디스크립터 시스템을 위한 혼합 H_2/H_∞ 제어기의 설계

論 文
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Mixed H_2/H_∞ Controller Design for Descriptor Systems

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Abstract - The descriptor system model has a high ability in representing dynamical systems. It can preserve physical parameters in the coefficient matrices, and describe the dynamic part, static part, and even the improper part of the system in the same form. The design of mixed H_2/H_∞ controllers for linear time-invariant descriptor systems is considered in this paper. Firstly, an H_2 and H_∞ synthesis problems for a descriptor system are presented separately in terms of linear matrix inequalities (LMIs) based on the bounded real lemma. Then, we show that the existence of a mixed H_2/H_∞ controller by which the H_2 norm of the second channel is minimized while keeping the H_∞ norm bound of the first channel less than γ is reduced to the linear objective minimization problem. The class of desired controllers that are assumed to have the same structure as the plant is parameterized by using the linearizing change of variables.

Key Words : Descriptor system, Robust control, mixed H_2/H_∞

1. Introduction

The descriptor system model has a high ability in representing dynamical systems. It can preserve physical parameters in the coefficient matrices, and describe the dynamic part, static part, and even the improper part of the system in the same form. Models of chemical processes, for example, typically consist of differential equations describing the dynamic balance of mass and energy (that describes static constraints on physical variables) while additional algebraic equations account for thermodynamic equilibrium relations, steady-state assumptions, empirical correlations, etc. (these are static and impulsive parts of the physical system). In other words, the descriptor model is much superior to the state-space one.

To design a controller for the descriptor system by using the general methods for the non-descriptor, requires modification of a descriptor system to a general state-space representation. That modification, however, necessarily causes a loss of information in the original descriptor system. In recent years, due to this fact, much work has been focused on analysis and design techniques for descriptor systems (see [1]~[3]). For linear systems many of the standard design techniques for state-space systems have been extended to descriptor systems. H_∞ controller design for descriptor systems was established by using a generalization of J-spectral factorization [4]. The approach, however, in [4] is restricted to the systems that satisfy the so-called DGKF assumptions [5]. These restrictions were overcome in [2] based on the linear matrix inequality (LMI) approach. However, the LMI-type conditions proposed in [2] contain equality constraints that may be no problem theoretically, but may cause a big trouble in checking conditions numerically. Recently, strict LMI conditions for H_∞ control of linear time-invariant descriptor systems have been proposed in [6] and [7], in which it was shown that the existence of an H_∞ controller that achieves a specified disturbance attenuation level, was reduced to the feasibility of a set of strict LMIs.

In this paper, we consider the mixed H_2/H_∞ control problem for descriptor systems. That is, the goal of this problem is to minimize bounds on the H_2-norm of the second channel ($T_2 : w_2 \rightarrow z_2$), while keeping the H_∞ norm bound of the first channel less than γ ($T_1 : w_1 \rightarrow z_1$), i.e.,

$$\min \| T_2 \|_2 \ \text{subject to} \ \| T_1 \|_\infty < \gamma$$

Generally, the H_2-norm constraint is imposed to specify desired performance requirements on the closed-loop system such as reducing the asymptotic output-variance against white noise input or

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the output energy against impulse inputs. The H_∞-norm constraint on $z_1 = T_1 w_1$ guarantees that the controller is robustly stabilizing against uncertainties $w_1 = \Delta_1 z_1$ with $\Delta_1 \in RH^\infty$ fulfilling the norm bound $\|\Delta_1\|_\infty \cdot \|T_1\|_\infty < 1$.

It is well known that the mixed H_2/H_∞ control problem has already been solved for steady-state model [8]. However, to the author's knowledge, the mixed H_2/H_∞ control problem for descriptor systems has not been well studied. Although this problem was dealt in [9], their LMI results contains equality constraints. The purpose of this paper is to present strict LMI conditions for the mixed H_2/H_∞ controller synthesis problem for descriptor systems on the assumption that the descriptor system has only *single* exogenous signal, that is, $w = w_1 = w_2$. The paper is organized as follows. Section 2 gives the necessary background and some results related to the stability of descriptor systems. In section 3, we propose an LMI formulation for the H_∞ and H_2 synthesis problem by means of an adaptation of the "linearizing change of variables"[10]. These results, in the sequel, show LMI algorithms of the mixed H_2/H_∞ control problem for descriptor systems. In section 4, we confirm the validity of the proposed method through two numerical examples.

2. Background

2.1 Linear Descriptor Systems

Let us consider a linear time-invariant descriptor system

$$E\dot{x} = Ax + Bw, z = Cx + Dw \tag{1}$$

where $x \in R^n$ is the descriptor variables, $w \in R^q$ is input, $z \in R^p$ is output, and , $E, A \in R^{n\times n}$, $B \in R^{n\times q}$, $C \in R^{p\times n}$, $D \in r^{p\times q}$ are constant matrices. The matrix E may be singular and we denote its rank by $r = rank E \leq n$. As a shorthand notation for system (1) we often write (E, A, B, C, D) (or (E, A, B, C) if $D = 0$).

The system (1) has a unique solution for any initial conditions and constraints input $w(\cdot)$ if $\det(sE - A) \neq 0$. In this case, (1) is said to be regular. On the other hand, a non-regular system always admits multiple solutions for the unforced ($w = 0$) homogenous initial value problem. For a regular system (1), the transfer matrix

$$G(s) := C(sE - A)^{-1}B + D \tag{2}$$

can be defined. The question of impulsive solutions of regular systems is usually studied in terms of the Weierstrass canonical form (WCF) of (E, A, B, C, D).

Theorem 1 [11]: Let (E, A, B, C, D) be regular. Then there exists an equivalent system $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \sim (E, A, B, C, D)$ with

$$\tilde{E} = \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix}, \tilde{A} = \begin{bmatrix} \Lambda & 0 \\ 0 & I_{n-r} \end{bmatrix}, \tilde{B} = \begin{bmatrix} B_s \\ B_f \end{bmatrix}, \quad \tilde{C} = [\ C_s \ \ C_f\ ], \tilde{D} = D \tag{3}$$

where matrix $\Lambda \in R^{r\times r}$ is of Jordan canonical form, and $N \in R^{(n-r)\times(n-r)}$ is nilpotent. □

Definition 1: The index ν of nilpotent N, i.e. $\nu := \min\{q \mid N^q = 0, q \in R\}$ is said to be the index of the linear descriptor system. Systems with $\nu \geq 2$ are called high index descriptor systems. □

If (1) is in WCF, i.e.

$$\begin{bmatrix} \dot{x}_s \\ N\dot{x}_f \end{bmatrix} = \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} x_s \\ x_f \end{bmatrix} + \begin{bmatrix} B_s \\ B_f \end{bmatrix} w \tag{4}$$

then the part x_s (slow mode) of the descriptor vector $x^T = [\ x_s^T \ \ x_f^T\ ]$ is governed by an ordinary differential equation while

$$x_f = -\sum_{i=0}^{\nu-1} \delta^{(i)}(t) N^{i+1} x_f(0-) - \sum_{i=0}^{\nu-1} N^i B_f w^{(i)} \tag{5}$$

represents impulsive (fast) mode of the system (1) (with $\delta(t)$ the Dirac delta and the superscript *(i)* means the *i*th derivative). We conclude that descriptor systems will have no impulsive solutions (for all $w(\cdot) \in L_2[0, \infty)$ and all the initial conditions) if and only if their index is one.

2.2 Stability Conditions

In this subsection, we describe the strict LMI conditions for determining the stability of the descriptor system (1) by using the results of [7].

Definition 2: The system (1) is said to be stable if it is regular and the finite eigenvalues of the matrix pair (E, A), that is, the solutions of $\det(sE - A) = 0$ lie on the left half of s-plane. □

To present a strict LMI condition, we introduce matrices $V, U \in R^{n\times(n-r)}$ which are of full column rank and composed of bases of $\mathrm{Ker}\,E$ and $\mathrm{Ker}\,E^T$, respectively.

Theorem 2 [7]: The system (1) is stable if and only if there exists a positive definite matrix $P \in R^{n\times n}$ and a matrix $S \in R^{(n-r)\times(n-r)}$ that satisfy the LMI

$$A(PE^T + VSU^T) + (PE^T + VSU^T)^T A^T < 0 \tag{6}$$

Proof: Refer to [7]. □

Since the LMI condition (6) has no equality constraints, it is highly tractable and reliable when we use recent software such as [12] for solving matrix inequalities.

3. Mixed H_2/H_∞ control problem for linear descriptor systems

We consider a generalized plant G that is a descriptor system

$$G:\begin{cases} E\dot{x} = Ax + B_1 w + B_2 u \\ z_\infty = C_1 x \\ z_2 = C_2 x \\ y = C_3 x \end{cases} \tag{7}$$

where $x \in R^n$ denotes the descriptor variables, $u \in R^{m_2}$ the control input, $w \in R^{m_1}$ the disturbance input, $z_\infty \in R^{p_1}$ the external output related to H_∞ control, $z_2 \in R^{p_2}$ the external output related to H_2 control, and $y \in R^{p_3}$ the measured output. A, B_i and C_i are constant matrices of appropriate dimensions and E is a possibly singular matrix having the same dimension as A. Notice that in the descriptor setup there is no loss of generality in not considering a direct fed-through from control/disturbance input to external/ measured output since the corresponding terms can be captured by additional descriptor variables ([3], refer example 1 of section 4).

The focus of the mixed H_2/H_∞ control problem is as follows: We want to find a dynamic output feedback controller K_E (with $\zeta \in R^{n_K}$) in descriptor form

$$K_E:\begin{cases} E_K\dot{\zeta} = A_K\zeta + B_K y \\ u = C_K\zeta + D_K y \end{cases} \tag{8}$$

such that

1. The closed-loop is regular and stable index one system. A system with these properties is said to be admissible [2].
2. The H_2 norm of the closed-loop transfer matrix $T_{cl2} : w \rightarrow z_2$ is minimized while keeping the H_∞ norm of the closed-loop transfer matrix $T_{cl\infty} : w \rightarrow z_\infty$ less than γ.

In this paper, we assume two conditions related to the controller, that is, $\mathrm{rank} E_K = r$ and $n_K = n$. Under this circumstance, it is always possible to make $E_K = E$ without lose of generality by using proper similarity transformation [7].

3.1 H_∞ control problem for descriptor systems

3.1.1 Problem Setup

The plant from (7)

$$G_\infty:\begin{cases} E\dot{x} = Ax + B_1 w + B_2 u \\ z_\infty = C_1 x \\ y = C_3 x \end{cases} \tag{9}$$

together with the controller (8) forms a closed-loop system as follows:

$$\begin{aligned} E_{cl}\dot{x}_c &= A_{cl}x_c + B_{cl}w \\ z_\infty &= C_{cl\infty} x_c,\ x_c^T = [\ x^T \ \ \zeta^T\ ] \end{aligned} \tag{10}$$

where

$$\begin{aligned} E_{cl} &= \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}, A_{cl} = \begin{bmatrix} A + B_2 D_K C_3 & B_2 C_K \\ B_K C_3 & A_K \end{bmatrix} \\ B_{cl} &= \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, C_{cl\infty} = [\ C_1 \ \ 0\ ] \end{aligned} \tag{11}$$

The following theorem is based on a LMI characterization of admissibility and H_∞ norm bound of the closed-loop system.

Theorem 3 [2](Generalized Bounded Real Lemma) A system $(E_{cl}, A_{cl}, B_{cl}, C_{cl\infty})$ is admissible and

$$\| G_{cl\infty} \|_\infty < \gamma, G_{cl\infty} := C_{cl\infty}(sE_{cl} - A_{cl})^{-1}B_{cl} \tag{12}$$

if and only if there exists matrix $\tilde{P}$ which satisfies

$$E_{cl}^T\tilde{P} = \tilde{P}^T E_{cl} \geq 0 \tag{13}$$

$$\begin{bmatrix} A_{cl}^T\tilde{P} + \tilde{P}^T A_{cl} & \tilde{P}^T B_{cl} & C_{cl\infty}^T \\ B_{cl}^T\tilde{P} & -\gamma I & 0 \\ C_{cl\infty} & 0 & -\gamma I \end{bmatrix} < 0 \tag{14}$$

Proof. This is a well-known result of LMI theory for descriptor systems. □

Theorem 3 represents a convenient tool for checking an H_∞ norm bound of a descriptor system since it only requires the computation of the solution of the LMIs (13), (14), i.e. the solution of a feasibility problem.

3.1.2 LMI Synthesis

In view of theorem 3, the existence of matrices A_K, B_K, C_K, D_K, and $\tilde{P}$ such that matrix inequalities (13), (14) hold true is sufficient for the H_∞ control problem. However (14) is nonlinear in these matrix variables and therefore difficult to solve. The idea in the following is to provide a linearizing change of variables along with the lines of [10] in order to end up with linear matrix inequalities instead of (13) and (14).

A possible solution $\tilde{P}$ of (14) is necessarily nonsingular, so we partition $\tilde{P}$ and $\tilde{P}^{-1}$ as

$$\tilde{P} = \begin{bmatrix} (QE + URV^T) & N \\ N^T & * \end{bmatrix} \tag{15-a}$$

$$\tilde{P}^{-1} = \begin{bmatrix} (PE^T + VSU^T) & M \\ M^T & * \end{bmatrix} \tag{15-b}$$

where $P, Q \in R^{n \times n}$ are symmetrical matrices, $S, R \in R^{(n-r)\times(n-r)}$ are nonsingular matrices, and

$U, V \in R^{n\times(n-r)}$ were already defined in theorem 2. And N, M are nonsingular matrices with proper dimensions. For the simplicity of expression, we temporarily adopt matrices X, Y such as

$$X = (PE^T + VSU^T),\ Y = (QE + URV^T) \tag{16}$$

Then from $\tilde{P}\tilde{P}^{-1} = I$, we obtain

$$\tilde{P}\Pi_1 = \Pi_2 \tag{17-a}$$

with

$$\Pi_1 = \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}, \Pi_2 = \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix} \tag{17-b}$$

Since Π_1 is nonsingular, a nonsingular congruence transformation of (13), (14) is possible. That is,

$$\Pi_1^T E_d^T \tilde{P} \Pi_1 = \Pi_1^T \tilde{P}^T E_d \Pi_1 \geq 0 \tag{18}$$

$$\Psi_{\Pi_1}^T \begin{bmatrix} A_d^T\tilde{P} + \tilde{P}^T A_d & \tilde{P}^T B_d & C_{cloo}^T \\ B_d^T \tilde{P} & -\gamma I & 0 \\ C_{cloc} & 0 & -\gamma I \end{bmatrix} \Psi_{\Pi_1} < 0$$
$$\Psi_{\Pi_1} := \mathrm{diag}(\ \Pi_1\ \ I\ I\ ) \tag{19}$$

Such a congruence transformation has been suggested in [10] in order to reveal the affine structure of underlying matrix inequalities. In order to carry out (18) and (19), we first define the change of variables as follows:

$$\begin{cases} \hat{D} = D_K \\ \hat{C} = C_K M^T + D_K C_3 X \\ \hat{B} = NB_K + YB_2 D_K \\ \hat{A} = NA_K M^T + NB_K C_3 X \quad + Y^T B_2 C_K M^T \end{cases}$$
$$+ Y^T (A + B_2 D_K C_3) X$$

(20)

Note that the new variables $\hat{A}, \hat{B}, \hat{C}$ have dimensions $n\times n, n\times m_2, p_3 \times n$, respectively. Then the direct calculation of inequalities (18) and (19) leads to (21) and (22).

$$\begin{bmatrix} EPE^T & E \\ E^T & E^T QE \end{bmatrix} \geq 0 \tag{21}$$

$$\begin{bmatrix} AX + X^T A^T + B_2\hat{C} + (B_2\hat{C})^T & \hat{A}^T + (A + B_2\hat{D}C_3) & * & * \\ \hat{A} + (A + B_2\hat{D}C_3)^T & A^T Y + Y^T A + \hat{B}C_3 + (\hat{B}C_3)^T & * & * \\ B_1^T & (Y^T B_1)^T & -\gamma I & * \\ C_1 X & C_1 & 0 & -\gamma I \end{bmatrix} < 0$$

(22)

By introducing full column matrices $E_L, E_R \in R^{n\times(n-r)}$ that satisfy $E = E_L E_R^T$, we can further simplify the inequality (21). That is, (21) can be rewritten as

$$\begin{bmatrix} E_L & 0 \\ 0 & E_R \end{bmatrix}\begin{bmatrix} E_R^T P E_R & I_r \\ I_r & E_L^T Q E_L \end{bmatrix}\begin{bmatrix} E_L^T & 0 \\ 0 & E_R^T \end{bmatrix} \geq 0 \tag{23}$$

We have $\mathrm{rank}(\tilde{E}_d^T \tilde{P}) = 2r$, and $E_R^T P E_R > 0, E_L^T Q E_L > 0$ can be assumed without loss of generality. Since the rank of a matrix is invariant under nonsingular congruence transformation, (21) can be written equivalently as

$$\begin{bmatrix} E_R^T P E_R & I_r \\ I_r & E_L^T Q E_L \end{bmatrix} > 0 \tag{24}$$

i.e., as strict inequality. The inequalities (22), (24) are LMIs in the matrix variables $\{P, Q, S, R, \hat{A}, \hat{B}, \hat{C}, \hat{D}\}$, and they constitute sufficient conditions for the existence of a controller by which the H_∞ norm condition is satisfied.

Next, in order to retrieve controller matrices A_K, B_K, C_K, D_K from the results of LMIs (22), (24) through (20), it is required to compute nonsingular matrices M, N. The following theorem is needed.

Theorem 4: From the previous development we can assume without loss of generality that S, R are nonsingular, and P, Q are symmetrical and satisfy $E_R^T P E_R > 0, E_L^T Q E_L > 0$. Then $X = (PE^T + VSU^T), Y = (QE + URV^T)$ are nonsingular.

Proof: Since $[E_L\ U], [E_R\ V]$ are nonsingular, we multiply $[E_R\ V]^T$ in the left side and $[E_L\ U]$ in the right side of X (or Y). Then we can obtain

$$\det X = \det[E_R\ V]^T X [E_L\ U]$$
$$= \det\begin{bmatrix} (E_R^T P E_R) E_L^T E_L & 0 \\ * & V^T V S U^T U \end{bmatrix} \neq 0 \qquad \square$$

Now we can determine the coefficient matrices of the controller (8).

Theorem 5: Consider a plant (9) and a controller (8). There exists a full order controller (that is $n_K = n$, and $\mathrm{rank} E_K = r$) such that the closed-loop system (10) is admissible and H_∞ norm is bounded by γ if and only if LMIs (22) and (24) admit solution $\{P, Q, S, R, \hat{A}, \hat{B}, \hat{C}, \hat{D}\}$. A controller (8) solving the H_∞ norm problem then is given by matrices A_K, B_K, C_K, D_K retrieving from (20) with nonsingular matrices M, N such that

$$MN^T = I - XY \tag{25}$$

Proof *Necessity*: If there exists a solution $\tilde{P}$ to the inequalities (13) and (14) for the closed-loop system (10), we always express it as $\tilde{P}\Pi_1 = \Pi_2$ with nonsingular matrices Π_1, Π_2 as in (17-b). Therefore a nonsingular congruence transformation (18) and (19) is possible. By introducing the nonlinear change of variables (20), we can confirm that the inequalities (18) and (19) become (24) and (22) with the result of theorem 4.

Sufficiency: Assume that we have a solution $\{P, Q, S, R, \hat{A}, \hat{B}, \hat{C}, \hat{D}\}$ to (22) and (24). In order to show the validity of the obtained controller (in other words, whether the H_∞ norm condition of the closed-loop system is satisfied), we need to establish nonsingular matrices Π_1, Π_2, $\tilde{P}$ as in (15)~(17). Looking at the left upper block of (17-a), M and N should be chosen as (25). From theorem 4, we infer that $I-XY$ is nonsingular. Hence we can always find square and nonsingular M and N satisfying (25). After defining Π_1, Π_2 as in (17-b), we observe that these matrices are nonsingular, and we set $\tilde{P}=\Pi_2\Pi_1^{-1}$ to obtain (17-a). With nonsingular matrices M, N, and the solutions $\{P, Q, S, R, \hat{A}, \hat{B}, \hat{C}, \hat{D}\}$ to LMIs (22) and (24), we can now determine controller matrices D_K, C_K, B_K, A_K in this order through (20). Consequently, using the relationship (20), it is possible to express the inequalities (21) and (22) by means of the matrices $\{P, Q, S, R, A_K, B_K, C_K, D_K\}$, i.e. to reverse the linearizing change of variables. Recalling that Π_1 is square and nonsingular, the congruence transformation (18) and (19) can be reversed, and by theorem 3 we can confirm that the H_∞ norm bound is satisfied with $\tilde{P}=\Pi_2\Pi_1^{-1}$ and controller matrices A_K, B_K, C_K, D_K. Hence the obtained controller indeed leads to $\|G_{cl\infty}\|_\infty<\gamma$. □

We can sum up the procedure for the controller computation as follows:

1) Compute the matrices U, V, E_L, E_R
2) Solve the LMIs (22), (24).
3) Compute the nonsingular matrices M, N, which satisfy (25), then carry out the computation of controller matrices A_K, B_K, C_K, D_K in this order.

3.2 H_2 Control for descriptor systems

Here, we think about an LMI approach for H_2 control of linear time-invariant descriptor systems. Let us consider the descriptor and its closed-loop system given in section 3:

$$G_2:\begin{cases} E\dot{x}=Ax+B_1w+B_2u \\ z_2=C_2x \\ y=C_3x \end{cases} \tag{26}$$

$$\begin{aligned} E_{cl}\dot{x}_c &= A_{cl}x_c+B_{cl}w \\ z_2 &= C_{cl2}x_c,\ C_{cl2}=[\,C_2\ \ 0\,] \end{aligned} \tag{27}$$

The H_2 norm for a descriptor system (27) is defined as

$$\begin{aligned} \|G_{cl2}(s)\|_2^2 &:= \|C_{cl2}(sE_{cl}-A_{cl})B_{cl}\|_2^2 \\ &= \left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\operatorname{trace}(G_{cl2}(-j\omega)^T G_{cl2}(j\omega))d\omega\right] \end{aligned} \tag{28}$$

which is finite if and only if

$$\lim_{s\to\infty} G_{cl2}(s)=0 \tag{29}$$

To ensure finiteness of the H_2 norm, we assume that the system (26) satisfies the following condition [12].

$$\ker C_2 \supseteq \ker E \tag{30}$$

Therefore the H_2 control problem is to obtain a controller by which the H_2 norm of the closed loop system is less than, say, ν. It is known that the H_2 norm of a descriptor system (28) can be computed as follows [9]:

$$\|G_{cl2}(s)\|_2^2=\operatorname{trace}(C_{cl2}E_{cl}L_cC_{cl2}^T)=\operatorname{trace}(C_{cl2}L_c^TE_{cl}^TC_{cl2}^T)<\nu \tag{31}$$

where L_c solves

$$A_{cl}L_c+L_c^TA_{cl}^T+B_{cl}B_{cl}^T=0 \tag{32-a}$$

$$E_{cl}L_c=L_c^TE_{cl}^T \tag{32-b}$$

Remark 1. The Lyapunov equation for a descriptor system is generally described as

$$AL^T+LA^T+BB^T=0,\ EL^T=LE^T\geq 0$$

We, however, have adopted the expression (32) in order to get a parallel with the H_∞ case, that is, (14). □

Since $L_c<L$ for any L satisfying

$$A_{cl}L_c+L_c^TA_{cl}^T+B_{cl}B_{cl}^T<0 \tag{33}$$

It is readily verified that $\|G_{cl2}\|_2^2<$ if and only if there exists $L>0$ satisfying (33) and $\operatorname{trace}(C_{cl2}L^TE_{cl}^TC_{cl2}^T)<\nu$. With an auxiliary parameter W, we obtain the following result. That is, $G_{cl2}(s)$ is stable and $\|G_{cl2}\|_2^2<$ if and only if there exists $\tilde{P}=L^{-1}$ and symmetric W such that

$$\begin{bmatrix} A_{cl}^T\tilde{P}+\tilde{P}^TA_{cl} & \tilde{P}^TB_{cl} \\ B_{cl}^T\tilde{P} & -I \end{bmatrix}<0 \tag{34-a}$$

$$\begin{bmatrix} \tilde{P}^TE_{cl} & C_{cl2}^T \\ C_{cl2} & W \end{bmatrix}>0 \tag{34-b}$$

$$\operatorname{trace}(W)<\nu \tag{34-c}$$

Now, we can derive a theorem, which is parallel to theorem 5 and related to the H_2 control problem.

Theorem 6 Consider a descriptor system (26) with (30) and a controller (8). There exits a full order controller (that is, $n_K = n$ and $\text{rank}\boldsymbol{E}_K = r$) such that the closed-loop system is admissible and the H_2 norm is bounded by ν if and only if LMIs (35) admits solution $\{\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{S}, \boldsymbol{R}, \hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{\boldsymbol{C}}, \hat{\boldsymbol{D}}\}$.

$$\begin{bmatrix} AX+X^TA^T+B_2\hat{C}+(B_2\hat{C})^T & * & * \\ \hat{A}+(A+B_2\hat{D}C_3)^T & A^TY+Y^TA+\hat{B}C_3+(\hat{B}C_3)^T & * \\ B_1^T & (Y^TB_1)^T & -I \end{bmatrix} < 0 \tag{35-a}$$

$$\begin{bmatrix} E_R^TPE_R & I_r & E_R^TPC_2^T \\ I_r & E_L^TQE_L & (E_R^TE_R)^{-1}E_R^TC_2^T \\ C_2PE_R & C_2E_R(E_R^TE_R)^{-1} & W \end{bmatrix} > 0 \tag{35-b}$$

$$\text{trace}(W) < \nu \tag{35-c}$$

where X, Y are given in (16).

Proof By simply applying the congruence transformation with $\text{diag}(\Pi_1 \ \ I)$ to (34-a), (35-a) can be obtained. To show (35-b), some matrix manipulations are needed. We, first, carry out the congruence transformation to (34-b). That is,

$$\begin{aligned} &\begin{bmatrix} \Pi_1^T & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \tilde{P}^TE_d & C_{d2}^T \\ C_{d2} & W \end{bmatrix}\begin{bmatrix} \Pi_1 & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} EPE^T & E & (C_2X)^T \\ E^T & E^TQE & C_2^T \\ C_2X & C_2 & W \end{bmatrix} > 0 \end{aligned} \tag{36}$$

where we used (18) and the following relationships that were obtained through (13).

$$\begin{aligned} &EX = EPE^T, \ Y^TE = E^TQE \\ &Y^TEX + NEM^T = E^T \end{aligned} \tag{37}$$

From Schur Complement, (36) is equivalent to

$$\begin{bmatrix} EPE^T & E \\ E^T & E^TQE \end{bmatrix} > \begin{bmatrix} (C_2X)^T \\ C_2^T \end{bmatrix} W^{-1} [\, C_2X \ \ C_2 \,] \tag{38}$$

The left side of (38) can be rewritten as follows:

$$\begin{bmatrix} E_L & U & 0 & 0 \\ 0 & 0 & E_R & V \end{bmatrix}\begin{bmatrix} E_R^TPE_R & 0 & I_r & 0 \\ 0 & 0 & 0 & 0 \\ I_r & 0 & E_L^TQE_L & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} E_L^T & 0 \\ U^T & 0 \\ 0 & E_R^T \\ 0 & V^T \end{bmatrix} \tag{39}$$

Since $[\, E_L \ \ U \,]$, $[\, E_R \ \ V \,]$ are nonsingular, we can check the following relationships:

$$[\, E_L \ \ U \,]\begin{bmatrix} (E_L^TE_L)^{-1}E_L^T \\ (U^TU)^{-1}U^T \end{bmatrix} = I \tag{40-a}$$

$$[\, E_R \ \ V \,]\begin{bmatrix} (E_R^TE_R)^{-1}E_R^T \\ (V^TV)^{-1}V^T \end{bmatrix} = I \tag{40-b}$$

Multiplying (39) by $\begin{bmatrix} E_L & U & 0 & 0 \\ 0 & 0 & E_R & V \end{bmatrix}^{-1}$ from the left and by its transpose from the right leads to

$$\begin{aligned} &\begin{bmatrix} E_R^TPE_R & I_r \\ I_r & E_L^TQE_L \end{bmatrix} > \\ &\begin{bmatrix} E_R^TPC_2^T \\ (E_R^TE_R)^{-1}E_R^TC_2^T \end{bmatrix} W^{-1} [\, C_2PE_R \ \ C_2E_R(E_R^TE_R)^{-1} \,] \end{aligned} \tag{41}$$

where we used (40). Again applying Schur complement to (41), (35-b) is obtained. We have shown that (34) are equivalent to (35). So the rest of the proof (that is, the existence of a controller) can be done in the same way as in theorem 5. □

3.3 LMI Conditions for Mixed H_2/H_∞ Control problem

The mixed H_2/H_∞ control problem is to design a controller (8) such that: (1) internally stabilizing the closed-loop system and (2) minimize H_2 norm of the closed-loop while satisfying the given H_∞ norm bound. Therefore, by combining the results of the previous subsections, we can easily obtain the following theorem:

Theorem 7 Consider a descriptor system (7) with (30). There exists a full order controller (8) such that the closed loop system is admissible and the H_∞ norm is bounded by γ and the H_2 norm is less than ν if and only if LMIs (22), (24), and (35) admit solution $\{\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{S}, \boldsymbol{R}, \hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{\boldsymbol{C}}, \hat{\boldsymbol{D}}\}$. Then one of the controller (8) solving the mixed H_2/H_∞ control problem is given by matrices A_K, B_K, C_K, D_K through (20) with nonsingular matrices M, N satisfying (25). □

As a result of this theorem, the mixed H_2/H_∞ control problem can be solved by minimizing ν under the LMI constraints. That is,

Minimize $\text{trace}(W)$ subject to LMIs (22), (24), (34-a), and (34-b)

Since this is so called the linear objective minimization problem, it is easily solved by using commercial software such as [11].

Remark 2. Although the controller (8) based on theorem 7 may be improper, we can always design a proper one by adding small perturbation to LMI variables without changing LMI conditions [7]. □

4. Numerical Examples

The first example is from [10] by which we can confirm that our result is compatible with the existing non-descriptor systems.

Example 1: Consider the three-state unstable plant*.

* In the output z_2, we omitted the term 'u' in order to meet condition (30):

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z_\infty &= C_1 x + u \\ z_2 &= C_2 x \\ y &= C_3 x + 2w \end{aligned} \tag{42-a}$$

$$A = \begin{bmatrix} 0 & 10 & 2 \\ -1 & 1 & 0 \\ 0 & 2 & -5 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{42}$$
$$C_1 = [\,1\ 0\ 0\,],\ C_2 = [\,0\ 1\ 1\,],\ C_3 = [\,0\ 1\ 0\,]$$

We transform (42) into (7), which has no direct fed-through terms from control/disturbance input to external/measured output. That is, by adopting extra variables such as $\zeta = 2w, \xi = u$, (42) can be rewritten as follows:

$$\begin{bmatrix} I_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\psi} = \begin{bmatrix} A & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \psi + \begin{bmatrix} B_1 \\ 2 \\ 0 \end{bmatrix} w + \begin{bmatrix} B_2 \\ 0 \\ 1 \end{bmatrix} u$$
$$\begin{aligned} z_\infty &= [\,C_1\ 0\ 1\,]\psi \\ z_2 &= [\,C_2\ 0\ 0\,]\psi \\ y &= [\,C_3\ 1\ 0\,]\psi \end{aligned} \tag{43-a}$$

with

$$\psi^T = [\,x^T\ \zeta\ \xi\,] \tag{43-b}$$

By using the computation result of [10], we consider the mixed H_2/H_∞ control problem as follows:

$$\text{Minimize} \| T_{w_2 z_2} \|_2 \text{ subject to} \| T_{w_\infty z_\infty} \|_\infty < 23.6$$

Solving LMIs (22), (24), and (35) with $\gamma = 23.6$ yields *6.46* as best constrained H_2 performance, which is slightly lower than the result of [10].

□

Next, we try to design a mixed H_2/H_∞ controller by using Theorem 7.

Example 2: Consider the following plant that is given by the descriptor format.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} w + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$
$$\begin{aligned} z_\infty &= [\,0\ 1\ 1\,]x + w + u \\ z_2 &= [\,1\ 1\ 0\,]x \\ y &= [\,1\ 0\ 1\,]x + w \end{aligned} \tag{44}$$

By applying Theorem 7, we obtain *1.86* as best constrained H_2 performance on the condition that H_∞ norm is less than *1.77*. The solutions of LMIs (22), (24), and (35) are as follows:

$$P = \begin{bmatrix} 2.216 & -1.539 & 1.638 & -.536 & -.470 \\ -1.539 & 3.794 & -2.249 & -.359 & -.793 \\ 1.638 & -2.249 & -3.60 \times 10^7 & 0 & 0 \\ -.536 & -.359 & 0 & -3.60 \times 10^7 & 0 \\ -.470 & -.793 & 0 & 0 & -3.60 \times 10^7 \end{bmatrix}$$

$$Q = \begin{bmatrix} 2.156 & .707 & -1.599 & .727 & 8.112 \times 10^{-5} \\ .707 & .577 & 1.011 & -1.194 & .607 \\ -1.599 & 1.011 & -3.60 \times 10^7 & 0 & 0 \\ .727 & -1.194 & 0 & -3.60 \times 10^7 & 0 \\ 8.112 \times 10^{-5} & .607 & 0 & 0 & -3.60 \times 10^7 \end{bmatrix}$$

$$S = \begin{bmatrix} 2.138 & -1.111 & -.285 \\ .751 & 1.628 & -.325 \\ -2.498 & 0 & 1.128 \end{bmatrix}, R = \begin{bmatrix} .836 & -.617 & -.303 \\ -1.066 & 1.011 & .303 \\ .303 & 0 & 1.443 \end{bmatrix}$$

And the coefficient matrices of the controller (8) become:

$$A_K = \begin{bmatrix} -.974 & -1.036 & .944 & 2.305 & -1.654 \\ .467 & -.566 & 3.162 & -1.771 & .549 \\ -667 & -1.819 & 5.257 & 2.127 & 11.681 \\ -.901 & .625 & 1.244 & -4.309 & 2.282 \\ 1.856 & 2.753 & -6.759 & .201 & -19.774 \end{bmatrix}$$
$$B_K = \begin{bmatrix} .630 \\ .402 \\ 1.256 \\ .221 \\ -1.803 \end{bmatrix}, C_K^T = \begin{bmatrix} -.598 \\ -.339 \\ 1.331 \\ 1.643 \\ -1.243 \end{bmatrix}, D_K = 0,$$
$$E_K = \text{diag}(1, 1, 0, 0, 0)$$

In this case, the order of the controller is ostensibly increased by *2*, because we augmented the original plant in order to eliminate "***D*** terms" in the performance and plant output.

5. Conclusion

We considered the mixed H_2/H_∞ control of linear descriptor systems. An LMI approach to the synthesis problem (based on a linearizing change of variables) reveals the relationship between the H_2 and H_∞ control problem of the descriptor systems. That is, by introducing extra LMI variables (V and U) which are determined only by matrix E, we could derive the generalized LMI conditions of descriptor systems that involve checking H_2 norm and H_∞ norm conditions of the different input-output channels. A controller computation procedure for the mixed H_2/H_∞ control problem based on the solution of linear matrix inequalities is provided. In this case, we assumed the controller was in descriptor form and had the same dimension as the plant. A Non-descriptor controller, however, can be easily derived by imposing some restrictions in advance such as normalized SVD representations on the plant and control system [14]. Finally we showed two examples to confirm the validity of the proposed LMI conditions for descriptor systems.

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