus \{M\}$ and $P = Q \cap R$. Then $TQ = RP$ (Corollary 2), and since $(J^n)_t$ is a $t$-invertible $t$-ideal of $T$, $(J^n)_tT_Q = ((J^n)_tT_Q)_t$, and hence (cf. Kang [13, Lemma 3.4])

$$(J^n)_tT_Q = \left(\left((IT)^n\right)_t\right)_tT_Q = \left(\left((IT)^n\right)_t\right)_tT_Q = ((IT)_tT_Q)_t = ((IT)_tT_Q)_t$$

$$= ((IR_P)^n)_t = (R^n)_t = (R^nR_P)_t = (aR_P)_t = aR_P = aT.$$ 

Also, since $I \not\subseteq M$, $aT \not\subseteq M$, and hence $(J^n)_tT_M = T_M = (aT)_T$. Thus $(J^n)_t = \cap_{Q \in X^1(T)}(J^n)_tT_Q = \cap_{Q \in X^1(T)}(aT)_T = aT$ (cf. Kang [13, Proposition 2.8]).
Assume that htM = 1, D is a field, and T is an AWFD. Let I be a t-invertible t-ideal of R. As in the beginning of the above proof, we may assume that I \not\subseteq M. Since I is t-invertible, II^{-1} \not\subseteq P for all P \in X^1(R), and hence II^{-1} \not\subseteq Q for all Q \in X^1(T) by Corollary 2. Hence IT is a t-invertible ideal of T. Also, since T is an AWFD and R is of type (□*), there are an integer n \geq 1 and a \in R such that 
((IT)_t^n)_t = (I^nT)_t = aT. Note that (I^n)_t is a t-ideal of R, and that for each P \in X^1(R) \setminus \{M\} and Q \in X^1(T) with Q \cap R = P (Corollary 2), (I^n R_P)_t = (I^n T_Q)_t = ((I^n T)_t T_Q)_t Kang [13, Lemma 3.4]. So by Kang [13, Proposition 2.8], we have

\[(I^n)_t = \cap_{P \in X^1(R)} (I^n)_t R_P = (I^n)_t R_M \cap (\cap \{(I^n)_t R_P | P \in X^1(R) \text{ and } P \neq M\}) \]
\[= R_M \cap (\cap \{(I^n)_t T_Q | Q \in X^1(T) \text{ and } Q \neq M\}) \]
\[= a R_M \cap (\cap \{a T_Q | Q \in X^1(T) \text{ and } Q \neq M\}) \]
\[= \cap_{P \in X^1(R)} a R_P = a R. \]

Hence R is an AWFD Anderson, Mott & Zafrullah [2, Theorem 3.4]. \hfill \Box

The proof of Theorem 4 yields the following theorem as a special case for n = 1 since R is a WFD if and only if R is weakly Krull and Cl(R) = 0 (cf. Anderson & Zafrullah [1, Theorem]).

**Theorem 5** (cf. Anderson, Chang & Park [5, Theorem 2.6]). If R is a pullback domain of type (□*), then R is a WFD if and only if htM = 1, D is a field, and T is a WFD.

**Remark 1.** Although some parts of the proofs of Theorems 1, 3, and 4 are the same as those of their counterparts in Anderson, Chang & Park [5], we give them here for the completeness.

We end this paper with an example which shows that Theorems 3, 4, and 5 do not hold without the assumption that R is of type (□*). However, we do not know if R being of type (□*) is best possible for Theorems 3, 4 and 5.

**Example 6.** Let K be a field of characteristic 0, X an indeterminate over K, and Y an indeterminate over the field K(X). Let \( \varphi : K(X^2)[Y] \to K(X) \) be the ring homomorphism determined by \( Y \mapsto X \), and let \( M = \text{ker}(\varphi) \). See the following pullback diagram.
\[
\begin{array}{ccc}
R = \varphi^{-1}(K) & \longrightarrow & K \\
\downarrow & & \downarrow \\
T = K(X^2)[Y] & \xrightarrow{\varphi} & T/M = K(X)
\end{array}
\]

(1) $M$ is a height-one maximal ideal of $T$ such that $T/M = K(X)$.
(2) $R$ is not of type $(\square^*)$.
(3) The map $\psi : \text{Spec}(T) \rightarrow \text{Spec}(R)$, given by $Q \mapsto Q \cap R$, is bijective.
(4) $\dim(R) = \dim(T) = 1$.
(5) $R$ is not a GWFD, while $T$ is a PID, $\text{ht}M = 1$, and $K$ is a field.

**Proof.** (1) Since $Y^2 - X^2 \in M$, $M \neq (0)$, and hence $M$ is a height-one maximal ideal of $T$ because $T$ is a PID. In particular, $\varphi(T) = T/M$ is a subfield of $K(X)$ containing $K(X^2)$ and $X$, and thus $T/M = K(X)$.

(2) Note that $U(T) = K(X^2) \setminus \{0\}$; so $Yu \not\in R$ for all $u \in U(T)$. For if $Yu \in R$, then $\varphi(Yu) = Xu = a \in K$, and thus $u = \frac{a}{X} \not\in K(X^2)$, a contradiction.

(3) and (4) Since $K$ is a field, $M$ is a maximal ideal of $R$. Hence if $P$ is a prime ideal of $R$ such that $P \neq M$, then there is a unique prime ideal $Q$ of $T$ such that $Q \cap R = P$ and $T_Q = R_P$ (cf. Fontana & Gabelli [9, p. 805]). This implies that $\psi$ is bijective and that $\dim(R) = \dim(T) = 1$ by (1) and the fact that $T$ is a PID.

(5) Let $Q = (Y - X^2)T$ and $P = Q \cap R$. Then $Y - X^2$ is a prime element of $T$, and hence $Q$ is a prime ideal of $T$. Assume that $P = \sqrt{fR}$ for some $f \in R$. Then $Q$ is a unique prime ideal of $T$ containing $f$ by (3) and (4); so $Q = \sqrt{fT}$. Since $T$ is a PID, there is a positive integer $n$ and $u \in U(T) = K(X^2) \setminus \{0\}$ such that $f = (Y - X^2)^nu$. Moreover, since $f \in R$, we have $\varphi(f) = (X - X^2)^nu = a \in K$, and hence

\[
u = \frac{a}{(X - X^2)^n}.
\]

However, since the characteristic of $K$ is $0$, $(X - X^2)^n \not\in K[X^2]$, and thus

\[
u = \frac{a}{(X - X^2)^n} \not\in K(X^2),
\]

a contradiction. Hence $P$ is not the radical of a principal ideal. Therefore $R$ is not a GWFD Anderson, Chang & Park [4, Theorem 2.2] because $\dim(R) = 1$. $\square$
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REFERENCES


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