ON A QUASI-SELF-SIMILAR MEASURE ON A SELF-SIMILAR SET ON THE WAY TO A PERTURBED CANTOR SET

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ABSTRACT. We find an easier formula to compute Hausdorff and packing dimensions of a subset composing a spectral class by local dimension of a self-similar measure on a self-similar Cantor set than that of Olsen. While we cannot apply this formula to computing the dimensions of a subset composing a spectral class by local dimension of a quasi-self-similar measure on a self-similar set on the way to a perturbed Cantor set, we have a set theoretical relationship between some distribution sets. Finally we compare the behaviour of a quasi-self-similar measure on a self-similar Cantor set with that on a self-similar set on the way to a perturbed Cantor set.

1. INTRODUCTION

Olsen [9] studied a formula to compute the Hausdorff and packing dimensions of the subset composing a multifractal spectral class of a self-similar set by a self-similar probability measure. He found the formula using some power equations essentially, so it is hard to find their solutions. We Baek [5] gave another method to find it using a set-theoretical relationship between a distribution set and a subset of same local dimension of a self-similar measure. We find it is more simpler than that of Olsen for the case of a self-similar Cantor set. Recently we Baek [6] also generalize Olsen's results to a perturbed Cantor set Baek [1, 2, 3, 4]. That is, we found a formula of computing the dimensions of the subset of same local dimension of a quasi-self-similar measure Baek [6] on a perturbed Cantor set. We find that the quasi self-similar measure in this paper plays a self-similar measure before its limit level. That is at the $n$-th level stage to construct a perturbed Cantor set, the $n$-th adjusted quasi-self-similar measure behaves like a self-similar measure on a self-similar set having $2^n$ contraction ratios. We need a generalized quasi-expansion...
of a point in the self-similar set to develop our theories which also need a strong law of large numbers. We naturally expected our easy computing method can be applied to that of a perturbed Cantor set, but in failure. However, we get many interesting facts of some relationship between quasi-distribution sets and generalized distribution sets (cf. Lee & Baek [8]).

2. Preliminaries

We recall the definition of a perturbed Cantor set Baek [1]. Let \( X_\phi = [0, 1] \). We obtain the left subinterval \( X_{1,1} \) and the right subinterval \( X_{1,2} \) of \( X_1 \) by deleting a middle open subinterval of \( X_i \) inductively for each \( i \in \{1, 2\}^n \), where \( n = 0, 1, 2, \ldots \). Let \( E_n = \bigcup_{i \in \{1, 2\}^n} X_i \). Then \( E_n \) is a decreasing sequence of closed sets. For each \( n \), we set \( |X_{i,1}|/|X_i| = a_{n+1} \) and \( |X_{i,2}|/|X_i| = b_{n+1} \) for all \( i \in \{1, 2\}^n \), where \( |X| \) denotes the length of \( X \). We assume that the contraction ratios \( a_n \) and \( b_n \) and gap ratios \( 1 - (a_n + b_n) \) are uniformly bounded away from 0. We call \( F = \cap_{n=0}^\infty E_n \) a perturbed Cantor set Baek [1]. For \( i \in \{1, 2\}^n \), \( X_i \) denotes a fundamental interval of the \( n \)-stage of construction of perturbed Cantor set and \( X_n(x) \) denotes the fundamental interval \( X_i \) containing \( x \in F \).

Let \( \mathbb{R} \) be the set of all real numbers and \( \mathbb{N} \) be the set of all natural numbers. We note that if \( x \in F \), then there is \( \sigma \in \{1, 2\}^\mathbb{N} \) such that

\[
\bigcap_{k=0}^\infty I_{\sigma|k} = \{x\} \quad \text{(Here} \, \, \sigma|k = i_1, i_2, \ldots, i_k \quad \text{where} \quad \sigma = i_1, i_2, \ldots, i_k, i_{k+1}, \ldots) \).
\]

Hereafter, we use \( \sigma \in \{1, 2\}^\mathbb{N} \) and \( x \in F \) as the same identity freely. For \( y \in \mathbb{R} \), we define a quasi-self-similar measure \( \mu_y \) on a perturbed Cantor set \( F \) to be a Borel probability measure on \( F \) satisfying

\[
\mu_y(X_i) = \frac{|X_i|^y}{\prod_{k=1}^m (a_k^y + b_k^y)}
\]

for \( m \in \mathbb{N} \) and \( i \in \{1, 2\}^m \).

For \( n \in \mathbb{N} \) we define a self-similar set \( F_n \) with contraction ratios generated by \( \{a_k, b_k\}_{k=1}^n \) by a perturbed Cantor set with \( a_{hn+k} = a_k \) and \( b_{hn+k} = b_k \) where \( h \in \mathbb{N} \) and \( k \in \{1, 2, \ldots, n\} \). Clearly, \( F_n \) is a self-similar set (on the way to a perturbed Cantor set \( F \)) having \( 2^n \) contraction ratios

\[
c_{i_1, i_2, \ldots, i_n} = d_{i_1}^{(1)} d_{i_2}^{(2)} \cdots d_{i_n}^{(n)} \quad \text{where} \quad d_{i_k}^{(k)} = \begin{cases} a_k & \text{for} \, \, i_k = 1 \\ b_k & \text{for} \, \, i_k = 2 \end{cases}.
\]
From now on, we write \( P_n(y) = (p_1, \ldots, p_n) \) where \( p_k = \frac{a_k^y}{a_k^y + b_k^y} \) and \( 1 \leq k \leq n \). We define an \( n \)-th adjusted quasi-self-similar measure \( \mu_y \) on a perturbed Cantor set \( F \) to be the measure \( \mu_y \) on the perturbed Cantor set \( F_n \). Clearly, \( \mu_y \) on \( F_n \) is a self-similar measure on \( F_n \) satisfying

\[
\mu_y(X_i) = r_{i_1}^{(1)} r_{i_2}^{(2)} \cdots r_{i_n}^{(n)} \quad \text{where} \quad r_{i_k}^{(k)} = \begin{cases} p_k & \text{for } i_k = 1 \\ 1 - p_k & \text{for } i_k = 2 \end{cases},
\]

\( i = i_1, \ldots, i_k, \ldots, i_n \) and \( 1 \leq k \leq n \).

We write \( E_{\alpha}^{P_n(y)} \) for the set of points at which the local dimension of \( \mu_y \) on \( F_n \) is exactly \( \alpha \), so that

\[
E_{\alpha}^{P_n(y)} = \{ x : \lim_{r \to 0} \frac{\log \mu_y(B_r(x))}{\log r} = \alpha \},
\]

where \( B_r(x) \) is a closed ball with center \( x \) and a positive radius \( r \). We write the above \( \mu_y \) on \( F_n \) as \( \gamma_{P_n(y)} \) from now on and note that \( \gamma_{P_n(y)} \) is a self-similar measure on a self-similar set \( F_n \).

Clearly, we see that a self-similar measure \( \mu \) on a self-similar Cantor set \( \text{(that is,} \ F_n = F_1 \text{)} \) satisfying \( \mu(X_1) = p \) is \( \gamma_p \).

We write \( \overline{E}_{\alpha}^{(p)} \) \( \left( \underline{E}_{\alpha}^{(p)} \right) \) for the set of points at which the lower (upper) local dimension of \( \gamma_p \) on a self-similar Cantor set \( F \) is exactly \( \alpha \), so that

\[
\underline{E}_{\alpha}^{(p)} = \{ x : \liminf_{r \to 0} \frac{\log \gamma_p(B_r(x))}{\log r} = \alpha \},
\]

\[
\overline{E}_{\alpha}^{(p)} = \{ x : \limsup_{r \to 0} \frac{\log \gamma_p(B_r(x))}{\log r} = \alpha \}.
\]

In particular, we write \( E_{\alpha}^{(p)} \) for the set of points at which the local dimension of \( \gamma_p \) on \( F \) is exactly \( \alpha \), so that

\[
E_{\alpha}^{(p)} = \underline{E}_{\alpha}^{(p)} \cap \overline{E}_{\alpha}^{(p)}.
\]

If \( 0 < p < 1 \), then there is \( y \in \mathbb{R} \) such that \( P_1(y) = p \). So we note that \( E_{\alpha}^{(p)} = E_{\alpha}^{P_1(y)} \). To get informations of the dimensions of \( E_{\alpha}^{P_n(y)} \) we need the following Proposition. We write the Hausdorff dimension of a set \( E \subset \mathbb{R} \) as \( \dim_H(E) \) and its packing dimension as \( \dim_P(E) \). The lower and upper local dimension of \( \mu \) at \( x \in \mathbb{R} \) are defined Falconer [7] by

\[
\dim_{loc} \mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r},
\]

\[
\overline{\dim}_{loc} \mu(x) = \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}.
\]
Proposition 1 (Falconer [7]). Let $E \subset \mathbb{R}$ be a Borel set and let $\mu$ be a finite measure.

(a) If $\text{dim}_{loc}\mu(x) \geq s$ for all $x \in E$ and $\mu(E) > 0$, then $\dim_H(E) \geq s$.
(b) If $\text{dim}_{loc}\mu(x) \leq s$ for all $x \in E$, then $\dim_H(E) \leq s$.
(c) If $\text{dim}_{loc}\mu(x) \geq s$ for all $x \in E$ and $\mu(E) > 0$, then $\dim_P(E) \geq s$.
(d) If $\text{dim}_{loc}\mu(x) \leq s$ for all $x \in E$, then $\dim_P(E) \leq s$.

Remark 1. If $A \subset E_{\alpha}^{P_n(y)}$ and $\gamma_{P_n(y)}(A) > 0$, then $\dim_H(A) = \dim_P(A) = \alpha$ from the above Proposition.

Lemma 2. Let $\mu$ be a finite measure on a perturbed Cantor set $F$ or $F_n$. Then for any $\alpha \geq 0$,

$$\lim_{r \to 0} \frac{\log \mu(B_r(x))}{\log r} = \alpha \quad \text{if and only if} \quad \lim_{m \to \infty} \frac{\log \mu(X_m(x))}{\log |X_m(x)|} = \alpha.$$ 

Proof. It is obvious from the fact that the contraction ratios are uniformly bounded away from 0. \qed

In this paper, we assume that $0 \log 0 = 0$ for convenience.

3. Main results

In this section we only consider subsets in $F_n$.

Remark 2. Let $y \in \mathbb{R}$ and $\alpha \geq 0$. Fix $n \in \mathbb{N}$. Put $P_n(y) = (p_1, \ldots, p_n)$ where

$$p_k = \frac{a_k^y}{a_k^y + b_k^y} \quad \text{and} \quad 1 \leq k \leq n.$$ 

With respect to $r_1, \ldots, r_n$ we can solve the equation

$$\alpha = \frac{\sum_{k=1}^n (r_k \log p_k + (1 - r_k) \log(1 - p_k))}{\sum_{k=1}^n (r_k \log a_k + (1 - r_k) \log b_k)} \equiv g(r_1, \ldots, r_n, P_n(y))$$

where

$$p_k = \frac{a_k^y}{a_k^y + b_k^y}.$$ 

Then there exists $z \in [-\infty, \infty]$ such that $P_n(z) = (r_1, \ldots, r_n)$ and $(r_1, \ldots, r_n)$ is a solution of the above equation $\alpha = g(r_1, \ldots, r_n, P_n(y))$. Since

$$\dim_H\left(E_{\alpha}^{P_n(y)}\right) = g(P_n(z), P_n(z))$$

holds Baek [5] for $n = 1$, we naturally expect that it holds also for $n \geq 2$. In this case, we consider a self-similar measure $\gamma_{P_n(y)}$ generated by $P_n(y)$ on a self-similar
set $F_n$ with contraction ratios generated by $\{a_k, b_k\}_{k=1}^n$. Later, we see that it is a wrong conjecture.

**Lemma 3.** Let

$$G(P_n(z), P_n(y)) = \frac{\sum_{\tau \in \{1,2\}^n} \tau \log p_\tau}{\sum_{\tau \in \{1,2\}^n} \tau \log c_\tau}$$

with

$$r_{i_1, i_2, \ldots, i_n} = s_{i_1}^{(1)} s_{i_2}^{(2)} \cdots s_{i_n}^{(n)} \quad \text{where} \quad s_{i_k}^{(k)} = \begin{cases} r_k & \text{for } i_k = 1 \\ 1 - r_k & \text{for } i_k = 2 \end{cases},$$

$$p_{i_1, i_2, \ldots, i_n} = q_{i_1}^{(1)} q_{i_2}^{(2)} \cdots q_{i_n}^{(n)} \quad \text{where} \quad q_{i_k}^{(k)} = \begin{cases} p_k & \text{for } i_k = 1 \\ 1 - p_k & \text{for } i_k = 2 \end{cases},$$

and

$$c_{i_1, i_2, \ldots, i_n} = d_{i_1}^{(1)} d_{i_2}^{(2)} \cdots d_{i_n}^{(n)} \quad \text{where} \quad d_{i_k}^{(k)} = \begin{cases} a_k & \text{for } i_k = 1 \\ b_k & \text{for } i_k = 2 \end{cases},$$

then $G(P_n(z), P_n(y)) = g(P_n(z), P_n(y))$.

**Proof.** It is immediate from the cancelation. \qed

If $x = \sigma \in \{1, 2\}^N$, then we can express $x$ or $\sigma$ as for $x_{k,j} \in \{1, 2\}$

$$x = ((x_{1,1}, x_{2,1}, \ldots, x_{k,1}, \ldots, x_{n,1}), (x_{1,2}, x_{2,2}, \ldots, x_{k,2}, \ldots, x_{n,2}), \ldots) \in F_n,$$

which we call a quasi-generalized expansion of $x$ in $F_n$. We denote by $n_{i_1, i_2, \ldots, i_n}(x|m)$ the number of times the $n$-tuple $(i_1, i_2, \ldots, i_n)$ occurs in the first $m$ places of the quasi-generalized expansion of

$$x = ((x_{1,1}, x_{2,1}, \ldots, x_{k,1}, \ldots, x_{n,1}), (x_{1,2}, x_{2,2}, \ldots, x_{k,2}, \ldots, x_{n,2}), \ldots) \in F_n.$$

For each $i \in \{1, 2\}^n$ and $s_i \in [0, 1]$ we define a generalized distribution set $F_n(\{s_i\}_{i \in \{1,2\}^n})$ containing the finite code $i$ in proportion $\{s_i\}_{i \in \{1,2\}^n}$ by

$$F_n(\{s_i\}_{i \in \{1,2\}^n}) = \{x \in F_n : \lim_{m \to \infty} \frac{n_i(x|m)}{m} = s_i \text{ for each } i \in \{1, 2\}^n\}.$$

We denote by $n_1(x_k|m)$ the number of times the digit 1 occurs in the first $k$, $m$ places of the quasi-generalized expansion of

$$x = ((x_{1,1}, x_{2,1}, \ldots, x_{k,1}, \ldots, x_{n,1}), (x_{1,2}, x_{2,2}, \ldots, x_{k,2}, \ldots, x_{n,2}), \ldots) \in F_n.$$

For $(r_1, \ldots, r_n) \in [0, 1]^n$, we define a quasi-distribution set $F_n^*(r_1, \ldots, r_n)$ containing the digit 1 in proportion $(r_1, \ldots, r_n)$ by

$$F_n^*(r_1, \ldots, r_n) = \{x \in F_n : \lim_{m \to \infty} \frac{n_1(x_k|m)}{m} = r_k \text{ for each } 1 \leq k \leq n\}.$$
Lemma 4. For \( i = i_1, i_2, \ldots, i_n \) and 
\[
  s_i = s_{i_1}^{(1)} s_{i_2}^{(2)} \cdots s_{i_n}^{(n)} \quad \text{where} \quad s_{i_k}^{(k)} = \begin{cases} 
  r_k & \text{for } i_k = 1 \\
  1 - r_k & \text{for } i_k = 2 
\end{cases}.
\]
\[
\{ x \in F_n : \lim_{m \to \infty} \frac{n_i(x|m)}{m} = s_i \quad \text{for each } i \in \{1, 2\}^n \} 
\subset \{ x \in F_n : \lim_{m \to \infty} \frac{n_1(x_k|m)}{m} = r_k \quad \text{for each } 1 \leq k \leq n \}.
\]

Proof. For \( m \in \mathbb{N} \),
\[
\frac{n_1(x_k|m)}{m} = \sum_{i_k=1}^{n_{i_1,i_2,\ldots,i_k,\ldots,i_n}(x|m)}.
\]

We easily obtain it from the limit of each term. \(\square\)

Remark 3. In the above Proof, for \( n = 3 \),
\[
\frac{n_1(x_1|m)}{m} = \frac{n_{111}(x|m)}{m} + \frac{n_{112}(x|m)}{m} + \frac{n_{121}(x|m)}{m} + \frac{n_{122}(x|m)}{m}.
\]
\[
\frac{n_1(x_2|m)}{m} = \frac{n_{111}(x|m)}{m} + \frac{n_{112}(x|m)}{m} + \frac{n_{211}(x|m)}{m} + \frac{n_{212}(x|m)}{m},
\]
\[
\frac{n_1(x_3|m)}{m} = \frac{n_{111}(x|m)}{m} + \frac{n_{121}(x|m)}{m} + \frac{n_{211}(x|m)}{m} + \frac{n_{221}(x|m)}{m}.
\]

Remark 4. Since from the strong law of large numbers (cf. Lee & Baek [8])
\[
\gamma_{\{s_i\}_{i \in \{1, 2\}^n}}(\{ x \in F_n : \lim_{m \to \infty} \frac{n_i(x|m)}{m} = s_i \quad \text{for each } i \in \{1, 2\}^n \}) = 1,
\]
we see that
\[
\gamma_{\{s_i\}_{i \in \{1, 2\}^n}}(\{ x \in F_n : \lim_{m \to \infty} \frac{n_1(x_k|m)}{m} = r_k \quad \text{for each } 1 \leq k \leq n \}) = 1.
\]

By the notation in the Preliminaries, we see that a self-similar measure \( \gamma_{\{s_i\}_{i \in \{1, 2\}^n}} \) in the above is \( \gamma_{P_n(z)} \) where \( P_n(z) = (r_1, \ldots, r_n) \). From now on, we write a generalized distribution set
\[
\{ x \in F_n : \lim_{m \to \infty} \frac{n_1(x|m)}{m} = s_i \quad \text{for each } i \in \{1, 2\}^n \}
\]
containing the finite codes \( i \) in proportion \( s_i \) in the above Lemma as \( F_n(P_n(z)) \).

Theorem 5. Fix \( n \in \mathbb{N} \) and consider a self-similar set \( F_n \) with contraction ratios generated by \( \{a_k, b_k\}_{k=1}^n \). Let \( y \in (-\infty, \infty) \) and consider a self-similar measure \( \gamma_{P_n(y)} \) on \( F_n \) where \( P_n(y) = (p_1, \ldots, p_n) \) and \( p_k = \frac{a_k^y}{a_k^y + b_k^y} \) for \( 1 \leq k \leq n \). Let \( z \in [-\infty, \infty] \) and consider
\[
g(P_n(z), P_n(y)) = \frac{\sum_{k=1}^n (r_k \log p_k + (1 - r_k) \log (1 - p_k))}{\sum_{k=1}^n (r_k \log a_k + (1 - r_k) \log b_k)}.
\]
where $P_n(z) = (r_1, \ldots, r_n)$ and $r_k = \frac{a_k^m}{a_k^m + b_k^m}$ for $1 \leq k \leq n$. Then

$$F_n^*(P_n(z)) \subset E_{g(P_n(z),P_n(y))}^{P_n(y)}.$$  

Proof. Let $x \in F_n^*(P_n(z))$. Then

$$\lim_{m \to \infty} \frac{\log \gamma_{P_n(y)}(c_m(x))}{\log |c_m(x)|} = \lim_{m \to \infty} \frac{\sum_{k=1}^{n} \left( n_1(x_k|m) \log p_k + (m - n_1(x_k|m) \log(1 - p_k) \right)}{\sum_{k=1}^{n} \left( n_1(x_k|m) \log a_k + (m - n_1(x_k|m) \log b_k \right)} = \frac{\sum_{k=1}^{n} \left( r_k \log p_k + (1 - r_k) \log(1 - p_k) \right)}{\sum_{k=1}^{n} \left( r_k \log a_k + (1 - r_k) \log b_k \right)} = g(P_n(z), P_n(y)),$$

\[\square\]

Corollary 6. $F_n(P_n(z)) \subset F_n^*(P_n(z)) \subset E_{g(P_n(z),P_n(y))}^{P_n(y)}$ where $z \in \mathbb{R}$, and $F_n(P_n(z)) \subset F_n^*(P_n(z)) \subset E_{g(P_n(z),P_n(y))}^{P_n(y)}$ where $z \in [-\infty, \infty]$ and $y \in \mathbb{R}$.

Proof. It is immediate from Lemma 4 and the above Theorem. \[\square\]

Remark 5. From now on, we will not designate the ranges of $z$ and $y$ if there is no confusion. That is, if we consider $E_{g(P_n(z),P_n(y))}^{P_n(y)}$ then $z \in \mathbb{R}$ and if we consider $E_{g(P_n(z),P_n(y))}^{P_n(y)}$ then $y \in \mathbb{R}$ and $z \in [-\infty, \infty]$.

Theorem 7. $\gamma_{P_n(z)}(F_n^*(P_n(z))) = \gamma_{P_n(z)}(E_{g(P_n(z),P_n(y))}^{P_n(y)}) = 1$. Further,

$$\dim_H \left( F_n^*(P_n(z)) \right) = g(P_n(z), P_n(z)) = \dim_p \left( F_n^*(P_n(z)) \right) \text{ and } \dim_H \left( E_{g(P_n(z),P_n(y))}^{P_n(y)} \right) = g(P_n(z), P_n(z)) = \dim_p \left( E_{g(P_n(z),P_n(y))}^{P_n(y)} \right).$$

Proof. It follows from the above Remark. That is, $\gamma_{P_n(z)}(F_n^*(P_n(z))) = 1$ follows from $F_n(P_n(z)) \subset F_n^*(P_n(z))$ and $\gamma_{P_n(z)}(F_n(P_n(z))) = 1$ from the strong law of large numbers. Further,

$$\dim_H \left( F_n^*(P_n(z)) \right) = g(P_n(z), P_n(z)) = \dim_p \left( F_n^*(P_n(z)) \right)$$

follows from the above Corollary and Remark 1 in the Preliminaries. Similarly, by Proposition 1, we have

$$\dim_H \left( E_{g(P_n(z),P_n(y))}^{P_n(y)} \right) = g(P_n(z), P_n(z)) = \dim_p \left( E_{g(P_n(z),P_n(y))}^{P_n(y)} \right).$$

\[\square\]
Remark 6. \[
F_n(P_n(z)) = F^*_n(P_n(z)) = E^P_{g(P_n(z), P_n(y))}
\]
for \( n = 1 \) (cf. Baek [5]). So
\[
\dim_H \left( E^P_{g(P_n(z), P_n(y))} \right) = g(P_n(z), P_n(z)) = \dim_p \left( E^P_{g(P_n(z), P_n(y))} \right)
\]
for \( n = 1 \). However, from the above Corollary and Theorem, we just find that \( g(P_n(z), P_n(z)) \) is a lower bound for the dimensions of \( E^P_{g(P_n(z), P_n(y))} \).

Theorem 8. If \( s \) is a real number satisfying
\[
\prod_{k=1}^n (a_k^s + b_k^s) = 1,
\]
then \( g(P_n(s), P_n(s)) = s \). Further, \( E^P_{s} = F_n \) and \( \dim_H(F_n) = \dim_p(F_n) = s \).

Proof. Put
\[
r_k = \frac{a_k^s}{a_k^s + b_k^s} \quad \text{in} \quad g(r_1, \ldots, r_n, r_1, \ldots, r_n).
\]
Then we easily see that
\[
g(P_n(s), P_n(s)) = \frac{s \left( \sum_{k=1}^n (r_k \log a_k + (1 - r_k) \log b_k) \right) - \sum_{k=1}^n \log(a_k^s + b_k^s)}{\sum_{k=1}^n (r_k \log a_k + (1 - r_k) \log b_k)} = s.
\]

Further, by Lemma 2 in the Preliminaries we easily see that \( E^P_{s} = F_n \) and
\[
\dim_H \left( E^P_{g(P_n(s), P_n(s))} \right) = g(P_n(s), P_n(s)) = \dim_p \left( E^P_{g(P_n(s), P_n(s))} \right).
\]

Proposition 9. Let
\[
H(P_n(y)) = \frac{\sum_{r \in \{1,2\}^n} r_r \log p_r}{\sum_{r \in \{1,2\}^n} r_r \log c_r}
\]
with
\[
r_{i_1, \ldots, i_n} = s_{i_1}^{(1)} \cdots s_{i_n}^{(n)} \quad \text{where} \quad s_{i_k}^{(k)} = \begin{cases} p_k^q a_k^\beta(q) & \text{for } i_k = 1 \\ (1 - p_k)^q b_k^\beta(q) & \text{for } i_k = 2 \end{cases},
\]
\[
p_{i_1, i_2, \ldots, i_n} = q_{i_1}^{(1)} q_{i_2}^{(2)} \cdots q_{i_n}^{(n)} \quad \text{where} \quad q_{i_k}^{(k)} = \begin{cases} p_k & \text{for } i_k = 1 \\ 1 - p_k & \text{for } i_k = 2 \end{cases},
\]
and \( c_{i_1, i_2, \ldots, i_n} = d_{i_1}^{(1)} d_{i_2}^{(2)} \cdots d_{i_n}^{(n)} \quad \text{where} \quad d_{i_k}^{(k)} = \begin{cases} a_k & \text{for } i_k = 1 \\ b_k & \text{for } i_k = 2 \end{cases} \).
Then the solution $q$ satisfying

$$H(P_n(y)) = \alpha \text{ and } \prod_{k=1}^{n} \left( p_k q a_k^{\beta(q)} + (1 - p_k) q b_k^{\beta(q)} \right) = 1$$

gives $\alpha q + \beta(q)$ as the dimensions of $E_{\alpha}^{P_n(y)} \subset F_n$.

**Proof.** It is immediate from (11.30) and (11.35) in Falconer [7].

**Remark 7.** In the above Proposition, if $q = 1$ then $\beta(q) = 0$ in the equation

$$\prod_{k=1}^{n} \left( p_k q a_k^{\beta(q)} + (1 - p_k) q b_k^{\beta(q)} \right) = 1.$$

Further, for $q = 1$ let $\alpha = H(P_n(y))$. Then by the above Proposition the dimensions of $E_{\alpha}^{P_n(y)}$ are $\alpha$. By the way, $\alpha = H(P_n(y)) = G(P_n(y), P_n(y)) = g(P_n(y), P_n(y))$ from Lemma 3. By the Theorem 7, we also see that the dimensions of $E_{\alpha}^{P_n(y)}$ are $g(P_n(y), P_n(y)) = \alpha$.

**Theorem 10.**

For $n = 1$, $E_{\alpha}^{P_n(y)} = E_{\alpha}^{P_n(z)}$.

For $n \geq 2$, in general, $E_{\alpha}^{P_n(y)} \neq E_{\alpha}^{P_n(z)}$.

Further, $\dim_H(E_{\alpha}^{P_n(y)}(P_n(z), P_n(y))) = \dim_p(E_{\alpha}^{P_n(y)}(P_n(z), P_n(y))) \geq g(P_n(z), P_n(z))$.

**Proof.** For $n = 1$, it follows from Baek [5]. For $n \geq 2$, it is immediate from the above Proposition and Lemma 3.

$$\dim_H(E_{\alpha}^{P_n(y)}(P_n(z), P_n(y))) = \dim_p(E_{\alpha}^{P_n(y)}(P_n(z), P_n(y))) \geq g(P_n(z), P_n(z))$$

follows from Remark 6.

**Remark 8.** In the above Proof, for $n \geq 2$ we cannot guarantee that

$$p_k q a_k^{\beta(q)} + (1 - p_k) q b_k^{\beta(q)} = 1$$

for each $1 \leq k \leq n$ in the above Proposition whereas $r_k + (1 - r_k) = 1$ for each $1 \leq k \leq n$ in Lemma 3. However, if we guarantee it,

$$\alpha = H(P_n(y)) = G(P_n(z), P_n(y)) = g(P_n(z), P_n(y))$$

where $P_n(z) = \{p_k q a_k^{\beta(q)}\}_{k=1}^{n}$ from Lemma 3. Then we easily see that

$$g(P_n(z), P_n(z)) = G(P_n(z), P_n(z)) = \alpha q + \beta(q),$$
which is the dimensions of $E_{g(P_n(z),P_n(y))}^{P_n(y)}$. But we know
\[ \dim_H \left( E_{g(P_n(z),P_n(z))}^{P_n(z)} \right) = g(P_n(z),P_n(z)) = \dim_p \left( E_{g(P_n(z),P_n(z))}^{P_n(z)} \right) \]
from Theorem 7. This gives many examples for $E_{g(P_n(z),P_n(y))}^{P_n(y)} \neq E_{g(P_n(z),P_n(z))}^{P_n(z)}$ for $n \geq 2$. But for $n = 1$, letting
\[ p_1^q a_1^{\beta(q)} + (1 - p_1)^q b_1^{\beta(q)} = 1 \quad \text{and} \]
\[ r_1 = p_1^q a_1^{\beta(q)} \quad \text{and} \]
\[ r_2 = 1 - r_1 = (1 - p_1)^q b_1^{\beta(q)} \]
in Lemma 3, we have $P_1(z) = r_1$ and $g(r_1, r_1) = \alpha q + \beta(q)$. Precisely, the solution $q$ satisfying
\[ H(P_1(y)) = H(p_1) = \alpha \quad \text{and} \quad \prod_{k=1}^{1} \left( p_k^q a_k^{\beta(q)} + (1 - p_k)^q b_k^{\beta(q)} \right) = 1 \]
gives $r_1 = p_1^q a_1^{\beta(q)}$ and $g(r_1, r_1) = \alpha q + \beta(q)$. Further, we see that
\[ g(P_n(z), P_n(y)) \geq g(P_n(z), P_n(z)) \]
from the Lagrange multiplier theorem. However, we also see it from the Proposition 1 and the above theorem, that is
\[ g(P_n(z), P_n(z)) \leq \dim_H \left( E_{g(P_n(z),P_n(y))}^{P_n(y)} \right) \leq g(P_n(z), P_n(y)). \]

**Theorem 11.** Let $s$ be a real number satisfying
\[ \prod_{k=1}^{n} (a_k^s + b_k^s) = 1 \quad \text{and let} \quad z \in [-\infty, \infty]. \]

Then for any $y \neq y'$ in $\mathbb{R}$,
\[ \text{for } n = 1, \quad E_{g(P_n(z),P_n(y))}^{P_n(y)} = E_{g(P_n(z),P_n(y'))}^{P_n(y')} \quad \text{if } y \neq s, \]
\[ \text{for } n \geq 2, \text{ we cannot guarantee} \]
\[ E_{g(P_n(z),P_n(y))}^{P_n(y)} = E_{g(P_n(z),P_n(y'))}^{P_n(y')} \quad \text{if } y \neq s. \]

**Proof.** It is immediate from the above Theorem and Baek [5]. \qed
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