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KRULL DIMENSION OF A COMPLETION

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ABSTRACT. We calculate $\dim \hat{A}$ which is a completion of a Noetherian ring A with respect to I-adic topology. We do not use localization but power series techniques.

Let R be a commutative ring (with 1). We denote by $\dim R$ the Krull dimension of R. Arnold [1, 2, 3, 4], and Kang & Park [9, 10, 11] there are many researches about topics relating dimension, completion and power series. In this note we calculate $\dim \hat{A}$, which is the completion of a Noetherian ring A with respect to I-adic topology. We do not use localization but power series techniques. For any Notherian local ring (A, m), let $\hat{A}$ be the $\mathfrak{m}$-adic completion of A. It is well known that $\dim A = \dim \hat{A}$ Atiyah & Macdonald [5, p. 122]. But this fact can not answer for arbitrary I-adic completion of a ring, although the ring is very simple, for example, a principal ideal domain. In this paper, we extend the above theorem and we calculate the dimension of a I-adic completion of a principal ideal domain. Let R be a Notherian ring, I an ideal of R such that

$$\bigcap_{n=1}^{\infty} I_n = (0),$$

and $\hat{R}$ the I-adic completion of R. Then

$$\hat{R} \cong \frac{R[[x_1, \ldots, x_n]]}{(x_1 - a_1, \ldots, x_n - a_n)}$$

by Greco & Salmon [8, p. 17] and Nagata [13, p. 55]. Even without the condition

$$\bigcap_{n=1}^{\infty} I_n = (0),$$

the same result can be obtained as Kang & Park [10, p. 5].

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Lemma 1. *Let R be a Notherian ring. And let $M, (b_1, \ldots, b_n)$ be ideals of R. Let $R[[x_1, \ldots, x_n]]$ be a power series ring over R. If M is minimal over $(b_1, \ldots, b_n)$ then the ideal $M + (x_1, \ldots, x_n)$ of $R[[x_1, \ldots, x_n]]$ is minimal over $(b_1, \ldots, b_n) + (x_1, \ldots, x_n)$.*

Proof. Suppose that there is a prime ideal P of $R[[x_1, \ldots, x_n]]$ such that

$$(b_1, \ldots, b_k) + (x_1, \ldots, x_n) \subset P \subset M + (x_1, \ldots, x_n).$$

Then we know that $(b_1, \ldots, b_k) \subset P \bigcap R \subset M$ implies $P \bigcap R = M$ since M is minimal over $(b_1, \ldots, b_k)$. Hence $P \supset M + (x_1, \ldots, x_n)$ implies $P = M + (x_1, \ldots, x_n)$. Therefore $M + (x_1, \ldots, x_n)$ is minimal over $(b_1, \ldots, b_n) + (x_1, \ldots, x_n)$. □

Lemma 2. *Let R be a Notherian ring. Let $R[[x_1, \ldots, x_n]]$ be a power series ring over R and let Q be a prime ideal of $R[[x_1, \ldots, x_n]]$. If $Q \supset (x_1 - a_1, \ldots, x_n - a_n)$ then $htQ \geq n$.*

Proof. We prove by using induction on the number of variables of the power series ring over R. When $n = 1, Q \supset (x - a)$.

Since $(x - a)$ is not a zero-divisor of $R[[x_1, \ldots, x_n]]$ by Brewer [6, p. 7], $htQ \neq 0$. We prove that $(x_1 - a_1, \ldots, x_n - a_n) \subset Q$ implies $htQ \geq n$.

Let

$$Q_0 = Q \bigcap R[[x_1, \ldots, x_{n-1}]].$$

Then we know that $(x_1 - a_1, \ldots, x_{n-1} - a_{n-1}) \subset Q_0$. Inductive hypothesis implies $htQ_0 \geq n - 1$. But $Q_0R[[x_1, \ldots, x_n]] \subsetneq Q$ since $x_n - a_n \notin Q_0\ R[[x_1, \ldots, x_n]]$. Hence $htQ \geq n$. □

Theorem 3. *Let R be a Notherian ring, $I = (a_1, \ldots, a_n)$ an ideal of R, and $\hat{R}$ the I-adic completion of R. Then*

$$\dim \hat{R} = \sup\{htM | M \in \max(R), M \supset I\}.$$

Proof. Let $\bar{Q_0} \subset \bar{Q_1} \subset, \ldots, \subset \bar{Q_l}$ be a maximal chain in $\hat{R}$. Since

$$\hat{R} \cong \frac{R[[x_1, \ldots, x_n]]}{(x_1 - a_1, \ldots, x_n - a_n)}, (x_1 - a_1, \ldots, x_n - a_n) \subset Q_0 \subset Q_1 \subset, \ldots, \subset Q_l$$

and Q_0 is minimal over $(x_1 - a_1, \ldots, x_n - a_n)$. And $Q_l = M + (x_1, \ldots, x_n)$ for some $M \in \max(R)$ and $(a_1, \ldots, a_n) \subset M$. Since Q_0 has height n, by lemma 2 and Krull's Generalized Principal Ideal Theorem, $n + l \geq ht(M + (x_1, \ldots, x_n))$.

Let ht$M = k$. By Theorem 153 of Nagata [13] there exist elements $b_1, \ldots, b_n \in R$ such that M is minimal over $(b_1, \ldots, b_n)$. By lemma 1, the ideal $M + (x_1, \ldots, x_n)$

of $R[[x_1,\dots,x_n]]$ is minimal over $(b_1,\dots,b_n)+(x_1,\dots,x_n)$. And by the generalized principal ideal theorem, $\mathrm{ht}\big(M+(x_1,\dots,x_n)\big) \le k+n$. Let $P_0 \subset P_1 \subset,\dots,\subset P_l = M$ be a maximal chain of M. Then

$$\begin{aligned} P_0[[x_1,\dots,x_n]] &\subset P_1[[x_1,\dots,x_n]] \subset,\dots,\subset P_k[[x_1,\dots,x_n]] \\ &\subset P_k[[x_1,\dots,x_n]]+(x_1) \subset P_k[[x_1,\dots,x_n]]+(x_1,x_2) \subset,\dots, \\ &\subset P_k[[x_1,\dots,x_n]]+(x_1,\dots,x_n) \end{aligned}$$

is a chain of length $k+n$. Hence $k+n \le ht\big(M+(x_1)\big)$. And we know $k+n = ht\big(M+(x_1)\big)$ and $l \le k$. Thus

$$\dim \hat{R} \le \sup\{htM \mid M \in \max(R), M \supset I\}.$$

To prove the other direction of inequality, we first consider a case when $\dim R < \infty$. We prove by induction on $\dim R$ that

$$\dim \hat{R} \ge \sup\{htM \mid M \in \max(R), M \supset I\}.$$

When $\dim R = 0$, we know that $\dim \hat{R} \ge 0$. Let $M_0 \in \max(R)$, and $M_0 \supset I$. Since R is a Notherian ring, $htM_0 < \infty$. Let

$$htM_0 = l \quad \text{and} \quad P_0 \subset P_1 \subset,\dots,\subset P_l = M_0$$

be a maximal chain of M_0. We may assume that $\dim R \ge 1$ and $l \ge 1$.

Let

$$A = \frac{R[[x_1,\dots,x_n]]}{P_1+(x_1-a_1,\dots,x_n-a_n)} = \frac{(R/P_1)[[x_1,\dots,x_n]]}{(x_1-\bar{a_1},\dots,x_n-\bar{a_n})}.$$

We can use the induction hypothesis since we have that $\dim R/P_1 < \dim R$. We have that

$$\begin{aligned} \dim A &= \sup\{ht(M/P_1) \mid M \in \max(R), \bar{M} \supset (\bar{a_1},\dots,\bar{a_n}) \text{ and } M \supset P_1\} \\ &= \sup\{ht(M/P_1) \mid M \in \max(R), M \supset P_1+(a_1,\dots,a_n)\}. \end{aligned}$$

And we know that $\dim A \ge ht(M_0/P_1) = l-1$ since $M_0 \supset P_1+(a_1,\dots,a_n)$.

Let

$$P_1+(x_1-a_1,\dots,x_n-a_n) \subset Q_0 \subset Q_1 \subset,\dots,\subset Q_{l-1} = M_0$$

be a chain of prime ideals of $R[[x_1,\dots,x_n]]$.

Suppose that Q_0 is minimal over $P_1+(x_1-a_1,\dots,x_n-a_n)$. Since

$$\frac{R[[x_1,\dots,x_n]]}{P_0+(x_1-a_1,\dots,x_n-a_n)} = \frac{R/P_0[[x_1,\dots,x_n]]}{(x_1-\bar{a_1},\dots,x_n-\bar{a_n})}, \ \bar{Q_0} = Q_0/P_0[[x_1,\dots,x_n]]$$

is minimal over $(x_1 - \bar{a_1}, \ldots, x_n - \bar{a_n})$. If we choose $b \in P_1 - P_0$, then b is not a zero devisor of the ring R/P_0. There exist natural number $k > 0$ and element $h \in (R/P_0)[[x_1, \ldots, x_n]]$ such that

$$h \notin \bar{Q_0} \text{ and } hb^k \in (x_1 - \bar{a_1}, \ldots, x_n - \bar{a_n}),$$

since $b \in \bar{Q_0}$. Plugging in $x_1 = \bar{a_1}, \ldots, x_n = \bar{a_n}$, we get $h(\bar{a_1}, \ldots, \bar{a_n})b^k = 0$ in

$$\frac{(R/P_0)[[x_1, \ldots, x_n]]}{(x_1 - \bar{a_1}, \ldots, x_n - \bar{a_n})}.$$

Since b is not a zero devisor of the ring $\widehat{R/P_0}$, $h(\bar{a_1}, \ldots, \bar{a_n}) = 0$ in

$$\frac{(R/P_0)[[x_1, \ldots, x_n]]}{(x_1 - \bar{a_1}, \ldots, x_n - \bar{a_n})}.$$

Then $h \in (x_1 - \bar{a_1}, \ldots, x_n - \bar{a_n})$, which is contrary to $h \notin \bar{Q_0}$. Hence Q_0 is not minimal over $P_1 + (x_1 - a_1, \ldots, x_n - a_n)$. Therefore $\dim \hat{R} \geq (l-1) + 1 = l$.

We proved that $\dim \hat{R} = \sup\{htM | M \in \max(R), M \supset I\}$ if $\dim R < \infty$. Let R be a Notherian ring with $\dim R = \infty$ and $M \in \max(R)$, and $M \supset I$. We know that $\dim R_M = \dim \widehat{R_M}$, where $\widehat{R_M}$ is the completion with M-adic topology Atiyah & Macdonald [5, p. 122]. And we know that $clR_M = \hat{R}_{\hat{M}}$, where clR_M is the closure of R_M in $\hat{R}_{\hat{M}}$ with respect to the $\hat{M}$-adic topology Boubaki [7, p. 205]. We have that

$$htM = \dim R_M = \dim \widehat{R_M} = \dim \widehat{\hat{R}_{\hat{M}}} = \dim \hat{R}_{\hat{M}}$$

since $\widehat{R_M}$ is complete with respect to the $\hat{M}$-adic topology. Since $\hat{M}$ is a maximal ideal of $\hat{R}$, we have that $htM \leq \dim \hat{R}$. We proved that

$$\dim \hat{R} = \sup\{htM | M \in \max(R), M \supset I\}.$$

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