*KSME International Journal, Vol. 18 No. 6, pp. 933~945, 2004* 

# Eigenstructure Assignment for Linear Systems with Probabilistic Uncertainties

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In this paper, S(stochastic)-eigenvalue concept and its S-eigenvector for linear continuous-time systems with probabilistic uncertainties is proposed. The proposed concept is concerned with the perturbation of eigenvalues due to the probabilistic variable parameters in the dynamic model of a plant. S-eigenstructure assignment scheme via the Sylvester equation approach based on the S-eigenvalue concept is also proposed. The proposed design schemes are illustrated by numerical examples, and applied to the longitudinal dynamics of open-loop-unstable aircraft with possible uncertainties in aerodynamic and thrust effects as well as separate dynamic pressure. These results explicitly characterize how S-eigenvalues in the complex plane may impose stability on S-eigenstructure assignment.

**Key Words :** Probabilistic Parameter Variation, Gaussian Distribution, S-Eigenvalue, S-Eigenvector, S-Stability, S-Eigenstructure Assignment

## 1. Introduction

In recent years, eigenstructure assignment has been applied to the design of various kinds of practical multivariable control systems, e.g., helicopters, aircraft, missiles, generator, voltage regulators and mechanical systems (Liu and Patton, 1998). The specified effect of eigenstructure assignment is achieved by assigning a certain set of eigenvalues and an associated set of eigenvectors to the closed-loop system. In general, the speed of response is determined by the assigned eigenvalues whereas the shape of the response is furnished by the assigned eigenvectors (Fahmy and Tantaway, 1984). Eigenstructure assignment is well-suited for incorporating the classical specifications on damping, settling time, and mode or disturbance decoupling into a modern multivariable control framework (Sobel and Lallman, 1989), and has been shown to be a useful tool for flight control design (Sobel et al., 1994). The eigenstructure assignment technique is used to design flight control laws for aircraft with many control efforts, and the technique together with suitable feedforward design can achieve static decoupling with internal stability, which is an important requirement in many flight control system (Lin, 1994; Sobel and Shapiro, 1985a; Sobel and Shapiro, 1985b).

In direct eigenstructure assignment techniques, the design parameters are the desired closed-loop eigenvalues and specified elements of the closed-loop eigenvectors. Once the design parameters are specified, the feedback control gains are uniquely determined. Therefore, given a set of specifications, the feedback control gains will provide the desired closed-loop transient res-

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ponse (or come as close to it as possible within the system constraints), but they might result in a system with poor stability robustness (Garg, 1991), i.e., a small change in the plant dynamics may cause the closed-loop system to go unstable. The designer is then faced with the dilemma of how to change the design specifications such that the resulting feedback system will also provide adequate stability robustness. Note that, in general, the designer does have a certain amount of freedom in choosing the design specifications. The designer rarely wants an exact value for a closed-loop eigenvalue or exact shape for a corresponding eigenvector. The specifications are rather in terms of desired regions for the closed-loop eigenvalues and acceptable sets of eigenvector shapes. The general eigenstructure assignment methodologies cannot guarantee stability robustness to parameter variations of a system. This problem is still unsolved, thus it is worthwhile to explore the extension of the conventional robust right eigenstructure assignment technique to the right and left eigenstructure assignment schemes.

The class of parameter perturbations which do not lend themselves to exact modeling is an important consideration in the design of control systems. These perturbations may be caused by aging of components, changes in environmental conditions, calibration errors, or by using an inexact but simpler model in design. To maintain at least stability of the overall system is desirable under such conditions. The two most popular approaches to deal with this problem are the robust control (Dorato, 1987) and random-parameter control techniques (Mohler and Kolodziej, 1980). Robust control uses a characterization of deterministic but unknown perturbations in terms of upper bounds on their allowable values. The degree of tolerance of a control system design to bounded but unknown perturbations has attracted the attention of many researchers, basically two approaches to the problem have emerged : frequency domain techniques which use the singular value decomposition and time domain techniques which use Lyapunov stability criteria (Anderson and Moore, 1971 ; Lethomaki et al., 1981 ; Patel et al, 1977 ; Safonov and Athans, 1977 ; Yedavalli et. al., 1985 ; Zhou and Khargonekar, 1987). Most of these works are concerned with the robustness of systems controlled by the infinite horizon linear quadratic regulator (Choi and Seo, 1999 ; Seo and Choi, 2003). An alternative way of treating the parameter perturbations is to use multiplicative noises or probabilistic parameters in the model which worked well in several cases such as modeling human operators in a control task (Levioson et al., 1968), control of macroeconomic models (Aoki, 1984), and modeling rounding errors of a digital computer used in control loop (Wingerden and Dekoning, 1984). Control system robustness is defined as the ability to maintain satisfactory stability or performance characteristics in the presence of all conceivable system parameter variations. While assured robustness may be viewed as an alternative to gain adaptation or scheduling to accommodate known parameter variations, more often it is seen as protection against uncertainties in plant specification. Consequently, a statistical description of control system robustness is consistent with what may be known about the structure and parameters of the plant's dynamic model (Stengel et al., 1991).

With the help of computers, singular-value analysis has extended the frequency-domain approach to multiinput/multioutput systems (Lethomaki et at., 1981 ; Sandell, 1979) ; however, guaranteed stability-bound estimates often are unduly conservative, and the relationship to parameter variations in the physical system is weak. Structured singular-value analysis (Doyle, 1982) reduces this conservatism somewhat, and alternate treatments of structured parameter variations have been proposed (Horowitz, 1982 ; Tahk and Speyer, 1987 ; Yedavalli and Liang, 1986), although these approaches remain deterministic. Reference (Yaz, 1988) uses the term "stochastic robustness" to describe a stability bound based on Lyapunov methods and parameter perturbations that are modeled as probabilistic sequences. This is a deterministic stability bound expressed in terms of the norm of a vector of noise variances. The notation of "probability of instability" was introduced in (Stengel, 1980) with application to the robustness of the Space Shuttle's flight control system. This method determines by the probability distributions of closed-loop eigenvalues, given the statistics of the variable parameters in the dynamic model of plant (Ray, 1992; Stengel and Ray, 1991). The probability that all these eigenvalues lie in the open left-half $s$ plane is the scalar measure of robustness. However, this takes no account of the effect of probabilistic information on parameter uncertainty. It is merely the deterministic evaluation of by Monte Carlo simulation. Thus, further study is required on the methods that deal with the effect of probabilistic information on parameter uncertainty stochastically.

In this paper, first, a novel eigenvalue and its corresponding eigenvector concept for linear systems with probabilistic uncertainties is also proposed. The proposed concept is concerned with the perturbation of eigenvalues due to the probabilistic variable parameters in the plant's dynamic model. The probability that all eigenvalues lie in the open left-half $s$ plane is the scalar measure of robustness. Second, the stability based on the proposed concept is presented on the appropriate random characteristics of perturbations to maintain the proper stability behavior of the overall system. Third, S-eigenstructure assignment scheme via a Sylvester equation approach based on the S-eigenvalue concept is proposed. The proposed design schemes are illustrated by numerical examples, and applied to the longitudinal dynamics of open-loop-unstable aircraft with possible uncertainties in aerodynamic and thrust effects as well as separate dynamic pressure. These results explicitly characterize how S-eigenvalues in the complex plane may impose stability on S-eigenstructure assignment.

## 2. S-Eigenvalue/Eigenvector

By $M_n(\mathbf{F})$ we denote the $n$-by-$n$ matrices over a field $\mathbf{F}$, usually the real numbers $\mathbf{R}$ or the complex numbers $\mathbf{C}$. Also the set (vector space) of all real-entried (respectively complex-entried) $n$ vectors is denoted by $\mathbf{R}^n$ (respectively $\mathbf{C}^n$, both interpreted as column vectors (Horn and Johnson, 1985). All matrices are with compatible dimensions if they are not explicitly stated.

Consider the linear time-invariant system with probabilistic parameters as follows:

$$\delta x = F(p)x + G(p)u \tag{1}$$

$$u = u_c - K(p)x \tag{2}$$

where $\delta x$ represents $\dot{x}(t)$ for continuous systems and $x(t+1)$ for discrete systems, $x \in \mathbf{R}^n$ the state vector, $u \in \mathbf{R}^m$ the control input vector. The matrices $F(p) \in \mathbf{R}^{n \times n}$ and $G(p) \in \mathbf{R}^{n \times m}$ are system and input matrices that may be Gaussian random parameter, $p$. $u_c$ is a command input vector, and, for simplicity, the $(m \times n)$ control gain matrix, $K(p)$, is assumed to be known. A linear state-feedback control law (2) is applied to the continuous system (1), then the closed-loop system representation is given by

$$\begin{aligned} \dot{x}(t) &= (F(p) - G(p)K(p))x(t) \\ &\triangleq A(p)x(t) \end{aligned} \tag{3}$$

where, the $n$ eigenvalues, $\rho_i(p)$, of the matrix $[F(p) - G(p)K(p)]$ determine closed-loop stability and can be determined as the roots of the determinant equation

$$\det(sI_n - [F(p) - G(p)K(p)]) = 0 \tag{4}$$

The $n$ eigenvalues of the closed-loop system can be represented as the mean of eigenvalues plus the perturbation terms, respectively.

$$\rho_i(p) = \rho_i^E + \tilde{\rho}_i(p), \; i = 1, \cdots, n \tag{5}$$

where, $\rho_i^E = E(\rho_i(p))$ and $\tilde{\rho}_i(p)$ denote the mean of $i$-th eigenvalue and the variation from the mean of $i$-th eigenvalue, respectively. The perturbation term in Eq. (5) reflect the eigenvalues variation due to the probabilistic parameter variation of the system matrix. Because root loci for individual parameter variations would follow classical configurations of root locus construction, with the heaviest density of roots in the vicinities of the nominal roots, the probabilistic distribution of eigenvalues may be assumed to be the Gaussian distribution. The basic justification of this statement is embodied in the central limit theorem: one of its numerous precise statements (differing in specific assumptions and details, but all essentially the same) is now stated.

**Theorem 1. Central Limit Theorem** (Maybeck, 1979)

Let $\{ \rho_i^k(p) \mid i=1, \cdots, n \}$ be a set of eigenvalues that are calculated from the production set, $\{ A^k(p) \}_{k=1}^N$, of a linear time-invariant system with probabilistic parameter variations on a large scale. Also, let each element of $\{ \tilde{\rho}_i^k(p) = \rho_i^k(p) - \rho_i^E \}_{k=1}^N$ be a $n$-vector which are identically distributed with means and covariance matrices $m^k$ and $P^k$, respectively. Define the random vector $y_N$ as their sum:

$$y_N = \sum_{i=1}^{N} \tilde{\rho}_i^k(p)$$

and also define $\tilde{\rho}_i(p)$ as the (zero-mean) normalized sum random variable:

$$\tilde{\rho}_i(p) = [P_{y_N y_N}]^{-1/2}[y_N - E[y_N]]$$

where

$$E[y_N] = \sum_{k=1}^{N} m^k,\ P_{y_N y_N} = \sum_{k=1}^{N} P^K,$$
$$\text{and } P^{-1/2} = (P^{1/2})^{-1}$$

where $P^{1/2}$ is defined as the $n$-by-$n$ matrix such that $P^{1/2}(P^{1/2})^T$. Then, in the limit as $N \to \infty$, $\tilde{\rho}_i(p)$ becomes a zero-mean Gaussian random $n$-vector with a covariance matrix equal to identity matrix:

$$\lim_{N \to \infty} f_{\tilde{\rho}_i(p)}(\zeta) = [(2\pi)^{n/2}] \exp\left\{ -\frac{1}{2} \zeta^T \zeta \right\}$$

The proof is going to show that $\tilde{\rho}_i(p)$ converges in distribution to a random variable having a standard normal distribution by showing that the moment generating function of $\tilde{\rho}_i(p)$ converges to the moment generating function of the standard normal distribution. The theorem states that if the eigenvalues are generated as the sum of eigenvalues of many identical system, the probabilistic distribution of eigenvalues (5) approaches a Gaussian distribution as more eigenvalues are summed. The eigenvalue-eigenvector equation of the closed-loop system (3) with Gaussian distribution eigenvalues can be defined as follows:

**Definition 1**. Let $A(p) \in M_n$ and $\phi(p) \in \mathbf{C}^n$. Consider

$$A(p)\phi(p) = \rho(p)\phi(p),\ \phi(p) \neq 0 \tag{6}$$

where $\rho(p)$ is a scalar. If a scalar $\rho(p)$ and a nonzero vector $\phi(p)$ happen to satisfy this equation, then $\rho(p)$ is called an "S-eigenvalue" of $A(p)$ and $\phi(p)$ is called an "S-eigenvector" of $A(p)$ associated with $\rho(p)$. Notice the two occur inextricably as a pair, and that an S-eigenvector cannot be the zero vector.

Suppose that $n$-eigenvalues can be plotted $l$ times for time interval $[t_0, t_f]$ on the complex plane, then $n \times l$-eigenvalues may be plotted on the complex plane. If all of $n \times l$-eigenvalues lie in the left-half $s$ plane, then the stability of closed-loop system is guaranteed. But, if some eigenvalues lie in the right-half $s$ plane, then closed-loop system has the probability of instability. Thus, the probability that all of these eigenvalues lie in the left-half $s$ plane is the scalar measure of robustness-stability. The density of these eigenvalues depicts the likelihood that eigenvalues vary from their mean values, this can be gained by plotting the probability density function corresponding eigenvalue on a three-dimensional complex plane. From Theorem 1, the probability density function of S-eigenvalue is assumed to be the Gaussian distribution as follows:

**Definition 2**. Let $A(p) \in M_n$ be a closed-loop system. For $A(p)$, the probability density function corresponding to the S-eigenvalue in a complex plane $\mathbf{C}$ is defined by:

- Case 1: Complex conjugate eigenvalue

$$\begin{aligned} \mathrm{pr}(\rho(p)) &= \frac{1}{2\pi |\Sigma|^{1/2}} \exp\left[ -\frac{1}{2}(P(p) - P^E)^T \Sigma^{-1} (P(p) - P^E) \right] \\ &= \frac{1}{2\pi |\Sigma|^{1/2}} \exp\left[ -\frac{1}{2} \tilde{P}^T(p) \Sigma^{-1} \tilde{P}(p) \right] \end{aligned} \tag{7}$$

- Case 2: Real eigenvalue

$$\mathrm{pr}(\sigma(p)) = \frac{1}{\sqrt{2\pi}\sigma_{\tilde{\sigma}}} \exp\left[ \frac{1}{2\sigma_{\tilde{\sigma}}^2} (\sigma(p) - E[\sigma(p)])^2 \right] \tag{8}$$

with $P^T(p)=[\tilde{\sigma}(p) \;\; \tilde{\omega}(p)]^T$ and $\Sigma=\mathrm{diag}(\sigma_{\tilde{\sigma}}^2, \sigma_{\tilde{\omega}}^2)$, where $\Sigma$ is a positive $(n\times n)$ matrix, $|\cdot|$ denotes the determinant of a matrix, and $\exp[\cdot]$ denotes exponential. The quantities

$$P^E=E[P(p)]=E\begin{bmatrix}\sigma(p)\\ \omega(p)\end{bmatrix}$$

and

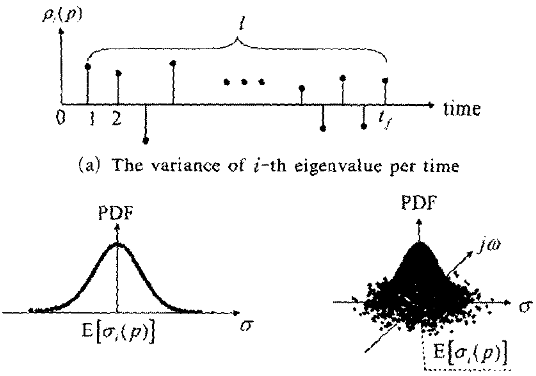
$$\Sigma=E[(P(p)-P^E)(P(p)-P^E)^T]$$

are the mean and covariance of the vector $P(p)$, respectively.

Figure 1 shows the probabilistic distribution of the $i$-th S-eigenvalue. The variation of $i$-th eigenvalue per time (Fig. 1(a)) can be represented to the concept as the probability density function on a three-dimensional complex plane in either case (real eigenvalue (Fig. 1(b)), complex conjugate eigenvalue (Fig. 1(c)), respectively.

Next, define the spectrum corresponding to S-eigenvalue as following:

**Definition 3.** The set of all $\rho(p)\in\mathbf{C}$ that are S-eigenvalues of $A(p)\in M_n$ is called the "S-spectrum (stochastic-spectrum)" of $A(p)$ and is denoted by $\sigma(A(p))$. The spectral radius of $A(p)$ is the non-negative real number $\rho(A)=\max\{|\rho(t)| : \rho(p)\in\sigma(A(p))\}$. This is the radius of the smallest disc centered at the origin in the complex plane that includes all the S-eigenvalues of $A(p)$.




(a) The variance of $i$-th eigenvalue per time

(b) Real eigenvalue (c) Complex conjugate eigenvalue

**Fig. 1** The probabilistic distribution of the S-eigenvalue

And, the S-eigenvector corresponding to S-eigenvalue can be stated as following theorem:

**Theorem 2.** Let $A(p)\in M_n$. For a given $\rho(p)\in(A(p))$, the set of all S-eigenvectors $\phi_i(p)\in\mathbf{C}^n$ satisfying $A_i(p)\phi_i(p)=\rho_i(p)\phi_i(p)$ is called the "S-eigenspace (stochastic-eigenspace)" of $A(p)$ corresponding to the S-eigenvalues.

$$\phi_i(p)=\gamma_i(p)\phi_i^E \tag{9}$$

where $\gamma_i(p)=\{I+(\rho_i(p)I-A(p))^{-1}(\tilde{A}(p)-\tilde{\rho}_i(p)I)\}$, $\phi_i^E=\phi_i(p)-\tilde{\phi}_i(p)$, $\phi_i^E=E[\phi_i(p)]$, and $\mathrm{VAR}[\tilde{\phi}_i(p)]=\phi_i^E(\phi_i^E)^T$.

**Proof**: Substitute $\rho_i(p)=\rho_i^E+\tilde{\rho}_i(p)$, $\phi_i(p)=\phi_i^E+\tilde{\phi}_i(p)$, and $A(p)=A^E+\tilde{A}(p)$ for Eq. (6),

$$(\rho_i^E I+\tilde{\rho}_i(p)I-A^E-\tilde{A}(p))(\phi_i^E+\tilde{\phi}_i(p))=0$$

and

$$(\rho_i^E I-A^E)\phi_i^E+(\tilde{\rho}_i(p)I-\tilde{A}(p))\phi_i^E + (\rho_i^E I+\tilde{\rho}_i(p)I-A^E-\tilde{A}(p))\tilde{\phi}_i(p)=0$$

From the characteristics of deterministic eigenvalue problem, $(\rho_i^E I-A^E)\phi_i^E=0$. The above equation is classified in terms of $\tilde{\phi}_i(p)$ as follows:

$$\tilde{\phi}_i(p)=(\rho_i(p)I-A(p))^{-1}(\tilde{A}(p)-\tilde{\rho}_i(p)I)\phi_i^E$$

Because of $\phi_i(p)=\phi_i^E+\tilde{\phi}_i(p)$, substitute $\{I+(\rho_i(p)I-A(p))^{-1}(\tilde{A}(p)-\tilde{\rho}_i(p)I)\}$ for $\gamma_i(p)$, then Eq. (9) can be obtained. Also, In such a case, the mean and covariance of S-eigenvector can be easily shown as follows:

$$E[\phi_i(p)]=\phi_i^E$$

$$\begin{aligned}\mathrm{VAR}\,[\tilde{\phi}_i(p)]&=E[\phi_i(p)\phi_i^T(p)]\\&=E[\gamma_i(p)\phi_i^E(\gamma_i(p)\phi_i^E)^T]\\&=\phi_i^E(\phi_i^E)^T\end{aligned}$$

The S-eigenvector to the mean eigenvector may be represented as the following absolute misalignment angle.

$$\theta_i(p)=\cos^{-1}\left(\frac{|\phi_i^T(p)\phi_i^E|}{\|\phi_i(p)\|_2\|\phi_i^E\|_2}\right) \tag{10}$$

where, $\theta_i(p)$ is a linear operator (Jonh, 1999) and geometrically identical with $\gamma_i(p)$ of Eq. (9). Theorem states that S-eigenvector rotating about

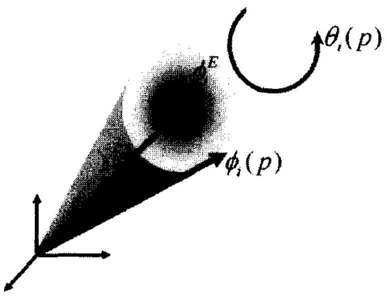



**Fig. 2** A geometric illustration of S-eigenvector on three-dimensional eigenspace

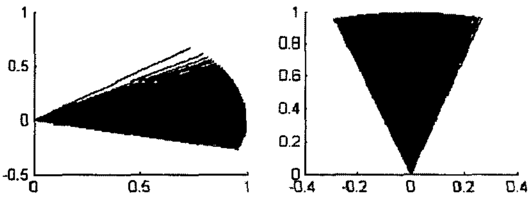



**Fig. 3** Eigenvector distribution corresponding to real eigenvalue of second order system

a fixed, arbitrary nominal eigenvector. The rotating vector in three-dimensional space sweeps out the surface of a cone centered on the axis rotation geometrically as depicts Fig. 2. For example, if the identity matrix is chosen for the nominal eigenvector in two-dimensional space, the eigenvector has a sector distribution in Fig. 3.

The modal matrix of S-eigenvector can be defined as follows:

**Definition 4**. A matrix $\Phi(p) \in \mathbf{C}^{n\times n}$ is called a "S-modal (stochastic-modal) matrix" of $A(p) \in M_n$ corresponding to $\rho(p) \in \sigma(A(p))$ if:

$$\Phi(p) = \Gamma(p)\,\Phi^E$$

where

$$\Gamma(p) = [\gamma_1(p)\ \ \gamma_2(p)\ \ \cdots\ \ \gamma_i(p)\ \ \cdots\ \ \gamma_n(p)]$$

$$\gamma_i(p) = \{I + (\rho_i(p)I - A(p)^{-1}(\tilde{A}(p) - \tilde{\rho}_i(p)I)\}$$

and

$$\Phi^E = \Phi(p) - \tilde{\Phi}(p)$$

## 3. S-Stability

It is well known that an LTI system is asymptotically stable if, and only if, deterministic eigenvalues are in the LHP of $\mathbf{C}$. If all of eigenvalues lie in the left-half $s$ plane, then the LHP stability of closed-loop system is guaranteed. But, the S-eigenvalue with the probabilistic distribution does not guaranteed the deterministic LHP stability criterion directly, because S-eigenvalues vary with the probabilistic uncertainty. First, in order to deal with probabilistic stability, define S-mean (stochastic-mean) of the S-eigenvalues as follows:

**Definition 5**. Let $\sigma(p)$ be a real variable of S-eigenvalues on the complex plane $\mathbf{C}$. The "S-mean (stochastic-mean)", $\mathrm{sm}(\sigma(p))$, over $\mathbf{C}$ is defined by:

$$\mathrm{sm}(\sigma(p)) = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} \sigma(p)\,dp \qquad (11)$$

where $\sigma(p) = \sigma^E + \tilde{\sigma}(p)$, $\sigma^E = E[\sigma(p)]$, and stochastic integrals of $\tilde{\sigma}(p)$ existed.

Definition 5 is defined to use that the real value of the eigenvalue determines the stability of the plant. The density of these eigenvalues depicts the likelihood that eigenvalues vary from their mean values, these means have only to exist on the left-half plane at least. Next, the probability of stability of the S-eigenvalue is defined as follows:

**Definition 6**. Let $\mathrm{pr}(\rho(p))$ be a given PDF corresponding to an S-eigenvalue in a complex plane $\mathbf{C}$. If $\mathrm{pr}(\rho(p))$ be a stochastic integrable function on the LHP $(-\infty, 0]$, then the "probability of stability" of LTI stochastic systems is defined by:

$$\mathrm{S} = \int_{-\infty}^{0} \mathrm{pr}(\rho(p))\,dp \qquad (12)$$

where $0 \le \mathrm{S} \le 1$. Notice that the probability of stability in the ergodic sense is given by:

$$\mathrm{S} = \lim_{J\to\infty} \frac{1}{Jn}\sum_{j=1}^{\infty} N(\sigma(p) \le 0) \qquad (13)$$

where $N(\cdot)$ is the number of cases for which all elements of $(\cdot)$ are less than or equal to zero, $n$

is the dimension of the system, and $J$ is the number of Monte Carlo evaluation.

Using definitions 5 and 6, the S-stability criterion based on the S-eigenvalue can be stated by the following theorem.

**Theorem 2** Let $\rho(p)$ be an S-eigenvalue of $A(p)$. Then the solution to $\det(\rho_i(p)I - A(p)) = 0$ is stochastically stable for all $t$ if and only if:

i) there exists $0 < \sigma_i^E \leq \infty$ such that

$$\mathrm{sm}\ (\sigma_i(p)) = -\sigma_i^E < 0$$

and moreover,

ii) there exists $\varepsilon > 0$ such that

$$\mathrm{P}\{\, |1 - S| > \varepsilon \} \rightarrow 0$$

as $J \rightarrow \infty$ for $\forall t,\ t \geq T_0$.

**Proof**: Condition i) states that the mean of the real value of eigenvalue is less than arbitrary negative value $-\sigma_i^E$. Thus, the core axis of the Gaussian distribution is located on the left-half plane at least. But, though condition i) is guaranteed by itself, the probability of instability remains still. In order to guarantee the stochastic stable, the probability of the instability approaches a zero in the process of repeating the simulations as depict in condition ii).

**Theorem 3** The sequence of random variables $\{X_n(\xi)\}$ converges in probability (Leon-Garcia, 1994) to the random variables $\{X(\xi)\}$ if, for any $\varepsilon > 0$:

$$\mathrm{P}[X_n(\xi) - X(\xi) > \varepsilon] \rightarrow 0 \text{ as } n \rightarrow \infty$$

**Example 1**. Considered a linearized second-order continuous controllable system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2\xi\omega_n & -\omega_n^2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix} u(t)$$

Suppose that the damping ratio ($\xi$) and natural frequency ($\omega_n$) are nominally 0.707 and 1, respectively, and that each may be a Gaussian-distributed random variable with standard deviation of 0.2. Both $\xi$ and $\omega$ uncertain and uncorrelated (i.e., $p = [\xi\ \omega_n]^T$), and It cause the open-loop system to unstable stochastically. The given system have two eigenvalues per time, these eigenvalues are represented to the combination of Gasussian distribution by the interrelation $\xi$ and $\omega_n$. The mean of the open-loop eigenvalues can be obtained by the final time $t_f = 5000$ as follows:

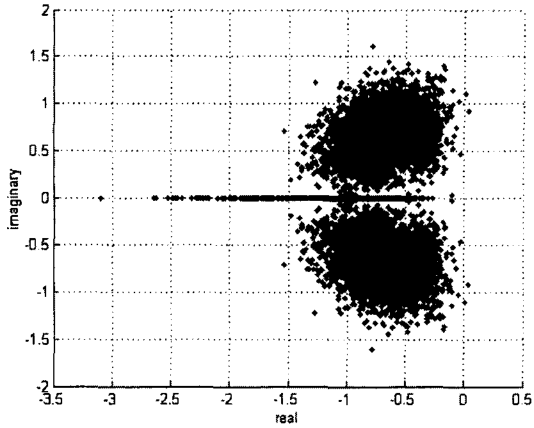



**Fig. 4** Open-loop eigenvalue distribution of Example 1 on two-dimensional complex plane

$$\rho_1 = -0.707 + 0.7072i$$

$$\rho_2 = -0.707 - 0.7072i$$

The probability of stability is $S = 0.96$. The mean of the open-loop eigenvalue are located on the left-half plane. Also, $P_{cip}$ approaches to the zero in the process of repeating the simulation. Thus, the given system is stochastically stable in the sense of theorem 3. The open-loop eigenvalue distribution of Example 1 on two-dimensional complex plane is shown in Fig. 4.

## 4. Stochastic-Eigenstructure Assignment

The specified effect of eigenstructure assignment is achieved by assigning a certain set of eigenvalues and an associated set of eigenvectors to the closed-loop system. First, from the previous definitions, the required S-eigenvalues could be established as follows:

$$\rho_i^d(p) = (\rho_i^E)^d + \tilde{\rho}_i^d(p)$$

where $E[\rho_i^d(p)]=(\rho_i^E)^d$, $E[(\rho_i^d(p)-(\rho_i^E)^d)(\rho_i^d(p)-(\rho_i^E)^d)^T]=(\sigma_i^d)^2$. And, if the mean of the required modal matrix, $(\Phi^E)^d$, is determined, the required S-modal matrix corresponding to the required S-eivenvalues could be established as follows :

$$\Phi^d(p)=\Gamma^d(p)(\Phi^E)^d$$

where

$$\Gamma^d(p)=[\gamma_1^d(p)\ \gamma_2^d()p \cdots \gamma_i^d(p) \cdots \gamma_n^d(p)]$$

$$\gamma_i^d(p)=\{I+(\rho_i^d(p)I-F(p))^{-1}(\tilde{F}(p)-\tilde{\rho}_i^d(p)I)\}$$

Our objective is to find the feedback-gain matrix $K(p)$ such that the closed-loop S-eigenvalues are obtained exactly, and that the required S-eigenvectors are assigned to the best possible set of eigenvectors with consistency of statistical fitting procedures.

**Theorem 4**. For a given set of $F(p)$, $G(p)$ matrices, which are LTI stochastic systems with inputs, and for a S-eigenvalues matrix $\Lambda(p)=\Lambda^E+\tilde{\Lambda}(p)$ and S-modal matrix $\Phi(p)=\Gamma(p)\Phi^E$, a parameter matrix $H(p)$ could be chosen by the following equation :

$$\begin{aligned}&\Lambda^E\Gamma(p)\Phi^E+\tilde{\Lambda}(p)\Gamma(p)\Phi^E\\&-\Gamma(p)\Phi^E F(p)=G(p)H(p)\end{aligned} \tag{14}$$

where $\Phi^E=[\phi_1^E\ \phi_2^E \cdots \phi_n^E]$, $\Lambda^E=\mathrm{diag}(\rho_1^E, \rho_2^E, \cdots, \rho_n^E)$, $\tilde{\Lambda}(p)=\mathrm{diag}(\tilde{\rho}_1(p), \tilde{\rho}_2(p), \cdots, \tilde{\rho}_n(p))$ and $H(p)=[h_1(p)\ h_2(p) \cdots h_n(p)]$.

**Proof**. If a state feedback $u(t)=-K(p)x(t)$ is applied to $\dot{x}(t)=F(p)x(t)+G(p)u(t)$, the closed-loop system becomes $\dot{x}(t)=(F(p)-G(p)K(p))x(t)$. The corresponding right S-eigenvalue problem is then defined by :

$$(F(p)-G(p)K(p))\phi_i(p)=\rho_i(p)\phi_i(p) \tag{15}$$

where $\phi_i(p)$ is the right S-eigenvector corresponding to the S-eigenvalue $\rho_i(p)$. The parameter vector $h_i(p)\in\mathbf{C}^m$ is defined by :

$$h_i(p)=K(p)\phi_i(p) \tag{16}$$

Then, Eq. (15) is put in the form of the Sylvester equation :

$$(F(p)-\rho_i(p)I)\phi_i(p)=G(p)h_i(p) \tag{17}$$

or :

$$\begin{aligned}&(F(p)-(\rho_i^E+\tilde{\rho}_i(p)I))\gamma_i(p)\phi_i^E\\&=G(p)h_i(p)\end{aligned} \tag{18}$$

The matrix form of Eq. (18) can be shown as Eq. (14).

Using Theorem 4, we can solve for $K(p)$ from the linear equation :

$$K(p)\Gamma(p)\Phi^E=H(p) \tag{19}$$

where the inverse matrix of $\Gamma(p)\Phi^E(=\Phi(p))$ is always existed, $K(p)$ consists of probabilistic elements due to variations of $\Gamma(p)\Phi^E$.

## 5. Simulation and Results

An example of the application is based on the longitudinal dynamics of an open-loop unstable aircraft (Stengel and Ray, 1991). The Forward-Swept-Wing Demonstrator's aerodynamic center is forward of its center of gravity, resulting in static instability. Possible uncertainties in aerodynamic and thrust effect as well as separate dynamic pressure ($\rho$ and $V$) effects lead to a 12-element parameter vector.

$$p=[\rho\ V\ f_{11}\ f_{12}\ f_{13}\ f_{22}\ f_{32}\ f_{33}\ g_{11}\ g_{12}\ g_{31}\ g_{32}]$$

Velocity ($V$) and air-density ($\rho$) are modeled as uniform parameter, the remaining terms are kinematics, due to gravity, identically zero or otherwise negligible. Each parameter perturbations are distributed around the nominal value and correlation is assumed to independent on each other. In terms of the element $p$, $F(p)$ and $G(p)$ are

$$F(p)=\begin{bmatrix}-2gf_{11} & \frac{\rho V^2 f_{12}}{2} & \rho V f_{13} & -g\\ \frac{-45}{V^2} & \frac{\rho V f_{22}}{2} & 1 & 0\\ 0 & \frac{\rho V^2 f_{32}}{2} & \rho f_{33} & 0\\ 0 & 0 & 1 & 0\end{bmatrix},$$

$$G(p)=\frac{\rho V^2}{2}\begin{bmatrix}g_{11} & g_{12}\\ 0 & 0\\ g_{31} & g_{32}\\ 0 & 0\end{bmatrix}$$

The state components represent forward velocity, angle of attack, pitch rate, and pitch angle. The principal control surfaces are the canard control surface and the thrust setting. The mean model and its eigenvalues for the given system are as follows:

$$F^E=\begin{bmatrix} -0.02 & -0.3 & -0.4 & -32.2 \\ -0.001 & -1.2 & 1 & 0 \\ 0 & 18. & -0.6 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$G^E=\begin{bmatrix} -0.04 & 35. \\ 0 & 0 \\ 0.2 & -0.2 \\ 0 & 0 \end{bmatrix}$$

$$\rho_{1-4}^E=[-5.1535 \quad -0.0102\pm0.057i \quad 3.3539]$$

For illustration, $\rho$ and $V$ are 10% standard deviation Gaussian uncertainties, and the remaining elements of $p$ are subject to independent 30% standard deviation Gaussian uncertainties. The open-loop eigenvalues distribution of the flight control application on two-dimensional complex plane is shown in Fig. 5.

Let the desired eigenvalues of the closed-loop system so that the natural frequency of the remaining eigenvalue can be three or five times as large as the one of a dominant eigenvalue as follows:

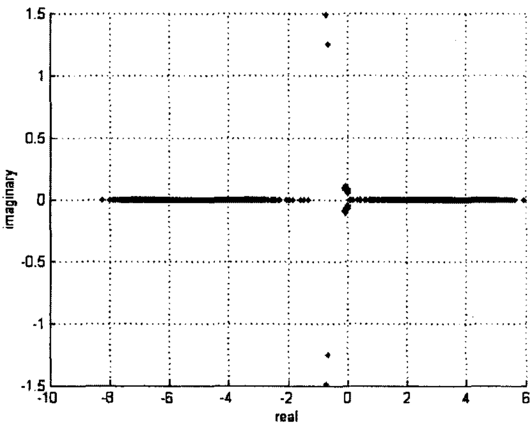



**Fig. 5** Open-loop eigenvalues distribution for the flight control application on two-dimensional complex plane

$$(\rho_1^E)^d=-5.1535,\ \sigma_{\delta_1}^d=0.6492$$

$$(\rho_2^E)^d=-4,\ \sigma_{\delta_2}^d=0.3525$$

$$(\rho_3^E)^d=-0.5+i,\ \Sigma_3^d=\mathrm{diag}(0.0102,\ 0.0099)$$

$$(\rho_4^E)^d=-0.5-i,\ \Sigma_4^d=\mathrm{diag}(0.0102,\ 0.0099)$$

where $(\rho_3^E)^d$ and $(\rho_4^E)^d$ are the eigenvalues on the damping ratio ($\xi=0.447$) and natural frequency ($\omega_n=1.12$) of the longitudinal short-period mode. The mean of desired modal matrix is selected to correspond with the desired eigenvalues as follows:

$$(\Phi^E)^d=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-i & 1+i \\ 0 & 0 & 1+i & 1-i \end{bmatrix}$$

The mean and covariance of the absolute misalignment angle using Eq. (10) are as follows.

$$\theta_1^d(p)\sim(0,\ 0)\ \text{rad}$$

$$\theta_2^d(p)\sim(0,\ 0)\ \text{rad}$$

$$\theta_3^d(p)\sim(0.0728,\ 0.0045)\ \text{rad}$$

$$\theta_4^d(p)\sim(0.0728,\ 0.0045)\ \text{rad}$$

According to the design procedure of the proposed algorithm, feedback gain matrix which consists of probabilistic elements can be obtained through the time interval [0, 5000]. The variation of the Frobenius norm $\left(\|K\|_F\equiv\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|k_{ij}|^2}\right)$ of feedback gain matrix is shown in Fig. 6. The mean of feedback gain matrix can be obtained as follows:

$$K^E=\begin{bmatrix} 0.4158 & 200.4464 & 28.3421 & -121.4804 \\ 0.0708 & 20.9026 & -2.4931 & -39.0587 \end{bmatrix}$$

The mean and covariance of the closed-loop eigenvalues can be obtained as follows:

$$(\rho_1^E)^a=-5.1525,\ \sigma_{\delta_1}^a=0.6444$$

$$(\rho_2^E)^a=-4.0005,\ \sigma_{\delta_2}^a=0.3474$$

$$(\rho_3^E)^a=-0.4982-0.9992i,$$

$$\Sigma_3^a=\mathrm{diag}(0.0099,\ 0.0101)$$

$$(\rho_4^E)^a=-0.4982+0.9992i,$$

$$\Sigma_4^a=\mathrm{diag}(0.0099,\ 0.0101)$$

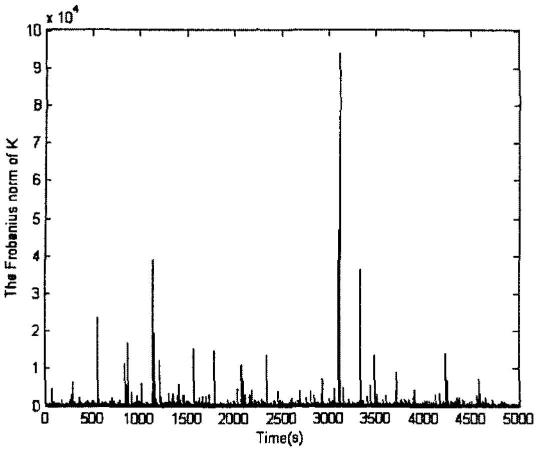



**Fig. 6** The variation of feedback gain matrix

The probability of stability of the closed-loop system is $S=1$. The mean matrix of the achievable modal matrix can be achieved in least square sense as follows :

$$(\Phi^E)^a=\begin{bmatrix} 0.541 & 0.8349 & 1 & 1 \\ 0.0608 & 0.1533 & -0.0004-0.0000i & -0.0004+0.0000i \\ -0.2249 & -0.4141 & -0.0017-0.0000i & -0.0017+0.0000i \\ 0.0476 & 0.1039 & 0.0013-0.0000i & 0.0013+0.0000i \end{bmatrix}$$

The mean and covariance of the absolute misalignment angle for closed-loop S-eigenvectors are as follows :

$$\theta_1^a(p)\sim(-1.0684e-012,\ 7.5822e-021)\ \text{rad}$$

$$\theta_2^a(p)\sim(-1.4593e-010,\ 1.8232e-015)\ \text{rad}$$

$$\theta_3^a(p)\sim(-1.2212e-006,\ 1.5576e-008)\ \text{rad}$$

$$\theta_4^a(p)\sim(1.2212e-006,\ 1.5576e-008)\ \text{rad}$$

These result show the perturbation of S-eigenvectors rarely to raise around the mean of achievable eigenvector. The closed-loop system, S-eigenvalues, and S-eigenvector are the following at each time $t=100$, 1000, 5000, respectively.

- At time $t=100$,

$$A_{100}^c=\begin{bmatrix} -4.7083 & -4369.7696 & 343.0898 & 6900.137 \\ -0.0001 & -8.8302 & 1 & 0 \\ -0.0583 & -60.4136 & -3.7714 & 77.6744 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\Lambda_{100}^c=\text{diag}\begin{bmatrix} -4.4246 \\ -3.7935 \\ -0.5460-0.9912i \\ -0.5460+0.9912i \end{bmatrix},$$

$$\Phi_{100}^c=\begin{bmatrix} 0.8263 & 0.7059 & 1 & 1 \\ 0.1475 & 0.2198 & -0.0003-0.0011i & -0.0003+0.0011i \\ -0.5302 & -0.6511 & -0.0011+0.0000i & -0.0011-0.0000i \\ 0.1198 & 0.1716 & 0.0004-0.0008i & 0.0004+0.0008i \end{bmatrix}$$

- At time $t=1000$,

$$A_{1000}^c=\begin{bmatrix} -2.7339 & -839.7266 & 140.9579 & 1828.6845 \\ -0.0001 & -1.3648 & 1 & 0 \\ -0.0367 & -13.8064 & -5.1817 & 16.5238 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\Lambda_{1000}^c=\text{diag}\begin{bmatrix} -4.2068 \\ -4.0024 \\ -0.5357+1.1953i \\ -0.5357-1.1953i \end{bmatrix},$$

$$\Phi_{1000}^c=\begin{bmatrix} 0.7422 & 0.8733 & 1 & 1 \\ 0.2171 & 0.1682 & -0.0005+0.0015i & -0.0005-0.0015i \\ -0.6168 & -0.4437 & -0.0022+0.0007i & -0.0022-0.0007i \\ 0.1466 & 0.1108 & 0.0011+0.0013i & 0.0011-0.0013i \end{bmatrix}$$

- At time $t=5000$,

$$A_{5000}^c=\begin{bmatrix} -8.2426 & -2422.354 & 290.6554 & 4920.5305 \\ -0.0001 & -1.3853 & 1 & 0 \\ -0.2131 & -73.2656 & -1.519 & 124.4201 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\Lambda_{5000}^c=\text{diag}\begin{bmatrix} -6.0427 \\ -3.9185 \\ -0.5929-0.8439i \\ -0.5929+0.8439i \end{bmatrix},$$

$$\Phi_{5000}^c=\begin{bmatrix} 0.8331 & 0.874 & 1 & 1 \\ 0.1146 & 0.1736 & -0.0005-0.0014i & -0.0005+0.0014i \\ -0.5338 & -0.4397 & -0.0015-0.0007i & -0.0015+0.0007i \\ 0.0883 & 0.1122 & 0.0014-0.0008i & 0.0014+0.0008i \end{bmatrix}$$

where, the upper script '$c$' denotes the closed-loop system, the eigenvalue-eigenvector problem is satisfied with the eigenvalues and eigenvectors at each time, respectively. The closed-loop eigenvalues distribution and its PDF for the flight control application on two-dimensional complex plane and three-dimensional probability density

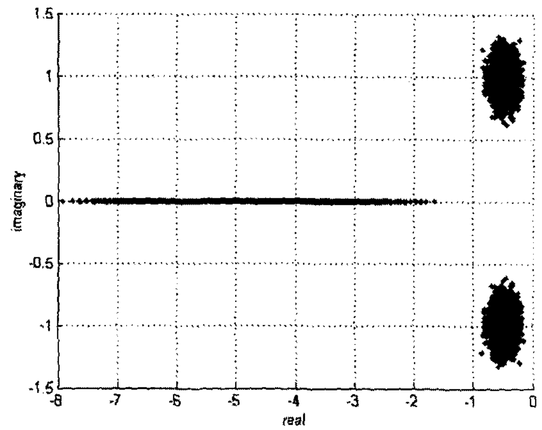



**Fig. 7** Closed-loop eigenvalues distribution for the flight control application on two-dimensional complex plone

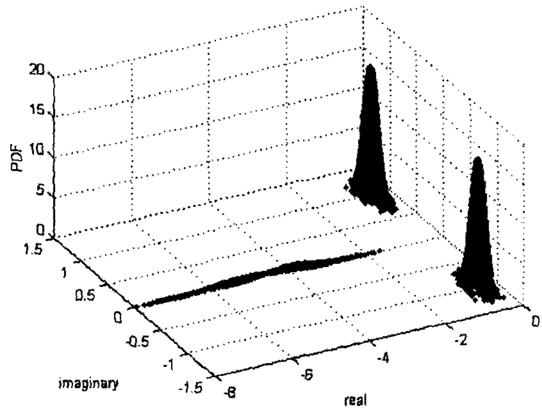



**Fig. 8** Closed-loop eigenvalues distribution for the flight control application on three-dimensional probability density complex plone

complex plane are shown in Fig. 7 and 8. All of eigenvalues are located on the left-half plane, and each eigenvalue varies with its mean value.

Figure 9 shows the closed-loop eigenvalues distribution on two-dimensional complex plane via general eigenstructure assignment as compared with the result of S-eigenstructure assignment scheme. In this case, the probability of stability of the closed-loop system is $S=0.9065$. The feedback gain matrix, closed-loop eigenvalues and its corresponding eigenvectors are as follows, respectively.

$$K=\begin{bmatrix} 0.5419 & 286.2739 & 20.8633 & -269.1261 \\ 0.1003 & 29.895 & -3.3662 & -56.8933 \end{bmatrix},$$

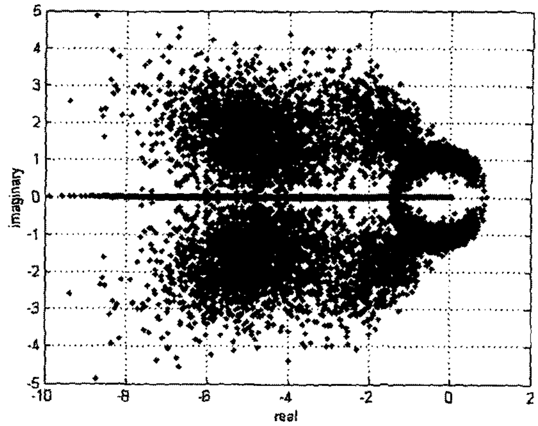



**Fig. 9** Closed-loop eigenvalues distribution via general eigenstructure assignment

$$\Lambda^a=\mathrm{diag}\begin{bmatrix} -5.1535 \\ -4 \\ -0.5-i \\ -0.5+i \end{bmatrix},$$

$$\Phi^a=\begin{bmatrix} 0.7675 & 0.8712 & 1 & 1 \\ 0.1545 & 0.1607 & -0.0006-0.0018i & -0.0006+0.0018i \\ -0.6107 & -0.4499 & -0.0021-0.0006i & -0.0021+0.0006i \\ 0.1185 & 0.1125 & 0.0014-0.0014i & 0.0014+0.0014i \end{bmatrix}$$

## 6. Conclusions

In this paper, first, the S-eigenvalue concept and its corresponding S-eigenvector pair for linear continuous-time systems with probabilistic uncertainty was proposed. The proposed concept is concerned with the perturbation of eigenvalues due to the probabilistic variable parameters in the dynamic model of a plant. Also, S-stability was presented on the appropriate random characteristics of perturbations to maintain the proper stability behavior of the overall system. Finally, S-eigenstructure assignment scheme via a Sylvester equation approach based on the S-eigenvalue concept was proposed. The proposed design schemes were applied to the longitudinal dynamics of open-loop-unstable aircraft with possible uncertainties in aerodynamic and thrust effects as well as separate dynamic pressure. These results explicitly characterized how S-eigenvalues in the complex plane may impose stability on the system.

## References


Anderson, B. D. O. and Moore, J. B., 1971, *Linear Optimal Control*, NJ : Prentice-Hall.

Aoki, M., 1984, "Control of Linear Discrete-Time Stochastic Dynamic Systems with Multiplicative Disturbances," *IEEE Transactions on Automatic Control*, Vol. AC-20, pp. 388~392.

Choi, J. W. and Seo, Y. B., 1999, "LQR Design with Eigenstructure Assignment Capability," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 35, No. 2, pp. 700~708.

Dorato, P., 1987, "A historical Review of Robust Control," *IEEE Control Systems Magazine*, Vol. 7, pp. 44~47.

Doyle, J. C., 1982, "Analysis of Feedback Systems with Structured Uncertainties," *IEE Proc.*, Vol. 129, Pt. D., No. 6, pp. 242~250.

Garg, S., 1991, "Robust Eigenspace Assignment Using Singular Value Sensitivities," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 2, pp. 416~424.

Fahmy, M. M. and Tantaway, H. S., 1984, "Eigenstructure Assignment via Linear State-Feedback Control," *International Journal of Control*, Vol. 40, No. 1, pp. 161~178.

Horn, R. A. and Johnson, C. R., 1985, *Matrix Analysis*, Cambridge University Press.

Horowitz, I., 1982, "Quantitative Feedback Theory," *IEE proc.*, Vol. 129, Pt. D., No. 6, pp. 215~226.

John, S. B., 1999, *Fundamentals of Linear State Space Systems*, WCB/ McGraw-Hill, pp. 147~193.

Leon-Garcia, A., 1994, *Probability and Random Processes for Electrical Engineering*, Addison-Wesley Publishing Company, Inc.

Lethomaki, N. A., Sandell, N. R. and Athans, M., 1981, "Robustness Results in Linear Quadratic Gaussian Based Multivariable Control Designs," *IEEE Transactions on Automatic Control*, Vol. AC-26, No. 1, pp. 75~92.

Levinson, W. L., Kleinman, D. L. and Baron, S., 1968, "A Model for Human Controller Remnant," *Bolt, Beranek and Newman, Inc., Tech. Rep. 1731.*

Lin, C. F., 1994, *Advanced Control Systems Design*, Englewood Cliffs, NJ, Prentice Hall, Inc.

Liu, G. P. and Patton, R. J., 1998, *Eigenstructure Assignment for Control System Design*, John Wiley & Sons.

Maybeck, P. S., 1979, *Stochastic Models, Estimation, and Control, Vol. 1*, Mathematics in Science and Engineering, Vol. 141, Academic Press, pp. 101~111.

Mohler, R. R. and Kolodziej, W. J., 1980, "An Overview of Stochastic Bilinear Control Process," *IEEE Transactions on Systems, Man and Cybernetics*, Vol. SMC-10, pp. 913~918.

Patel, R. V., Toda, M. and Sridhar, B., 1977, "Robustness of Linear Quadratic State Feedback Designs in the Presence of System Uncertainty," *IEEE Transactions on Automatic Control*, Vol. AC-22, pp. 945~949.

Ray, L. R., 1992, "Robust Linear-Optimal Control Laws for Active Suspension Systems," *Journal of Dynamic Systems, Measurement, and Control*, Vol. 114, pp. 592~597.

Safonov, M. G. and Athans, M., 1977, "Gains and Phase Margin for Multiloop LQG Regulators," *IEEE Transactions on Automatic Control*, Vol. AC-22, pp. 173-179.

Sandell, N. R. Jr., 1979, "Recent Developments in the Robustness Theory of Multivariable Systems," *Office of Naval Research*, Rep. No. ONR-CR215-271-1F.

Seo, Y. B. and Choi, J. W., 2003, *Frequency Domain Properties of EALQR with Indefinite Weighting Matrix*, *KSME International Journal*, Vol. 17, No. 3, pp. 329~335.

Sobel, K. M. and Lallman, F. J., 1989, "Eigenstructure Assignment for the Control of Highly Augmented Aircraft," *Journal of Guidance, Control, and Dynamics*, Vol. 12, No. 3, pp. 318~324.

Sobel, K. M. and Shapiro, E. Y., 1985a, "A Design Methodology for Pitch Pointing Flight Control Systems," *Journal of Guidance, Control and Dynamics*, Vol. 8, No. 2, pp. 181~187.

Sobel, K. M. and Shapiro, E. Y., 1985b, "Eigenstructure Assignment for Design of Multimode Flight Control Systems," *IEEE Control Systems Magazine*, pp. 9~15.

Sobel, K. M., Shapiro, E. Y. and Andry, A. N., 1994, "Eigenstructure Assignment," *International Journal of Control*, Vol. 59, No. 1, pp. 13~37.

Stengel, R. F., 1980, "Some Effects of Parameter Variations on the Lateral-Directional Stability of Aircraft," *Journal of Guidance, Control and Dynamics*, Vol. 3, No. 2, pp. 124~131.

Stengel, R. F. and Ray, L. R., 1991, "Stochastic Robustness of Linear Time-Invariant Control Systems," *IEEE Transactions on Automatic Control*, Vol. 36, No. 1, pp. 82~87.

Tahk, M. and Speyer, J. L., 1987, "Modelling of Parameter Variations and Asymptotic LQG Synthesis," *IEEE Transactions on Automatic Control*, Vol. AC-32, No. 9, pp. 793~801.

Wingerden, A. J. M. and Dekoning, W. L., 1984, "The Influence of Finite Word Length on Digital Optimal Control," *IEEE Transactions on Automatic Control*, Vol. AC-29, pp. 385~391.

Yaz, E., 1988, "Deterministic and Stochastic Robustness Measures for Discrete Systems," *IEEE Transactions on Automatic Control*, Vol. 33, No. 10, pp. 952~955.

Yedavalli, R. K., Banda, S. S. and Ridgely, D. B., 1985, "Time-Domain Stability Robustness Measures for Linear Regulators," *Journal of Guidance, Control, and Dynamics*, Vol. 8, pp. 520~524.

Yedavalli, R. K. and Liang, Z., 1986, "Reduced Conservatism in Stability Robustness Bounds by State Transformation," *IEEE Transactions on Automatic Control*, Vol. AC-31, No. 9, pp. 863~866.

Zhou, K. and Khargonekar, P. P., 1987, "Stability Robustness Bounds for Linear State-Space Models with Structured Uncertainty," *IEEE Transactions on Automatic Control*, Vol. AC-32, pp. 621~623.