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FUZZY SUBRINGS OF FUNDAMENTAL RINGS

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ABSTRACT. H_v-rings first were introduced by Vougiouklis in 1990. The largest class of algebraic systems satisfying ring-like axioms is the H_v-ring. Let R be an H_v-ring and γ_R^* the smallest equivalence relation on R such that the quotient R/γ_R^*, the set of all equivalence classes, is a ring. In this case R/γ_R^* is called the fundamental ring. In this short communication, we study the fundamental rings with respect to the product of two fuzzy subsets.

1. INTRODUCTION

In 1971, Rosenfeld [8] applied the concept of fuzzy set theory to algebra and introduced the concept of fuzzy subgroup of a group. Sherwood [9] defined the direct product of fuzzy subgroups, Osmer [6] and Ray [7] investigated this concept, also you can see Davvaz [2, 3]. In 1982, Liu [5] defined and studied fuzzy subrings as well as fuzzy ideals.

Vougiouklis in the Fourth AHA Congress 1990 Vougiouklis [11] introduced the notion of H_v-structures and then some researchers followed him. Davvaz [1, 4] defined the concepts of fuzzy H_v-ideals and fuzzy H_v-subrings which are a generalization of the concepts of fuzzy ideals and fuzzy subrings. Davvaz [4] used the definition of a fuzzy H_v-subring and defined the product of fuzzy H_v-subrings. Let R be an H_v-ring and γ_R^* the smallest equivalence relation on R such that the quotient R/γ_R^*, the set of all equivalence classes, is a ring. In this case R/γ_R^* is called the fundamental ring. In this short communication, we study the fundamental rings with respect to the product of two fuzzy subsets.

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2. Basic Definitions

In this section we recall some basic definitions.

Definition 2.1. Let X be a non-empty set. A fuzzy subset μ of X is a function $\mu : X \longrightarrow [0,1]$. Let X, Y be non-empty sets and μ, λ fuzzy subsets of X, Y, respectively. The direct product $\mu \times \lambda$ is usually defined by $(\mu\times\lambda)(x,y) = \min\{\mu(x), \lambda(y)\}$ for all $x \in X$ and $y \in Y$.

Definition 2.2 (Liu [5]). Let A be an ordinary ring and $\mu : A \longrightarrow [0,1]$ be a fuzzy subset of A. Then μ is called a fuzzy subring of A if it satisfies the following conditions:

(1) $\min\{\mu(x),\mu(y)\} \leq \mu(x+y)$ for all x, y in A,
(2) $\mu(x) \leq \mu(-x)$ for all x in A,
(3) $\min\{\mu(x),\mu(y)\} \leq \mu(xy)$ for all x, y in A.

Let μ be any fuzzy subring of A and 0 be the additive identity of A. Then it is easy to verify the following: $\mu(x) \leq \mu(0)$ and $\mu(x) = \mu(-x)$ for all $x \in A$.

Definition 2.3 (Vougiouklis [12]). A hyperstructure is a non-empty set R together with a function $* : R \times R \longrightarrow \mathcal{P}^*(R)$ called hyperoperation, where $\mathcal{P}^*(R)$ is the set of all non-empty subsets of R. A hyperstructure $(R, *)$ is called an H_v-group if the follwoing axioms hold:

(1) $(x * y) * z \cap x * (y * z) \neq \emptyset$ for all $x, y, z \in R$,
(2) $a * R = R * a = R$ for all $a \in R$.

An H_v-ring is a multivalued system $(R, +, \cdot)$ satisfying the ring-like axioms in the following way:

(1) $(R, +)$ is an H_v-group,
(2) $(R, \cdot)$ is an H_v-semigroup, *i. e.*, $(x \cdot y) \cdot z \cap x \cdot (y \cdot z) \neq \emptyset$ for all $x, y, z \in R$,
(3) $\cdot$ is weak distributive with respect to $+$, *i. e.*, $x \cdot (y + z) \cap (x \cdot y + x \cdot z) \neq \emptyset$ and $(x + y) \cdot z \cap (x \cdot z + y \cdot z) \neq \emptyset$ for all $x, y, z \in R$.

Let A and B be two H_v-rings. Then in $A \times B$ we can define two hyperoperations as follows:

$$(a_1, b_1) \oplus (a_2, b_2) = \{(a, b) | a \in a_1 + a_2,\ b \in b_1 + b_2\},$$
$$(a_1, b_1) \odot (a_2, b_2) = \{(a, b) | a \in a_1 \cdot a_2,\ b \in b_1 \cdot b_2\}.$$

Then $A \times B$ is an H_v-ring. We call this H_v-ring the external direct product of A, B.

Definition 2.4 (Davvaz [4]). Let $(R, +, \cdot)$ be an H_v-ring and μ a fuzzy subset of R. Then μ is said to be a fuzzy H_v-subring of R, if the following axioms hold:

(1) $\min\{\mu(x), \mu(y)\} \leq \inf_{\alpha \in x+y}\{\mu(\alpha)\}$ for all $x, y \in R$,

(2) for all $x, a \in R$ there exists $y \in R$ such that $x \in a + y$ and $\min\{\mu(a), \mu(x)\} \leq \mu(y)$,

(3) for all $x, a \in R$ there exists $z \in R$ such that $x \in z+a$ and $\min\{\mu(a), \mu(x)\} \leq \mu(z)$,

(4) $\min\{\mu(x), \mu(y)\} \leq \inf_{\alpha \in x \cdot y}\{\mu(\alpha)\}$ for all $x, y \in R$.

3. Fundamental Relations And Fuzzy Subrings

Let $(R, +, \cdot)$ be an H_v-ring. The relation γ_R^* is the smallest equivalence relation on R such that the quotient R/γ_R^*, the set of all equivalenc classes, is a ring. γ_R^* is called the fundamental relation on R, and R/γ_R^* is called the fundamental ring. If $\mathcal{U}$ denotes the set of all finite polynomials of elements of R, over $\mathbb{N}$ (the set of all natural numbers), then a relation γ_R can be defined on R whose transitive closure is the fundamental relation γ_R^* (see Vougiouklis [12]). The relation γ_R is as follows: For x, y in R we write $x\gamma_R y$ if and only if $\{x, y\} \subseteq u$ for some $u \in \mathcal{U}$. Suppose $\gamma_R^*(a)$ is the equivalence class containing $a \in R$. Then both the sum $\oplus$ and the product $\odot$ on R/γ_R^* are defined as follows:

$$\gamma_R^*(a) \oplus \gamma_R^*(b) = \gamma_R^*(c), \text{ for all } c \in \gamma_R^*(a) + \gamma_R^*(b),$$
$$\gamma_R^*(a) \odot \gamma_R^*(b) = \gamma_R^*(d), \text{ for all } d \in \gamma_R^*(a) \cdot \gamma_R^*(b).$$

Definition 3.1. Let R be an H_v-ring and μ a fuzzy subset of R. The fuzzy subset $\mu_{\gamma_R^*} : R/\gamma_R^* \longrightarrow [0, 1]$ is defined as follows:

$$\mu_{\gamma_R^*}(\gamma_R^*(x)) = \sup_{a \in \gamma_R^*(x)} \{\mu(a)\}.$$

Theorem 3.2 (Davvaz [4]). *Let R be an H_v-ring and μ be a fuzzy H_v-subring of R. Then $\mu_{\gamma_R^*}$ is a fuzzy subring of the ring R/γ_R^*.*

The kernel of the canonical map $\varphi : R \longrightarrow R/\gamma_R^*$ is called the core of R and is denoted by ω_R. Here we also denote by ω_R the zero element of R/γ_R^*, (see Spartalis & Vougiouklis [10], Vougiouklis [11, 12]).

Theorem 3.3 (Vougiouklis [12]). *Let A, B be H_v-rings. Let γ_A^*, γ_B^* and $\gamma_{A\times B}^*$ are fundamental relations on A, B and $A \times B$, respectively. Then*

$$(A \times B)/\gamma_{A\times B}^* \cong A/\gamma_A^* \times B/\gamma_B^*.$$

Theorem 3.4 (Davvaz [4]). *Let A, B be H_v-rings and let γ_A^*, γ_B^* and $\gamma_{A\times B}^*$ be fundamental relations on A, B and $A\times B$, respectively. If μ, λ are fuzzy H_v-subrings of A, B respectively, then we have*

$$(\mu \times \lambda)_{\gamma_{A\times B}^*} = \mu_{\gamma_A^*} \times \lambda_{\gamma_B^*}.$$

Theorem 3.5. *Let μ, λ be fuzzy subsets of H_v-rings A and B, respectively. If $\mu \times \lambda$ is a fuzzy H_v-subring of $A \times B$, then at least one of the following two statements must be held:*

(1) $\lambda_{\gamma_B^*}(\omega_B) \geq \mu_{\gamma_A^*}(\gamma_A^*(a))$ *for all* $a \in A$,
(2) $\mu_{\gamma_A^*}(\omega_A) \geq \lambda_{\gamma_B^*}(\gamma_B^*(b))$ *for all* $b \in B$.

Proof. Suppose $\mu \times \lambda$ is a fuzzy H_v-subring of $A \times B$. Then by Theorem 3.2, $(\mu \times \lambda)_{\gamma_{A\times B}^*}$ is a fuzzy subring of $(A \times B)/\gamma_{A\times B}^*$. Using Theorem 3.4, we have $(\mu \times \lambda)_{\gamma_{A\times B}^*} = \mu_{\gamma_A^*} \times \lambda_{\gamma_B^*}$. By contraposition, suppose that none of the statements (1) and (2) holds. Then we can find $a_0 \in A$ and $b_0 \in B$ such that

$$\mu_{\gamma_A^*}(\gamma_A^*(a_0)) > \lambda_{\gamma_B^*}(\omega_B) \quad \text{and} \quad \lambda_{\gamma_B^*}(\gamma_B^*(b_0)) > \mu_{\gamma_A^*}(\omega_A).$$

Now, we have

$$\begin{aligned}(\mu_{\gamma_A^*} \times \lambda_{\gamma_B^*})(\gamma_A^*(a_0),\ \gamma_B^*(b_0)) &= \min\{\mu_{\gamma_A^*}(\gamma_A^*(a_0)),\ \lambda_{\gamma_B^*}(\gamma_B^*(b_0))\} \\ &> \min\{\mu_{\gamma_A^*}(\omega_A),\ \lambda_{\gamma_B^*}(\omega_B)\} \\ &= (\mu_{\gamma_A^*} \times \lambda_{\gamma_B^*})(\omega_A,\ \omega_B).\end{aligned}$$

On the other hand, it can be easily verified that a fuzzy subring of a ring attains its supremum at zero element, and so we have

$$(\mu_{\gamma_A^*} \times \lambda_{\gamma_B^*})(\omega_A,\ \omega_B) \geq (\mu_{\gamma_A^*} \times \lambda_{\gamma_B^*})(\gamma_A^*(a_0),\ \gamma_B^*(b_0)).$$

Thus $\mu_{\gamma_A^*} \times \lambda_{\gamma_B^*}$ is not a fuzzy subring of $A/\gamma_A^* \times B/\gamma_B^*$. Therefore either $\lambda_{\gamma_B^*}(\omega_B) \geq \mu_{\gamma_A^*}(\gamma_A^*(a))$ for all $a \in A$ or $\mu_{\gamma_A^*}(\omega_A) \geq \lambda_{\gamma_B^*}(\gamma_B^*(b))$ for all $b \in B$. □

Theorem 3.6. *Let μ, λ are fuzzy subsets of H_v-rings A, B, respectively, such that $\mu \times \lambda$ is a fuzzy H_v-subring of $A \times B$. If $\mu_{\gamma_A^*}(\gamma_A^*(a)) \leq \lambda_{\gamma_B^*}(\omega_B)$ for all $a \in A$, then $\mu_{\gamma_A^*}$ is a fuzzy subring of A/γ_A^*.*

Proof. Suppose $x, y \in A$, then we have

$$\begin{aligned}\mu_{\gamma_A^*}(\gamma_A^*(x) \oplus \gamma_A^*(y)) &= \min\{\mu_{\gamma_A^*}(\gamma_A^*(x) \oplus \gamma_A^*(y)),\ \lambda_{\gamma_B^*}(\omega_B \oplus \omega_B)\}\\ &= (\mu_{\gamma_A^*} \times \lambda_{\gamma_B^*})((\gamma_A^*(x), \omega_B) \oplus (\gamma_A^*(y), \omega_B))\\ &\geq \min\{(\mu_{\gamma_A^*} \times \lambda_{\gamma_B^*})(\gamma_A^*(x), \omega_B), (\mu_{\gamma_A^*} \times \lambda_{\gamma_B^*})(\gamma_A^*(y), \omega_B)\}\\ &= \min\{\min\{\mu_{\gamma_A^*}(\gamma_A^*(x)),\ \lambda_{\gamma_B^*}(\omega_B)\},\\ &\qquad\qquad \min\{\mu_{\gamma_A^*}(\gamma_A^*(y)),\ \lambda_{\gamma_B^*}(\omega_B)\}\}\\ &= \min\{\mu_{\gamma_A^*}(\gamma_A^*(x)),\ \mu_{\gamma_A^*}(\gamma_A^*(y))\}.\end{aligned}$$

Also, we have

$$\begin{aligned}\mu_{\gamma_A^*}(-\gamma_A^*(x)) &= \min\{\mu_{\gamma_A^*}(-\gamma_A^*(x)),\ \lambda_{\gamma_B^*}(\omega_B)\}\\ &= (\mu_{\gamma_A^*} \times \lambda_{\gamma_B^*})(-\gamma_A^*(x),\ \omega_B)\\ &= (\mu_{\gamma_A^*} \times \lambda_{\gamma_B^*})(-(\gamma_A^*(x),\ \omega_B))\\ &\geq (\mu_{\gamma_A^*} \times \lambda_{\gamma_B^*})(\gamma_A^*(x),\ \omega_B)\\ &= \min\{\mu_{\gamma_A^*}(\omega_A^*(x)),\ \lambda_{\gamma_B^*}(\omega_B)\}\\ &= \mu_{\gamma_A^*}(\gamma_A^*(x)).\end{aligned}$$

Similarly, we have

$$\mu_{\gamma_A^*}(\gamma_A^*(x) \odot \gamma_A^*(y)) \geq \min\{\mu_{\gamma_A^*}(\gamma_A^*(x)),\ \mu_{\gamma_A^*}(\gamma_A^*(y))\}.$$

Therefore $\mu_{\gamma_A^*}$ is a fuzzy subgring of A/γ_A^*. □

Corollary 3.7. *Let μ, λ be fuzzy subsets of H_v-rings A, B, respectively, such that $\mu \times \lambda$ is a fuzzy H_v-subring of $A \times B$. If $\lambda_{\gamma_B^*}(\gamma_B^*(b)) \leq \mu_{\gamma_A^*}(\omega_A)$ for all $b \in B$, then $\lambda_{\gamma_B^*}$ is a fuzzy subring of B/γ_B^*.*

Proof. The proof is similar to the proof of Theorem 3.6. □

Corollary 3.8. *Let μ, λ be fuzzy subsets of H_v-rings A, B, respectively. If $\mu \times \lambda$ is a fuzzy H_v-subring of $A \times B$, then either $\mu_{\gamma_A^*}$ is a fuzzy subring of A/γ_A^* or $\lambda_{\gamma_B^*}$ is a fuzzy subring of B/γ_B^*.*

Proof. The proof follows from Theorems 3.5, 3.6 and Corollary 3.7. □

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