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VECTOR VARIATIONAL INEQUALITY PROBLEMS WITH GENERALIZED $C(x)$-L-PSEUDOMONOTONE SET-VALUED MAPPINGS

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ABSTRACT. In this paper, we introduce new monotone concepts for set-valued mappings, called generalized $C(x)$-L-pseudomonotonicity and weakly $C(x)$-L-pseudomonotonicity. And we obtain generalized Minty-type lemma and the existence of solutions to vector variational inequality problems for weakly $C(x)$-L-pseudomonotone set-valued mappings, which generalizes and extends some results of Konnov & Yao [11], Yu & Yao [20], and others Chen & Yang [6], Lai & Yao [12], Lee, Kim, Lee & Cho [16] and Lin, Yang & Yao [18].

1. INTRODUCTION

Let X and Y be real Banach spaces with norms $\| \ \|_X$ and $\| \ \|_Y$, respectively, and $L(X,Y)$ the space of all bounded linear mappings from X into Y. A nonempty subset P of X is called a convex cone if $\lambda P + P \subset P$ for all $\lambda \geq 0$. A cone P is said to be pointed if $P \cap (-P) = \{0\}$, and proper if it is properly contained in X. The partial order $\leq$ on X induced by a pointed cone P is defined as $x \leq y$ if and only if $y - x \in P$ for x, $y \in X$, in which case P is called a positive cone in X. A linear order is a partial order induced by a convex cone. An ordered Banach space (X,P) consists of a real Banach space X and a pointed convex cone P with the linear order induced by P. The weak order $\not<$ on an ordered Banach space (X,P) with non-empty interior $\operatorname{int} P$ of P is defined as $x \not< y$ if only if $y - x \notin \operatorname{int} P$ for x, $y \in X$. Let $T : K \to 2^{L(X,Y)}$ be a set-valued mapping from a nonempty convex subset K of X into $2^{L(X,Y)}$, and $C : K \to 2^Y$ a set-valued mapping such that $C(x)$ is a closed convex solid cone of Y.

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After Chen & Cheng [4] proposed the infinite version of finite dimensional vector variational inequality problems introduced by Giannessi [9], and applied it to vector optimization problems, there have been intensive researches on the solutions to vector variational inequality problems and their applications, see Baiocchi & Capelo [1], Chang, Thompson & Yuan [2], Chen [3], Chen & Cheng [4], Chen & Craven [5], Chen & Yang [6], Giannessi [9], Kinderlehrer & Stampacchia [10], Konnov & Yao [11], Lai & Yao [12], Lee & Lee [13], Lee & Lee [14, 15], Lee, Kim, Lee & Cho [16], Lee, Kim, Lee & Yen [17], Lin, Yang & Yao [18] and Yu & Yao [20].

Especially, Konnov & Yao [11] considered the existence of solutions to the following generalized vector variational inequality problems (GVVIP) for $C(x)$-pseudomonotone set-valued mappings;

(GVVIP). Find $\bar{x} \in K$ such that for $y \in K$ there exists $\bar{t} \in T(\bar{x})$ such that

$$\langle \bar{t}, y - \bar{x} \rangle \notin -\operatorname{int} C(\bar{x}).$$

Lin, Yang & Yao [18] also considered the existence of solutions to the (GVVIP) under the assumption of the generalized $C(x)$-pseudomonotonicity and the generalized hemicontinuity of the defining mapping T.

In this paper, we introduce two generalized monotone concepts which can be called as a weakly $C(x)$-L-pseudomonotonicity and a generalized $C(x)$-L-pseudomonotonicity for set-valued mappings.

Next, we consider a generalized Minty-type lemma, and the existence of solutions to the following generalized vector variational inequality problems (**LGVVIP**) for weakly and generalized $C(x)$-L-pseudomonotone set-valued mappings, which is a generalized form of (**GVVIP**).

(LGVVIP). Find a $y \in K$ such that for each $x \in K$, there exists a $z \in \operatorname{seg}[x, y)$ such that for each $w \in \operatorname{seg}[z, y]$ there exists a $t \in T(w)$ satisfying

$$\langle t, x - w \rangle \notin -\operatorname{int} C(w),$$

where the line segement $\operatorname{seg}[x, y)$ denotes the set $\{r = tx + (1-t)y : t \in (0, 1]\}$.

2. SOME MONOTONE SET-VALUED MAPPINGS

Definition 2.1. Let X and Y be Banach spaces, K a nonempty convex subset of X and $T : K \to 2^{L(X,Y)}$ a mapping. Let $C : K \to 2^Y$ be a set-valued mapping such that for each $x \in K$, $C(x)$ is a closed convex pointed cone with $C(x) \neq \emptyset$.

(1) T is $C(x)$-pseudomonotone on K if for every pair of points x, $y \in K$ and for all $t \in T(x)$, $t' \in T(y)$, we have

$$\langle t', x - y\rangle \notin -\operatorname{int} C(y) \quad \text{implies} \quad \langle t, x - y\rangle \notin -\operatorname{int} C(y).$$

(2) T is $C(x)$-L-pseudomonotone on K if for every pair of points x, $y \in K$, there exists a point $z \in \operatorname{seg}[x, y)$ such that for each point $w \in \operatorname{seg}[z, y)$ and for all $t \in T(x)$, $t' \in T(w)$, we have

$$\langle t', x - w\rangle \notin -\operatorname{int} C(w) \quad \text{implies} \quad \langle t, x - w\rangle \notin -\operatorname{int} C(w).$$

(3) T is weakly $C(x)$-pseudomonotone on K if for every pair of points x, $y \in K$ and for every $t' \in T(y)$, we have

$$\langle t', x - y\rangle \notin -\operatorname{int} C(y) \quad \text{implies} \quad \langle t, x - y\rangle \notin -\operatorname{int} C(y) \quad \text{for some } t \in T(x).$$

(4) T is weakly $C(x)$-L-pseudomonotone on K if for every pair of points x, $y \in K$, there exists a point $z \in \operatorname{seg}[x, y)$ such that for each point $w \in \operatorname{seg}[z, y)$ and for every $t' \in T(w)$, we have

$$\langle t', x - w\rangle \notin -\operatorname{int} C(w) \quad \text{implies} \quad \langle t, x - w\rangle \notin -\operatorname{int} C(w) \quad \text{for some } t \in T(x).$$

(5) (Lin, Yang & Yao [18], Schaible [19]) T is generalized $C(x)$-pseudomonotone on K if, for every pair of points x, $y \in K$, there exists $t' \in T(y)$ such that

$$\langle t', x - y\rangle \notin -\operatorname{int} C(y)$$

implies that there exists $t \in T(x)$ such that $\langle t, x - y\rangle \notin -\operatorname{int} C(y)$.

(6) T is generalized $C(x)$-L-pseudomonotone on K if, for every pair of points x, $y \in K$, there exists $z \in \operatorname{seg}[x, y)$ such that for each point $w \in \operatorname{seg}[z, y)$, there exists $t' \in T(w)$ such that $\langle t', x - w\rangle \notin -\operatorname{int} C(w)$ implies that there exists $t \in T(x)$ such that $\langle t, x - w\rangle \notin -\operatorname{int} C(w)$.

Definition 2.2. Let K be a nonempty convex subset of a Banach space X. A set-valued mapping $T : K \to 2^{L(X,Y)}$ is said to be hemicontinuous on K if its restriction to any line segments $\operatorname{seg}[x, y]$ in K is upper semicontinuous with respect to the weak topology on $L(X, Y)$.

3. Minty-type lemmas

Now we consider Minty-type lemmas for generalized $C(x)$-L-pseudomonotone and weakly $C(x)$-L-pseudomonotone set-valued mappings which are useful on the consideration of the existence of solutions to (**LGVVIP**).

Lemma 3.1. *Let X, Y be Banach spaces and K a nonempty convex subset of X. Let $T : K \to 2^{L(X,Y)}$ be a generalized $C(x)$-L-pseudomonotone and hemicontinuous set-valued mapping.*

Then the following are equivalent;

(a) *For every pair of distinct points $x, y \in K$, there exists a $z \in \operatorname{seg}[x, y)$ such that for each $w \in \operatorname{seg}[x, y)$, there exists a $t \in T(w)$ satisfying*

$$\langle t, x - w\rangle \notin -\operatorname{int} C(w).$$

(b) *For every pair of distinct points $x, y \in K$, there exists a $z \in \operatorname{seg}[x, y)$ such that for each $w \in \operatorname{seg}[x, y)$, there exists a $t \in T(x)$ satisfying*

$$\langle t, x - w\rangle \notin -\operatorname{int} C(w).$$

Proof. (a)$\Longrightarrow$(b) Let x and y be distinct points of K. Suppose that there exists a $z_1 \in \operatorname{seg}[x, y)$ such that for each $w \in \operatorname{seg}[z_1, y)$, there exists a $t' \in T(w)$ satisfying

$$\langle t', x - w\rangle \notin -\operatorname{int} C(w).$$

Since $T : K \to 2^{L(X,Y)}$ is generalized $C(x)$-L-pseudomonotone, there exists a $z_2 \in \operatorname{seg}[x, y)$ such that for each $w \in \operatorname{seg}[z_2, y)$ there exists $t' \in T(w)$ such that

$$\langle t', x - w\rangle \notin -\operatorname{int} C(w)$$

implies that there exists $t \in Tx$ such that $\langle t, x - w\rangle \notin -\operatorname{int} C(w)$.

Choose $z \in K$ with $\operatorname{seg}[z, y) = \operatorname{seg}[z_1, y) \cap \operatorname{seg}[z_2, y)$. Then for each $w \in \operatorname{seg}[z, y)$, there exists a $t \in T(x)$, we have $\langle t, x - w\rangle \notin -\operatorname{int} C(w)$.

(b)$\Longrightarrow$(a) Let z be such a point in (b). Suppose that there exists $w \in \operatorname{seg}[z, y)$ such that for all $t' \in T(w)$,

$$\langle t', x - w\rangle \in -\operatorname{int} C(w).$$

For each $n \in \mathbb{N}$, set $w_n = \dfrac{1}{n}x + \left(1 - \dfrac{1}{n}\right) w$. Then, by the hemicontinuity of T at w, there exists $n_0 \in \mathbb{N}$ such that $\langle t, x - w\rangle \in -\operatorname{int} C(w)$ for all $t \in T(w_n)$, $n \geq n_0$. Since

$\frac{1}{n}\langle t, x-w\rangle \in -\operatorname{int} C(w)$ for $n \geq n_0$ and

$$\frac{1}{n}(x-w) = \frac{1}{n}x + \left(1-\frac{1}{n}\right)w - w = w_n - w,$$

we have

$$\langle t, w_n - w\rangle \in -\operatorname{int} C(w) \text{ for all } t \in T(w_n),\ n \geq n_0.$$

This contradicts (b). □

As corollaries, we can obtain the following Minty-type lemmas for weakly $C(x)$-L-pseudomonotone and for weakly $C(x)$-pseudomonotone set-valued mappings, and weakly $C(x)$-pseudomonotone single-valued mappings.

Lemma 3.2. *Let X, Y be Banach spaces and K a nonempty convex subset of X. Let $T : K \to 2^{L(X,Y)}$ be a weakly $C(x)$-L-pseudomonotone and hemicontinuous set-valued mapping.*

Then the following are equivalent;

(a) *For every pair of distinct points x, $y \in K$, there exists a $z \in \operatorname{seg}[x,y)$ such that for each $w \in \operatorname{seg}[z,y)$ and $t' \in T(w)$, we have*

$$\langle t', x-w\rangle \notin -\operatorname{int} C(w).$$

(b) *For every pair of distinct points x, $y \in K$, there exists a $z \in \operatorname{seg}[x,y)$ such that for each $w \in \operatorname{seg}[z,y)$ and for all $t \in T(x)$*

$$\langle t, x-w\rangle \notin -\operatorname{int} C(w).$$

Proof. By the similar method to Lemma 3.1, it can be proved. □

Corollary 3.3 (Generalized Linearization Lemma: Konnov & Yao [11]). *Let X, Y be Banach spaces and K be a nonempty convex subset of X. Let $T : K \to 2^{L(X,Y)}$ be a weakly $C(x)$-pseudomonotone and hemicontinuous set-valued mappings.*

Then the following are equivalent;

(I) *There exists an $x \in K$ such that for $y \in K$, there exists a $t' \in T(x)$ such that $(t', y-x) \notin -\operatorname{int} C(x)$.*

(II) *There exists an $x \in K$ such that for $y \in K$, there exists a $t'' \in T(y)$ such that $(t'', y-x) \notin -\operatorname{int} C(x)$.*

4. Existence results of solutions

Let K be a nonempty subset of a topological vector space X. Then a set-valued mapping $F : K \to 2^X$ is said to be a Knaster-Kuratowski-Mazurkiewicz mapping (in short, KKM-mapping) if for each nonempty finite subset N of K, $coN \subset F(N)$, where coN is the convex hull of N.

Lemma 4.1 (Dugundji & Granas [7]). *A Banach space X is reflexive if and only if every closed convex bounded subset of X is weakly compact.*

The following F-KKM theorem in Fan [8] is essential in our results.

Theorem 4.2. *Let X be a topological vector space, $K \subset X$ an arbitrary subset, and $G : K \to 2^X$ a KKM-mapping. If all the sets $G(x)$ are closed in X and if at least one is compact, then $\bigcap\{G(x) : x \in K\} \neq \emptyset$.*

For weakly $C(x)$-L-pseudomonotone set-valued mappings, we obtain the following existence result of solutions to (**LGVVIP**).

Theorem 4.3. *Let X be a real reflexive Banach space, Y a real Banach space, and K a nonempty closed convex bounded subset of X. Let $C : X \to 2^Y$ be a set-valued mapping such that for each $x \in K$, $C(x)$ is a proper closed convex cone with $\operatorname{int} C(x) \neq \emptyset$ and $W : K \to 2^Y$ be a set-valued mapping defined by $W(x) = Y \backslash (-\operatorname{int} C(x))$ such that the graph $Gr(W) := \{(x,y) | x \in K,\ y \in W(x)\}$ of W is weakly closed in $X \times Y$. Let $T : K \to 2^{L(X,Y)}$ be a weakly $C(x)$-L-pseudomonotone and hemicontinuous mapping. Then (**LGVVIP**) is solvable.*

Proof. Define a set-valued mapping $F : K \to 2^K$ by

$F(x) = \{y \in K :$ there exists a $z \in \text{seg}\,[x,y)$ such that for each $w \in \text{seg}[z,y)$ and for some $t \in T(w)$, $\langle t, x - w\rangle \notin -\operatorname{int} C(w)\}$.

Note that $F(x)$ is a nonempty set for each $x \in K$, since $x \in F(x)$. The proof is divided into the following five steps.

(i) F is a KKM-mapping on K. Assume to the contrary, there exist

$$x_1, x_2, \dots, x_m \in K$$

such that

$$co\{x_1, x_2, \dots, x_m\} \not\subset \bigcup_{i=1}^{m} F(x_i).$$

Put

$$w := \sum_{i=1}^{m} \alpha_i x_i \notin \bigcup_{i=1}^{m} F(x_i)$$

for all nonnegative real numbers α_i, $1 \le i \le m$, with

$$\sum_{i=1}^{m} \alpha_i = 1.$$

Then for all $t \in T(w)$, $\langle t, x_i - w\rangle \in -\operatorname{int} C(w)$, $i = 1, 2, \cdots, m$. Since $\sum_{i=1}^{m} \alpha_i = 1$ and each $\langle t, x_i - w\rangle$ is an element of the convex set $-\operatorname{int} C(w)$, $\sum_{i=1}^{m} \alpha_i \langle t, x_i - w\rangle$ belongs to $-\operatorname{int} C(w)$.

On the other hand,

$$\sum_{i=1}^{m} \alpha_i \langle t, x_i - w\rangle = \left\langle t, \sum_{i=1}^{m} \alpha_i x_i - w \right\rangle = \langle t, w - w\rangle = 0.$$

Therefore $0 \in -\operatorname{int} C(w)$ and so $0 \in \operatorname{int} C(w)$, which means $C(w) = Y$. This is a contradiction to the fact that $C(w)$ is a proper cone of Y. Thus F is a KKM mapping.

(ii) Define a mapping $G : K \to 2^K$ by

$$\begin{aligned} G(x) = \{y \in K :\ &\text{there exists a } z \in \operatorname{seg}[x, y) \text{ such that for each} \\ &w \in \operatorname{seg}[z, y) \text{ and for all } t \in T(x),\ \langle t, x - w\rangle \notin -\operatorname{int} C(w)\}, \end{aligned}$$

then $G(x)$ is weakly closed for each $x \in K$. In fact, suppose that $\{y_\alpha\}_{\alpha \in I}$ is a net in $G(x)$ weakly converging to y in K. Then for each $\alpha \in I$, there exists a $z_\alpha \in \operatorname{seg}[x, y_\alpha)$ such that for each $w_\alpha \in \operatorname{seg}[z_\alpha, y_\alpha)$ and for all $t \in T(x)$, $\langle t, x - w_\alpha\rangle \notin -\operatorname{int} C(w_\alpha)$. Choose a real net $\{k_\alpha\}$ in $[0, 1]$ such that

$$z_\alpha = k_\alpha x + (1 - k_\alpha) y_\alpha.$$

Without loss of generality, from the compactness of $[0, 1]$ we can assume that there exists a limit $k_0 = \lim_{\alpha \in I} k_\alpha$ in $[0, 1]$. Hence $\{z_\alpha\}$ weakly converges to $z_0 = k_0 x + (1 - k_0) y$ in K. If $k_0 = 1$, since $\operatorname{seg}[z_0, y)$ is empty it was done. Let $w \in \operatorname{seg}[z_0, y)$ and let $w = s z_0 + (1 - s) y$ for some $s \in (0, 1]$. For each $\alpha \in I$, let $w_\alpha = s z_\alpha + (1 - s) y_\alpha$, then $\langle t, x - w_\alpha\rangle \notin -\operatorname{int} C(w_\alpha)$ for all $t \in T(w)$ and $w_\alpha \to w$ weakly in K. Hence for $t \in T(x)$

$$\begin{aligned} \langle t, x - w\rangle &= \langle t, x - \lim_{\alpha \in I} w_\alpha\rangle \\ &= \lim_{\alpha \in I} \langle t, x - w_\alpha\rangle. \end{aligned}$$

Therefore the net $\{(w_\alpha, \langle t, x - w_\alpha\rangle)\}_\alpha$ in the graph $Gr(W)$ of W weakly converges to $(w, \langle t, x - w\rangle)$ in $X \times Y$. Since $Gr(W)$ is weakly closed in $X \times Y$, $(w, \langle t, x - w\rangle)$ belongs to $Gr(W)$, that is, $\langle t, x - w\rangle \notin -\operatorname{int} C(w)$. Thus $y \in G(x)$, which implies that $G(x)$ is weakly closed.

(iii) G is a KKM-mapping and $\bigcap_{x\in K} F(x) = \bigcap_{x\in K} G(x)$. Indeed, since T is weakly $C(x)$-L-pseudomonotone, $F(x) \subset G(x)$ for $x \in K$, and then G is a KKM-mapping. By Lemma 3.2, $\bigcap_{x\in K} F(x) = \bigcap_{x\in K} G(x)$.

(iv) $\bigcap_{x\in K} F(x)$ is nonempty. In fact, since X is a reflexive Banach space, by Lemma 4.1, K is weakly compact and the weakly closed subset $G(x)$ of K is also weakly compact. From F-KKM theorem, it follows that $\bigcap_{x\in K} G(x) \neq \emptyset$. Therefore by the step (iii), $\bigcap_{x\in K} F(x) \neq \emptyset$, that is, there exists $y \in K$ such that for each $x \in K$, there exists $z \in \operatorname{seg}[x, y)$ such that for each $w \in \operatorname{seg}[z, y)$

$$\langle t, x - w\rangle \notin -\operatorname{int} C(w) \quad \text{for some} \quad t \in T(w).$$

(v) If $y \in \bigcap_{x\in K} F(x)$, then y is a solution of (LGVVIP). Let $y \in \bigcap_{x\in K} F(x)$ and $x \in K$, then we show that $\langle t, x - y\rangle \notin -\operatorname{int} C(y)$ for some $t \in T(y)$. Assume to the contrary, for all $t \in T(y)$,

$$\langle t, x - y\rangle \in -\operatorname{int} C(y).$$

Choose a sequence $\{y_n\}$ such that

$$y_n = \frac{1}{n}x + \left(1 - \frac{1}{n}\right) y$$

for $n \in \mathbb{N}$. Since T is hemicontinuous there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$\langle t_n, x - y\rangle \in -\operatorname{int} C(y) \quad \text{for all} \quad t_n \in T(y_n).$$

On the other hand, since $y \in F(x)$, there exists $n_1 \in \mathbb{N}$ such that if $n \geq n_1$ then there exists $t_n \in T(y_n)$ satisfying the following;

$$\langle t_n, x - y_n\rangle \notin -\operatorname{int} C(y_n).$$

Since $y_n \to y$ in K and $\langle t_n, x - y_n\rangle \to \langle t_n, x - y\rangle$ weakly in Y, we have

$$\Big(y_n, \langle t_n, x - y_n\rangle\Big) \longrightarrow \Big(y, \langle t_n, x - y\rangle\Big) \text{ weakly in } X \times Y.$$

Therefore $\Big(y, \langle t_n, x - y\rangle\Big) \in Gr(W)$, which is weakly closed in $X \times Y$. Hence $\langle t_n, x - y\rangle \notin -\operatorname{int} C(y)$. This is a contradiction to the fact that for $n = \max\{n_0, n_1\}$

$$\langle t_n, x - y\rangle \in -\operatorname{int} C(y) \quad \text{for all} \quad t_n \in T(y_n).$$

Thus there exists $t \in T(y)$ such that $\langle t, x - y\rangle \notin -\operatorname{int} C(y)$. □

For generalized $C(x)$-L-pseudomonotone set-valued mappings, we obtain the following existence result of solutions to the (**LGVVIP**).

Theorem 4.4. *Let $T : K \to 2^{L(X,Y)}$ be a generalized $C(x)$-L-pseudomonotone and hemicontinuous mapping, and other conditions be same as Theorem 4.3. Then (**LGVVIP**) is also solvable.*

Proof. Define a set valued mapping F as that in the proof of Theorem 4.3, which is a KKM-mapping. Define a mapping $G : K \to 2^K$ by

$$G(x) = \{y \in K : \text{there exist a } z \in \operatorname{seg}[x, y) \text{ such that for each } w \in \operatorname{seg}[z, y) \text{ and} \\ \text{for some } t \in T(x),\ \langle t, x - w\rangle \notin -\operatorname{int} C(w)\}.$$

To prove that $G(x)$ is weakly closed in K for each $x \in X$, it suffices to follow the step (ii) of Theorem 4.3 provided that "for all $t \in T(x)$" is replaced by "for some $t \in T(x)$".

For step(iii), by the part (a)$\Longrightarrow$(b) of Lemma 3.1, $F(x) \subset G(x)$ for $x \in K$ and so G is a KKM-mapping. And Lemma 3.1 implies that

$$\bigcap_{x\in K} F(x) = \bigcap_{x\in K} G(x).$$

If we choose the same step (iv) and step (v) of Theorem 4.3, the proof is complete. □

As a main corollary, we obtain the following main result of Konnov & Yao [11].

Theorem 4.5 (Konnov & Yao [11]). *Let X and Y be real Banach spaces. Let K be a nonempty weakly compact convex subset of X. Let $C : K \to 2^Y$ be such that for each $x \in K$, $C(x)$ is a proper closed convex cone with $\operatorname{int} C(x) \neq \emptyset$, and $W : K \to 2^Y$ be defined by $W(x) = Y\backslash(-\operatorname{int} C(x))$ such that the graph $Gr(W)$ of W is weakly closed in $X \times Y$. Suppose that $T : K \to 2^{L(X,Y)}$ is $C(x)$-pseudomonotone and hemicontinuous on K. Suppose also that T has nonempty values. Then, there exists a solution to the (GVVIP).*

We note that T is said to be generalized v-coercive on K if there exists a weakly compact subset B of X and $y_0 \in B \cap K$, such that, for every $t \in T(x)$,

$$\langle t, y_0 - x\rangle \in -\operatorname{int} C(x) \quad \text{for all} \quad x \in K\backslash B.$$

We obtain the following corollary.

Theorem 4.6 (Konnov & Yao [11]). *Let X, Y, C, W, and $Gr(W)$ be the same as in Theorem 4.5. Let K be a nonempty closed convex subset of X. Suppose that $T : K \to 2^{L(X,Y)}$ is $C(x)$-pseudomonotone, generalized v-coercive, and hemicontinuous on K. Suppose also that T has nonempty values. Then, the (GVVIP) has a solution.*

We obtain the following main result of Lin, Yang & Yao [18] as corollaries.

Theorem 4.7 (Lin, Yang & Yao [18]). *Let X and Y be real Banach space. Let K be a nonempty weakly compact convex subset of X. Let $C : K \to 2^Y$ be such that, for each $x \in K$, $C(x)$ is a proper closed convex solid cone; and let $W : K \to 2^Y$ be defined by $W(x) = Y \backslash (-\operatorname{int} C(x))$, such that the graph $Gr(W)$ of W is weakly closed in $X \times Y$. If $T : K \to 2^{L(X,Y)}$ is generalized $C(x)$-pseudomonotone, nonempty compact-valued, and hemicontinuous on K, then (GVVIP) has a solution.*

Corollary 4.8 (Lin, Yang & Yao [18]). *Let X and Y be real Banach space. Let K be a nonempty weakly compact convex subset of X. Let $C : K \to 2^Y$ be such that, for each $x \in K$, $C(x)$ is a proper closed convex solid cone; and let $W : K \to 2^Y$ be defined by $W(x) = Y \backslash (-\operatorname{int} C(x))$, such that the graph $Gr(W)$ of W is weakly closed in $X \times Y$. If $T : K \to 2^{L(X,Y)}$ is generalized $C(x)$-pseudomonotone, nonempty compact-valued, and upper semicontinuous from line segments in K, then (GVVIP) has a solution.*

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HILBERT-SCHMIDT INTERPOLATION ON $Ax = y$ IN A TRIDIAGONAL ALGEBRA ALG$\mathcal{L}$

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ABSTRACT. Given vectors x and y in a separable Hilbert space $\mathcal{H}$, an interpolating operator is a bounded operator A such that $Ax = y$. In this article, we investigate Hilbert-Schmidt interpolation problems for vectors in a tridiagonal algebra. We show the following: Let $\mathcal{L}$ be a subspace lattice acting on a separable complex Hilbert space $\mathcal{H}$ and let $x = (x_i)$ and $y = (y_i)$ be vectors in $\mathcal{H}$. Then the following are equivalent:

(1) There exists a Hilbert-Schmidt operator $A = (a_{ij})$ in Alg$\mathcal{L}$ such that $Ax = y$.
(2) There is a bounded sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ and

$$\begin{aligned} y_1 &= \alpha_1 x_1 + \alpha_2 x_2 \\ &\vdots \\ y_{2k} &= \alpha_{4k-1} x_{2k} \\ y_{2k+1} &= \alpha_{4k} x_{2k} + \alpha_{4k+1} x_{2k+1} + \alpha_{4k+1} x_{2k+2} \end{aligned}$$

for $k \in \mathbb{N}$.

1. INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A}$ be a subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of all operators acting on $\mathcal{H}$. Suppose that X and Y are specified, not necessarily in the algebra. Under what conditions can we expect there to be a solution of the operator equation $AX = Y$, where the operator A is required to lie in $\mathcal{A}$? We refer to such a question as an interpolation problem. The 'n-vector interpolation problem', asks for an operator A such that $Ax_i = y_i$ for fixed finite collections $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_n\}$. The n-vector interpolation problem was considered for a C^*-algebra $\mathcal{U}$ by Kadison [6]. In case $\mathcal{U}$ is a nest algebra, the (one-vector) interpolation problem was solved by Lance [7]: his result was extended by Hopenwasser [2] to the case that $\mathcal{U}$ is a CSL-algebra. Munch [8] obtained conditions for interpolation in case A is required to lie in the ideal of Hilbert-Schmidt operators in a nest algebra.

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Hopenwasser [3] once again extended the interpolation condition to the ideal of Hilbert-Schmidt operators in a CSL-algebra. Hopenwasser's paper also contains a sufficient condition for interpolation n-vectors, although necessity was not proved in that paper.

We establish some notations and conventions. A commutative subspace lattice $\mathcal{L}$, or CSL $\mathcal{L}$ is a strongly closed lattice of pairwise-commuting projections acting on a Hilbert space $\mathcal{H}$. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. If $\mathcal{L}$ is CSL, Alg$\mathcal{L}$ is called a CSL-algebra. The symbol Alg$\mathcal{L}$ is the algebra of all bounded operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$. Let x and y be two vectors in a Hilbert space $\mathcal{H}$. Then $\langle x, y\rangle$ means the inner product of the vectors x and y. Let M be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{M}$ means the closure of M and $\overline{M}^{\perp}$ the orthogonal complement of $\overline{M}$. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers.

2. RESULTS

Let $\mathcal{H}$ be a separable complex Hilbert space with a fixed orthonormal basis $\{e_1, e_2, \ldots\}$. Let $x_1, x_2, \ldots, x_n$ be vectors in $\mathcal{H}$. Then $[x_1, x_2, \ldots, x_n]$ means the closed subspace generated by the vectors $x_1, x_2, \ldots, x_n$. Let $\mathcal{L}$ be the subspace lattice generated by the subspaces $[e_{2k-1}], [e_{2k-1}, e_{2k}, e_{2k+1}]$ $(k = 1, 2, \ldots)$. Then the algebra Alg$\mathcal{L}$ is called a tridiagonal algebra which was introduced by Gilfeather & Larson [1]. These algebras have been found to be useful counterexample to a number of plausible conjectures.

Let $\mathcal{A}$ be the algebra consisting of all bounded operators acting on $\mathcal{H}$ of the form

$$\begin{pmatrix} * & * & & & \\ & * & & & \\ & * & * & * & \\ & & & * & \\ & & & * & \ddots \end{pmatrix}$$

with respect to the orthonormal basis $\{e_1, e_2, \ldots\}$, where all non-starred entries are zero. It is easy to see that Alg$\mathcal{L} = \mathcal{A}$.

We consider interpolation problems for the above tridiagonal algebra Alg$\mathcal{L}$.

Theorem 1. *Let Alg$\mathcal{L}$ be the tridiagonal algebra on a Hilbert space $\mathcal{H}$ and let $x = (x_i)$ and $y = (y_i)$ be vectors in $\mathcal{H}$. Then the following are equivalent:*

(1) *There exists a Hilbert-Schmidt operator $A = (a_{ij})$ in Alg$\mathcal{L}$ such that $Ax = y$.*
(2) *There is a bounded sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ and*

$$\begin{aligned} y_1 &= \alpha_1 x_1 + \alpha_2 x_2 \\ &\vdots \\ y_{2k} &= \alpha_{4k-1} x_{2k} \\ y_{2k+1} &= \alpha_{4k} x_{2k} + \alpha_{4k+1} x_{2k+1} + \alpha_{4k+2} x_{2k+2} \textit{ for } k \in \mathbb{N}. \end{aligned}$$

Proof. Suppose that A is a Hilbert-Schmidt operator $A = (a_{ij})$ in Alg$\mathcal{L}$ such that $Ax = y$. Let $\alpha_n = a_{ij}$ for $n = i + j - 1$ and $\{e_n\}$ is the standard orthonormal basis for $\mathcal{H}$. Since A is Hilbert-Schmidt, $\sum_i \|Ae_i\|^2 < \infty$. Hence

$$\begin{aligned} \sum_i \|Ae_i\|^2 &= \sum_i \sum_j |\langle Ae_i, e_j \rangle|^2 \\ &= \sum_{k=1}^{\infty} \langle Ae_{2k-1}, e_{2k-1} \rangle + \sum_{k=1}^{\infty} \langle Ae_{2k}, (e_{2k-1} + e_{2k} + e_{2k+1}) \rangle \\ &= \sum_{k=1}^{\infty} |\alpha_{4k-3}|^2 + \sum_{k=1}^{\infty} (|\alpha_{4k-2}|^2 + |\alpha_{4k+1}|^2 + |\alpha_{4k}|^2) \\ &= \sum_{k=1}^{\infty} |\alpha_k|^2 < \infty. \end{aligned}$$

Since $Ax = y$,

$$\begin{aligned} y_1 &= \alpha_1 x_1 + \alpha_2 x_2 \\ &\vdots \\ y_{2k} &= \alpha_{4k-1} x_{2k} \\ y_{2k+1} &= \alpha_{4k} x_{2k} + \alpha_{4k+1} x_{2k+1} + \alpha_{4k+1} x_{2k+2}. \end{aligned}$$

Conversely, assume that there is a bounded sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that

$$\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$$

and

$$\begin{aligned} y_1 &= \alpha_1 x_1 + \alpha_2 x_2 \\ &\vdots \\ y_{2k} &= \alpha_{4k-1} x_{2k} \\ y_{2k+1} &= \alpha_{4k} x_{2k} + \alpha_{4k+1} x_{2k+1} + \alpha_{4k+2} x_{2k+2}. \end{aligned}$$

Let A be a matrix with $a_{ij} = \alpha_n$ for $i + j - 1 = n$. Then A is a Hilbert-Schmidt operator. Since

$$\begin{aligned} y_1 &= \alpha_1 x_1 + \alpha_2 x_2 \\ y_{2k} &= \alpha_{4k-1} x_{2k} \\ &\vdots \\ y_{2k+1} &= \alpha_{4k} x_{2k} + \alpha_{4k+1} x_{2k+1} + \alpha_{4k+2} x_{2k+2}, \end{aligned}$$

$Ax = y$. □

Theorem 2. *Let Alg$\mathcal{L}$ be the tridiagonal algebra on a Hilbert space $\mathcal{H}$ and let $x_i = (x_j^{(i)})$ and $y_i = (y_j^{(i)})$ be vectors in $\mathcal{H}$ for $i = 1, 2, \dots, n$. Then the following are equivalent:*

(1) *There exists a Hilbert-Schmidt operator $A = (a_{ij})$ in Alg$\mathcal{L}$ such that $Ax_i = y_i$ for all $i = 1, 2, \dots, n$.*
(2) *There is a bounded sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ and*

$$\begin{aligned} y_1^{(i)} &= \alpha_1 x_1^{(i)} + \alpha_2 x_2^{(i)} \\ &\vdots \\ y_{2k}^{(i)} &= \alpha_{4k-1} x_{2k}^{(i)} \\ y_{2k+1}^{(i)} &= \alpha_{4k} x_{2k}^{(i)} + \alpha_{4k+1} x_{2k+1}^{(i)} + \alpha_{4k+2} x_{2k+2}^{(i)} \text{ for } k \in \mathbb{N}, \end{aligned}$$

for all $i = 1, 2, \dots, n$.

Proof. Suppose that A is a Hilbert-Schmidt operator $A = (a_{ij})$ in Alg$\mathcal{L}$ such that $Ax_i = y_i$ for all $i = 1, 2, \dots, n$. Let $\alpha_n = a_{ij}$ for $n = i + j - 1$ and $\{e_n\}$ is the standard orthonormal basis for $\mathcal{H}$. Since A is Hilbert-Schmidt, $\sum_i \|Ae_i\|^2 < \infty$.

Hence

$$\begin{aligned}\sum_i \|Ae_i\|^2 &= \sum_i \sum_j |\langle Ae_i, e_j\rangle|^2 \\ &= \sum_{k=1}^{\infty} \langle Ae_{2k-1}, e_{2k-1}\rangle + \sum_{k=1}^{\infty} \big\langle Ae_{2k}, (e_{2k-1} + e_{2k} + e_{2k+1})\big\rangle \\ &= \sum_{k=1}^{\infty} |\alpha_{4k-3}|^2 + \sum_{k=1}^{\infty} (|\alpha_{4k-2}|^2 + |\alpha_{4k+1}|^2 + |\alpha_{4k}|^2) \\ &= \sum_{k=1}^{\infty} |\alpha_k|^2 < \infty.\end{aligned}$$

So $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$. Since $Ax_i = y_i$ for all $i = 1, 2, \ldots, n$,

$$\begin{aligned} y_1^{(i)} &= \alpha_1 x_1^{(i)} + \alpha_2 x_2^{(i)} \\ &\vdots \\ y_{2k}^{(i)} &= \alpha_{4k-1} x_{2k}^{(i)} \\ y_{2k+1}^{(i)} &= \alpha_{4k} x_{2k}^{(i)} + \alpha_{4k+1} x_{2k+1}^{(i)} + \alpha_{4k+2} x_{2k+2}^{(i)} \text{ for } k \in \mathbb{N}, \end{aligned}$$

for all $i = 1, 2, \ldots, n$.

Conversely, assume that there is a bounded sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that

$$\sum_{k=1}^{\infty} |\alpha_k|^2 < \infty$$

and

$$\begin{aligned} y_1^{(i)} &= \alpha_1 x_1^{(i)} + \alpha_2 x_2^{(i)} \\ &\vdots \\ y_{2k}^{(i)} &= \alpha_{4k-1} x_{2k}^{(i)} \\ y_{2k+1}^{(i)} &= \alpha_{4k} x_{2k}^{(i)} + \alpha_{4k+1} x_{2k+1}^{(i)} + \alpha_{4k+2} x_{2k+2}^{(i)} \text{ for } k \in \mathbb{N}, \end{aligned}$$

for all $i = 1, 2, \ldots, n$. Let A be a matrix with $a_{ij} = \alpha_n$ for $i + j - 1 = n$. Then A is a Hilbert-Schmidt operator. Since

$$\begin{aligned} y_1^{(i)} &= \alpha_1 x_1^{(i)} + \alpha_2 x_2^{(i)} \\ &\vdots \\ y_{2k}^{(i)} &= \alpha_{4k-1} x_{2k}^{(i)} \\ y_{2k+1}^{(i)} &= \alpha_{4k} x_{2k}^{(i)} + \alpha_{4k+1} x_{2k+1}^{(i)} + \alpha_{4k+2} x_{2k+2}^{(i)} \text{ for } k \in \mathbb{N}, \end{aligned}$$

for all $i = 1, 2, \ldots, n$, $Ax_i = y_i$. □

By the similar way with the above, we have the following.

Theorem 3. *Let AlgL be the tridiagonal algebra on a Hilbert space $\mathcal{H}$ and let $x_i = (x_j^{(i)})$ and $y_i = (y_j^{(i)})$ be vectors in $\mathcal{H}$ for $i = 1, 2, \ldots$. Then the following are equivalent:*

(1) *There exists a Hilbert-Schmidt operator $A = (a_{ij})$ in AlgL such that $Ax_i = y_i$ for all $i = 1, 2, \ldots$.*

(2) *There is a bounded sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ and*

$$\begin{aligned} y_1^{(i)} &= \alpha_1 x_1^{(i)} + \alpha_2 x_2^{(i)} \\ &\vdots \\ y_{2k}^{(i)} &= \alpha_{4k-1} x_{2k}^{(i)} \\ y_{2k+1}^{(i)} &= \alpha_{4k} x_{2k}^{(i)} + \alpha_{4k+1} x_{2k+1}^{(i)} + \alpha_{4k+2} x_{2k+2}^{(i)} \text{ for } k \in \mathbb{N}, \end{aligned}$$

for all $i = 1, 2, \ldots$.

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