HILBERT-SCHMIDT INTERPOLATION ON $Ax = y$ IN A TRIDIAGONAL ALGEBRA $\text{Alg} \mathcal{L}$

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ABSTRACT. Given vectors $x$ and $y$ in a separable Hilbert space $\mathcal{H}$, an interpolating operator is a bounded operator $A$ such that $Ax = y$. In this article, we investigate Hilbert-Schmidt interpolation problems for vectors in a tridiagonal algebra. We show the following: Let $\mathcal{L}$ be a subspace lattice acting on a separable complex Hilbert space $\mathcal{H}$ and let $x = (x_i)$ and $y = (y_i)$ be vectors in $\mathcal{H}$. Then the following are equivalent:

1. There exists a Hilbert-Schmidt operator $A = (a_{ij})$ in $\text{Alg} \mathcal{L}$ such that $Ax = y$.
2. There is a bounded sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ and

$$y_1 = \alpha_1 x_1 + \alpha_2 x_2$$

$$\vdots$$

$$y_{2k} = \alpha_{4k-1} x_{2k}$$

$$y_{2k+1} = \alpha_{4k} x_{2k} + \alpha_{4k+1} x_{2k+1} + \alpha_{4k+2} x_{2k+2}$$

for $k \in \mathbb{N}$.

1. INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A}$ be a subalgebra of the algebra $B(\mathcal{H})$ of all operators acting on $\mathcal{H}$. Suppose that $X$ and $Y$ are specified, not necessarily in the algebra. Under what conditions can we expect there to be a solution of the operator equation $AX = Y$, where the operator $A$ is required to lie in $\mathcal{A}$? We refer to such a question as an interpolation problem. The 'n-vector interpolation problem', asks for an operator $A$ such that $Ax_i = y_i$ for fixed finite collections $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$. The n-vector interpolation problem was considered for a $C^*$-algebra $\mathcal{U}$ by Kadison [6]. In case $\mathcal{U}$ is a nest algebra, the (one-vector) interpolation problem was solved by Lance [7]: his result was extended by Hopenwasser [2] to the case that $\mathcal{U}$ is a CSL-algebra. Munch [8] obtained conditions for interpolation in case $A$ is required to lie in the ideal of Hilbert-Schmidt operators in a nest algebra.

Received by the editors January 1, 2004.

2000 Mathematics Subject Classification. 47L35.

Key words and phrases. Hilbert-Schmidt Interpolation, CSL-Algebra, Tridiagonal Algebra, $\text{Alg} \mathcal{L}$.
Hopenwasser [3] once again extended the interpolation condition to the ideal of Hilbert-Schmidt operators in a CSL-algebra. Hopenwasser's paper also contains a sufficient condition for interpolation $n$-vectors, although necessity was not proved in that paper.

We establish some notations and conventions. A commutative subspace lattice $\mathcal{L}$, or CSL $\mathcal{L}$ is a strongly closed lattice of pairwise-commuting projections acting on a Hilbert space $\mathcal{H}$. We assume that the projections 0 and 1 lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. If $\mathcal{L}$ is CSL, $\text{Alg}\mathcal{L}$ is called a CSL-algebra. The symbol $\text{Alg}\mathcal{L}$ is the algebra of all bounded operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$. Let $x$ and $y$ be two vectors in a Hilbert space $\mathcal{H}$. Then $\langle x, y \rangle$ means the inner product of the vectors $x$ and $y$. Let $M$ be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{M}$ means the closure of $M$ and $\overline{M}^\perp$ the orthogonal complement of $\overline{M}$. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers.

2. Results

Let $\mathcal{H}$ be a separable complex Hilbert space with a fixed orthonormal basis $\{e_1, e_2, \ldots \}$. Let $x_1, x_2, \ldots, x_n$ be vectors in $\mathcal{H}$. Then $[x_1, x_2, \ldots, x_n]$ means the closed subspace generated by the vectors $x_1, x_2, \ldots, x_n$. Let $\mathcal{L}$ be the subspace lattice generated by the subspaces $[e_{2k-1}, e_{2k}, e_{2k+1}]$ ($k = 1, 2, \ldots$). Then the algebra $\text{Alg}\mathcal{L}$ is called a tridiagonal algebra which was introduced by Gilfeather & Larson [1]. These algebras have been found to be useful counterexample to a number of plausible conjectures.

Let $\mathcal{A}$ be the algebra consisting of all bounded operators acting on $\mathcal{H}$ of the form

$$
\begin{pmatrix}
* & * \\
* & * & * \\
* & * \\
& * & \ddots
\end{pmatrix}
$$

with respect to the orthonormal basis $\{e_1, e_2, \ldots \}$, where all non-starred entries are zero. It is easy to see that $\text{Alg}\mathcal{L} = \mathcal{A}$.

We consider interpolation problems for the above tridiagonal algebra $\text{Alg}\mathcal{L}$.
Theorem 1. Let AlgL be the tridiagonal algebra on a Hilbert space $\mathcal{H}$ and let $x = (x_i)$ and $y = (y_i)$ be vectors in $\mathcal{H}$. Then the following are equivalent:

1. There exists a Hilbert-Schmidt operator $A = (a_{ij})$ in AlgL such that $Ax = y$.
2. There is a bounded sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ and

\[
\begin{align*}
  y_1 &= \alpha_1 x_1 + \alpha_2 x_2 \\
  \vdots & \nonumber \\
  y_{2k} &= \alpha_{4k-1} x_{2k} \\
  y_{2k+1} &= \alpha_{4k} x_{2k} + \alpha_{4k+1} x_{2k+1} + \alpha_{4k+2} x_{2k+2} \text{ for } k \in \mathbb{N}.
\end{align*}
\]

Proof. Suppose that $A$ is a Hilbert-Schmidt operator $A = (a_{ij})$ in AlgL such that $Ax = y$. Let $\alpha_n = a_{ij}$ for $n = i + j - 1$ and $\{e_n\}$ is the standard orthonormal basis for $\mathcal{H}$. Since $A$ is Hilbert-Schmidt, $\sum_i \|Ae_i\|^2 < \infty$. Hence

\[
\sum_i \|Ae_i\|^2 = \sum_i \sum_j |\langle Ae_i, e_j \rangle|^2 = \sum_{k=1}^{\infty} \langle Ae_{2k-1}, e_{2k-1} \rangle + \sum_{k=1}^{\infty} \langle Ae_{2k}, (e_{2k-1} + e_{2k} + e_{2k+1}) \rangle
\]

\[
= \sum_{k=1}^{\infty} |\alpha_{4k-3}|^2 + \sum_{k=1}^{\infty} (|\alpha_{4k-2}|^2 + |\alpha_{4k+1}|^2 + |\alpha_{4k}|^2)
\]

\[
= \sum_{k=1}^{\infty} |\alpha_k|^2 < \infty.
\]

Since $Ax = y$,

\[
\begin{align*}
  y_1 &= \alpha_1 x_1 + \alpha_2 x_2 \\
  \vdots & \nonumber \\
  y_{2k} &= \alpha_{4k-1} x_{2k} \\
  y_{2k+1} &= \alpha_{4k} x_{2k} + \alpha_{4k+1} x_{2k+1} + \alpha_{4k+2} x_{2k+2}.
\end{align*}
\]

Conversely, assume that there is a bounded sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that

\[
\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty
\]
and

\[ y_1 = \alpha_1 x_1 + \alpha_2 x_2 \]
\[ \vdots \]
\[ y_{2k} = \alpha_{4k-1} x_{2k} \]
\[ y_{2k+1} = \alpha_{4k} x_{2k} + \alpha_{4k+1} x_{2k+1} + \alpha_{4k+2} x_{2k+2} \]

Let \( A \) be a matrix with \( a_{ij} = \alpha_n \) for \( i + j - 1 = n \). Then \( A \) is a Hilbert-Schmidt operator. Since

\[ y_1 = \alpha_1 x_1 + \alpha_2 x_2 \]
\[ y_{2k} = \alpha_{4k-1} x_{2k} \]
\[ \vdots \]
\[ y_{2k+1} = \alpha_{4k} x_{2k} + \alpha_{4k+1} x_{2k+1} + \alpha_{4k+2} x_{2k+2} \]

\[ Ax = y. \]

\[ \square \]

**Theorem 2.** Let Alg\( \mathcal{C} \) be the tridiagonal algebra on a Hilbert space \( \mathcal{H} \) and let \( x_i = (x_j^{(i)}) \) and \( y_i = (y_j^{(i)}) \) be vectors in \( \mathcal{H} \) for \( i = 1, 2, \ldots, n \). Then the following are equivalent:

1. There exists a Hilbert-Schmidt operator \( A = (a_{ij}) \) in Alg\( \mathcal{C} \) such that \( Ax_i = y_i \) for all \( i = 1, 2, \ldots, n \).
2. There is a bounded sequence \( \{\alpha_n\} \) in \( \mathbb{C} \) such that \( \sum_{n=1}^{\infty} |\alpha_n|^2 < \infty \) and

\[ y_1^{(i)} = \alpha_1 x_1^{(i)} + \alpha_2 x_2^{(i)} \]
\[ \vdots \]
\[ y_{2k}^{(i)} = \alpha_{4k-1} x_{2k}^{(i)} \]
\[ y_{2k+1}^{(i)} = \alpha_{4k} x_{2k}^{(i)} + \alpha_{4k+1} x_{2k+1}^{(i)} + \alpha_{4k+2} x_{2k+2}^{(i)} \]

for all \( i = 1, 2, \ldots, n \).

**Proof.** Suppose that \( A \) is a Hilbert-Schmidt operator \( A = (a_{ij}) \) in Alg\( \mathcal{C} \) such that \( Ax_i = y_i \) for all \( i = 1, 2, \ldots, n \). Let \( \alpha_n = a_{ij} \) for \( n = i + j - 1 \) and \( \{e_n\} \) is the standard orthonormal basis for \( \mathcal{H} \). Since \( A \) is Hilbert-Schmidt, \( \sum_i \|Ae_i\|^2 < \infty \).
Hence
\[ \sum_i \|Ae_i\|^2 = \sum_i \sum_j |\langle Ae_i, e_j \rangle|^2 \]
\[ = \sum_{k=1}^{\infty} \langle Ae_{2k-1}, e_{2k-1} \rangle + \sum_{k=1}^{\infty} \langle Ae_{2k}, (e_{2k-1} + e_{2k} + e_{2k+1}) \rangle \]
\[ = \sum_{k=1}^{\infty} |\alpha_{4k-3}|^2 + \sum_{k=1}^{\infty} (|\alpha_{4k-2}|^2 + |\alpha_{4k+1}|^2 + |\alpha_{4k}|^2) \]
\[ = \sum_{k=1}^{\infty} |\alpha_k|^2 < \infty. \]

So \( \sum_{n=1}^{\infty} |\alpha_n|^2 < \infty \). Since \( Ax_i = y_i \) for all \( i = 1, 2, \ldots, n \),
\[ y_1^{(i)} = \alpha_1 x_1^{(i)} + \alpha_2 x_2^{(i)} \]
\[ \vdots \]
\[ y_{2k}^{(i)} = \alpha_{4k-1} x_{2k}^{(i)} \]
\[ y_{2k+1}^{(i)} = \alpha_{4k} x_{2k+1}^{(i)} + \alpha_{4k+1} x_{2k+1}^{(i)} + \alpha_{4k+2} x_{2k+2}^{(i)} \]
for \( k \in \mathbb{N} \), for all \( i = 1, 2, \ldots, n \).

Conversely, assume that there is a bounded sequence \( \{\alpha_n\} \) in \( \mathbb{C} \) such that
\[ \sum_{k=1}^{\infty} |\alpha_k|^2 < \infty \]
and
\[ y_1^{(i)} = \alpha_1 x_1^{(i)} + \alpha_2 x_2^{(i)} \]
\[ \vdots \]
\[ y_{2k}^{(i)} = \alpha_{4k-1} x_{2k}^{(i)} \]
\[ y_{2k+1}^{(i)} = \alpha_{4k} x_{2k+1}^{(i)} + \alpha_{4k+1} x_{2k+1}^{(i)} + \alpha_{4k+2} x_{2k+2}^{(i)} \]
for \( k \in \mathbb{N} \), for all \( i = 1, 2, \ldots, n \). Let \( A \) be a matrix with \( a_{ij} = \alpha_n \) for \( i + j - 1 = n \). Then \( A \) is a Hilbert-Schmidt operator. Since
\[ y_1^{(i)} = \alpha_1 x_1^{(i)} + \alpha_2 x_2^{(i)} \]
\[ \vdots \]
\[ y_{2k}^{(i)} = \alpha_{4k-1} x_{2k}^{(i)} \]
\[ y_{2k+1}^{(i)} = \alpha_{4k} x_{2k+1}^{(i)} + \alpha_{4k+1} x_{2k+1}^{(i)} + \alpha_{4k+2} x_{2k+2}^{(i)} \]
for \( k \in \mathbb{N} \),
for all $i = 1, 2, \ldots, n$, $Ax_i = y_i$. \hfill \square

By the similar way with the above, we have the following.

**Theorem 3.** Let $\text{Alg} \mathcal{L}$ be the tridiagonal algebra on a Hilbert space $\mathcal{H}$ and let $x_i = (x_j^{(i)})$ and $y_i = (y_j^{(i)})$ be vectors in $\mathcal{H}$ for $i = 1, 2, \ldots$. Then the following are equivalent:

1. There exists a Hilbert-Schmidt operator $A = (a_{ij})$ in $\text{Alg} \mathcal{L}$ such that $Ax_i = y_i$ for all $i = 1, 2, \ldots$.
2. There is a bounded sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ and
   
   $y_1^{(i)} = \alpha_1 x_1^{(i)} + \alpha_2 x_2^{(i)}$

   $\vdots$

   $y_{2k}^{(i)} = \alpha_{4k-1} x_{2k}^{(i)}$

   $y_{2k+1}^{(i)} = \alpha_{4k} x_{2k}^{(i)} + \alpha_{4k+1} x_{2k+1}^{(i)} + \alpha_{4k+2} x_{2k+2}^{(i)}$ for $k \in \mathbb{N}$,

   for all $i = 1, 2, \ldots$.

**References**

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