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HILBERT-SCHMIDT INTERPOLATION ON $Ax = y$ IN A TRIDIAGONAL ALGEBRA ALG$\mathcal{L}$

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ABSTRACT. Given vectors x and y in a separable Hilbert space $\mathcal{H}$, an interpolating operator is a bounded operator A such that $Ax = y$. In this article, we investigate Hilbert-Schmidt interpolation problems for vectors in a tridiagonal algebra. We show the following: Let $\mathcal{L}$ be a subspace lattice acting on a separable complex Hilbert space $\mathcal{H}$ and let $x = (x_i)$ and $y = (y_i)$ be vectors in $\mathcal{H}$. Then the following are equivalent:

(1) There exists a Hilbert-Schmidt operator $A = (a_{ij})$ in Alg$\mathcal{L}$ such that $Ax = y$.
(2) There is a bounded sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ and

$$\begin{aligned} y_1 &= \alpha_1 x_1 + \alpha_2 x_2 \\ &\vdots \\ y_{2k} &= \alpha_{4k-1} x_{2k} \\ y_{2k+1} &= \alpha_{4k} x_{2k} + \alpha_{4k+1} x_{2k+1} + \alpha_{4k+1} x_{2k+2} \end{aligned}$$

for $k \in \mathbb{N}$.

1. INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A}$ be a subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of all operators acting on $\mathcal{H}$. Suppose that X and Y are specified, not necessarily in the algebra. Under what conditions can we expect there to be a solution of the operator equation $AX = Y$, where the operator A is required to lie in $\mathcal{A}$? We refer to such a question as an interpolation problem. The 'n-vector interpolation problem', asks for an operator A such that $Ax_i = y_i$ for fixed finite collections $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_n\}$. The n-vector interpolation problem was considered for a C^*-algebra $\mathcal{U}$ by Kadison [6]. In case $\mathcal{U}$ is a nest algebra, the (one-vector) interpolation problem was solved by Lance [7]: his result was extended by Hopenwasser [2] to the case that $\mathcal{U}$ is a CSL-algebra. Munch [8] obtained conditions for interpolation in case A is required to lie in the ideal of Hilbert-Schmidt operators in a nest algebra.

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Hopenwasser [3] once again extended the interpolation condition to the ideal of Hilbert-Schmidt operators in a CSL-algebra. Hopenwasser's paper also contains a sufficient condition for interpolation n-vectors, although necessity was not proved in that paper.

We establish some notations and conventions. A commutative subspace lattice $\mathcal{L}$, or CSL $\mathcal{L}$ is a strongly closed lattice of pairwise-commuting projections acting on a Hilbert space $\mathcal{H}$. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. If $\mathcal{L}$ is CSL, Alg$\mathcal{L}$ is called a CSL-algebra. The symbol Alg$\mathcal{L}$ is the algebra of all bounded operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$. Let x and y be two vectors in a Hilbert space $\mathcal{H}$. Then $\langle x, y\rangle$ means the inner product of the vectors x and y. Let M be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{M}$ means the closure of M and $\overline{M}^{\perp}$ the orthogonal complement of $\overline{M}$. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers.

2. RESULTS

Let $\mathcal{H}$ be a separable complex Hilbert space with a fixed orthonormal basis $\{e_1, e_2, \ldots\}$. Let $x_1, x_2, \ldots, x_n$ be vectors in $\mathcal{H}$. Then $[x_1, x_2, \ldots, x_n]$ means the closed subspace generated by the vectors $x_1, x_2, \ldots, x_n$. Let $\mathcal{L}$ be the subspace lattice generated by the subspaces $[e_{2k-1}], [e_{2k-1}, e_{2k}, e_{2k+1}]$ $(k = 1, 2, \ldots)$. Then the algebra Alg$\mathcal{L}$ is called a tridiagonal algebra which was introduced by Gilfeather & Larson [1]. These algebras have been found to be useful counterexample to a number of plausible conjectures.

Let $\mathcal{A}$ be the algebra consisting of all bounded operators acting on $\mathcal{H}$ of the form

$$\begin{pmatrix} * & * & & & \\ & * & & & \\ & * & * & * & \\ & & & * & \\ & & & * & \ddots \end{pmatrix}$$

with respect to the orthonormal basis $\{e_1, e_2, \ldots\}$, where all non-starred entries are zero. It is easy to see that Alg$\mathcal{L} = \mathcal{A}$.

We consider interpolation problems for the above tridiagonal algebra Alg$\mathcal{L}$.

Theorem 1. *Let Alg$\mathcal{L}$ be the tridiagonal algebra on a Hilbert space $\mathcal{H}$ and let $x = (x_i)$ and $y = (y_i)$ be vectors in $\mathcal{H}$. Then the following are equivalent:*

(1) *There exists a Hilbert-Schmidt operator $A = (a_{ij})$ in Alg$\mathcal{L}$ such that $Ax = y$.*
(2) *There is a bounded sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ and*

$$\begin{aligned} y_1 &= \alpha_1 x_1 + \alpha_2 x_2 \\ &\vdots \\ y_{2k} &= \alpha_{4k-1} x_{2k} \\ y_{2k+1} &= \alpha_{4k} x_{2k} + \alpha_{4k+1} x_{2k+1} + \alpha_{4k+2} x_{2k+2} \text{ for } k \in \mathbb{N}. \end{aligned}$$

Proof. Suppose that A is a Hilbert-Schmidt operator $A = (a_{ij})$ in Alg$\mathcal{L}$ such that $Ax = y$. Let $\alpha_n = a_{ij}$ for $n = i + j - 1$ and $\{e_n\}$ is the standard orthonormal basis for $\mathcal{H}$. Since A is Hilbert-Schmidt, $\sum_i \|Ae_i\|^2 < \infty$. Hence

$$\begin{aligned} \sum_i \|Ae_i\|^2 &= \sum_i \sum_j |\langle Ae_i, e_j \rangle|^2 \\ &= \sum_{k=1}^{\infty} \langle Ae_{2k-1}, e_{2k-1} \rangle + \sum_{k=1}^{\infty} \big\langle Ae_{2k}, (e_{2k-1} + e_{2k} + e_{2k+1}) \big\rangle \\ &= \sum_{k=1}^{\infty} |\alpha_{4k-3}|^2 + \sum_{k=1}^{\infty} (|\alpha_{4k-2}|^2 + |\alpha_{4k+1}|^2 + |\alpha_{4k}|^2) \\ &= \sum_{k=1}^{\infty} |\alpha_k|^2 < \infty. \end{aligned}$$

Since $Ax = y$,

$$\begin{aligned} y_1 &= \alpha_1 x_1 + \alpha_2 x_2 \\ &\vdots \\ y_{2k} &= \alpha_{4k-1} x_{2k} \\ y_{2k+1} &= \alpha_{4k} x_{2k} + \alpha_{4k+1} x_{2k+1} + \alpha_{4k+1} x_{2k+2}. \end{aligned}$$

Conversely, assume that there is a bounded sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that

$$\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$$

and

$$y_1 = \alpha_1 x_1 + \alpha_2 x_2$$
$$\vdots$$
$$y_{2k} = \alpha_{4k-1} x_{2k}$$
$$y_{2k+1} = \alpha_{4k} x_{2k} + \alpha_{4k+1} x_{2k+1} + \alpha_{4k+2} x_{2k+2}.$$

Let A be a matrix with $a_{ij} = \alpha_n$ for $i + j - 1 = n$. Then A is a Hilbert-Schmidt operator. Since

$$y_1 = \alpha_1 x_1 + \alpha_2 x_2$$
$$y_{2k} = \alpha_{4k-1} x_{2k}$$
$$\vdots$$
$$y_{2k+1} = \alpha_{4k} x_{2k} + \alpha_{4k+1} x_{2k+1} + \alpha_{4k+2} x_{2k+2},$$

$Ax = y$. □

Theorem 2. *Let Alg$\mathcal{L}$ be the tridiagonal algebra on a Hilbert space $\mathcal{H}$ and let $x_i = (x_j^{(i)})$ and $y_i = (y_j^{(i)})$ be vectors in $\mathcal{H}$ for $i = 1, 2, \dots, n$. Then the following are equivalent:*

(1) *There exists a Hilbert-Schmidt operator $A = (a_{ij})$ in Alg$\mathcal{L}$ such that $Ax_i = y_i$ for all $i = 1, 2, \dots, n$.*
(2) *There is a bounded sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ and*

$$y_1^{(i)} = \alpha_1 x_1^{(i)} + \alpha_2 x_2^{(i)}$$
$$\vdots$$
$$y_{2k}^{(i)} = \alpha_{4k-1} x_{2k}^{(i)}$$
$$y_{2k+1}^{(i)} = \alpha_{4k} x_{2k}^{(i)} + \alpha_{4k+1} x_{2k+1}^{(i)} + \alpha_{4k+2} x_{2k+2}^{(i)} \text{ for } k \in \mathbb{N},$$

for all $i = 1, 2, \dots, n$.

Proof. Suppose that A is a Hilbert-Schmidt operator $A = (a_{ij})$ in Alg$\mathcal{L}$ such that $Ax_i = y_i$ for all $i = 1, 2, \dots, n$. Let $\alpha_n = a_{ij}$ for $n = i + j - 1$ and $\{e_n\}$ is the standard orthonormal basis for $\mathcal{H}$. Since A is Hilbert-Schmidt, $\sum_i \|Ae_i\|^2 < \infty$.

Hence

$$\begin{aligned}\sum_i \|Ae_i\|^2 &= \sum_i \sum_j |\langle Ae_i, e_j\rangle|^2 \\ &= \sum_{k=1}^{\infty} \langle Ae_{2k-1}, e_{2k-1}\rangle + \sum_{k=1}^{\infty} \big\langle Ae_{2k}, (e_{2k-1} + e_{2k} + e_{2k+1})\big\rangle \\ &= \sum_{k=1}^{\infty} |\alpha_{4k-3}|^2 + \sum_{k=1}^{\infty} (|\alpha_{4k-2}|^2 + |\alpha_{4k+1}|^2 + |\alpha_{4k}|^2) \\ &= \sum_{k=1}^{\infty} |\alpha_k|^2 < \infty.\end{aligned}$$

So $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$. Since $Ax_i = y_i$ for all $i = 1, 2, \dots, n$,

$$\begin{aligned}y_1^{(i)} &= \alpha_1 x_1^{(i)} + \alpha_2 x_2^{(i)} \\ &\vdots \\ y_{2k}^{(i)} &= \alpha_{4k-1} x_{2k}^{(i)} \\ y_{2k+1}^{(i)} &= \alpha_{4k} x_{2k}^{(i)} + \alpha_{4k+1} x_{2k+1}^{(i)} + \alpha_{4k+2} x_{2k+2}^{(i)} \text{ for } k \in \mathbb{N},\end{aligned}$$

for all $i = 1, 2, \dots, n$.

Conversely, assume that there is a bounded sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that

$$\sum_{k=1}^{\infty} |\alpha_k|^2 < \infty$$

and

$$\begin{aligned}y_1^{(i)} &= \alpha_1 x_1^{(i)} + \alpha_2 x_2^{(i)} \\ &\vdots \\ y_{2k}^{(i)} &= \alpha_{4k-1} x_{2k}^{(i)} \\ y_{2k+1}^{(i)} &= \alpha_{4k} x_{2k}^{(i)} + \alpha_{4k+1} x_{2k+1}^{(i)} + \alpha_{4k+2} x_{2k+2}^{(i)} \text{ for } k \in \mathbb{N},\end{aligned}$$

for all $i = 1, 2, \dots, n$. Let A be a matrix with $a_{ij} = \alpha_n$ for $i + j - 1 = n$. Then A is a Hilbert-Schmidt operator. Since

$$\begin{aligned}y_1^{(i)} &= \alpha_1 x_1^{(i)} + \alpha_2 x_2^{(i)} \\ &\vdots \\ y_{2k}^{(i)} &= \alpha_{4k-1} x_{2k}^{(i)} \\ y_{2k+1}^{(i)} &= \alpha_{4k} x_{2k}^{(i)} + \alpha_{4k+1} x_{2k+1}^{(i)} + \alpha_{4k+2} x_{2k+2}^{(i)} \text{ for } k \in \mathbb{N},\end{aligned}$$

for all $i = 1, 2, \ldots, n$, $Ax_i = y_i$. □

By the similar way with the above, we have the following.

Theorem 3. *Let AlgL be the tridiagonal algebra on a Hilbert space* $\mathcal{H}$ *and let* $x_i = (x_j^{(i)})$ *and* $y_i = (y_j^{(i)})$ *be vectors in* $\mathcal{H}$ *for* $i = 1, 2, \ldots$. *Then the following are equivalent:*

(1) *There exists a Hilbert-Schmidt operator* $A = (a_{ij})$ *in AlgL such that* $Ax_i = y_i$ *for all* $i = 1, 2, \ldots$.

(2) *There is a bounded sequence* $\{\alpha_n\}$ *in* $\mathbb{C}$ *such that* $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ *and*

$$\begin{aligned} y_1^{(i)} &= \alpha_1 x_1^{(i)} + \alpha_2 x_2^{(i)} \\ &\vdots \\ y_{2k}^{(i)} &= \alpha_{4k-1} x_{2k}^{(i)} \\ y_{2k+1}^{(i)} &= \alpha_{4k} x_{2k}^{(i)} + \alpha_{4k+1} x_{2k+1}^{(i)} + \alpha_{4k+2} x_{2k+2}^{(i)} \text{ for } k \in \mathbb{N}, \end{aligned}$$

for all $i = 1, 2, \ldots$.

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