Characteristics of the Eigenvalue Sensitivities to the Change of Element Correction Factors for Beams

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Some characteristics of the eigenvalue sensitivities have been found for beams in the paper. For cantilever beams and simply supported beams, the sensitivities of the eigenvalues to the stiffness correction factor of one element are equal and opposite to the sensitivities to the mass correction factor of the symmetrically positioned element. For beams with other boundary conditions, however, the relationship does not hold. The relationship has been proven analytically for simply supported beams.

Key Words : Eigenvalue Sensitivity, Beam, Element Correction Factor

1. Introduction

Finite Element (FE) analysis is widely used to predict the dynamic responses of mechanical systems and structures subject to dynamic loading. The predicted responses may differ from the experimentally measured ones and there have been active researches on finite element model updating (Kim et al., 2003) so that the predicted responses based on the model agree with the measured ones. The related researches are surveyed (Mottershead and Friswell, 1993) and summarized (Friswell and Mottershead, 1995) in references. One of the approaches to model updating is the sensitivity analysis (Jeong et al., 2003). In the approach the sensitivities of the model responses, for example eigenvalues (natural frequencies) and eigenvectors (mode shapes) of the FE model, to changes in the updating parameters are calculated. And the updating parameters of the model are modified according to the sensitivities. Material properties, physical dimensions or joint parameters can be selected as updating parameters (Mottershead et al., 2000). Another choice is element correction factors which are multiplied to each element mass and stiffness matrices to modify the FE model (Imregun et al., 1995). This paper investigates some characteristics of the eigenvalue sensitivities to element correction factors for beams with various boundary conditions. And the observed characteristics are proven analytically for simply supported beams.

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2. Sensitivity of Eigenvalues

2.1 Element correction factors

This section explains element correction factors which are used to update FE models. Element correction factors can be defined for the mass and stiffness matrices of each element. P_{mj} and P_{kj} represent the mass and stiffness correction factors for the j-th element, respectively. Element mass and stiffness matrices are modified proportionally to the correction factors as follows.

$$[M_{ej}]=(1+P_{mj})[M_{ej}]_0 \tag{1}$$

$$[K_{ej}]=(1+P_{kj})[K_{ej}]_0 \tag{2}$$

where $[M_{ej}]$ and $[K_{ej}]$ represent the mass and stiffness matrices of the j-th element, respectively, and the subscript 0 initial matrices. The element matrices $[M_{ej}]$ and $[K_{ej}]$ have the same sizes as the whole system matrices, with zeros outside the corresponding positions, and rows and columns deleted for fixed boundary conditions. Then the system mass and stiffness matrices are expressed by the summation of element matrices as follows.

$$[M]=\sum_{j=1}^{N}[M_{ej}]=\sum_{j=1}^{N}(1+P_{mj})[M_{ej}]_0 \quad (3)$$

$$[K]=\sum_{j=1}^{N}[K_{ej}]=\sum_{j=1}^{N}(1+P_{kj})[K_{ej}]_0 \quad (4)$$

where N is the number of elements.

2.2 Sensitivity of the eigenvalues for a cantilever beam

It is known that the sensitivity of the eigenvalue (square of the natural frequency) of mode i, λ_i to change in the updating parameter θ_j is expressed by Eq. (5)

$$\frac{\partial\lambda_i}{\partial\theta_j}=\phi_i^T\left(\frac{\partial[K]}{\partial\theta_j}-\lambda_i\frac{\partial[M]}{\partial\theta_j}\right)\phi_i \quad (5)$$

where ϕ_i represents the mass normalized eigenvector of mode i (Wittrick, 1962). If we take element correction factors as updating parameters and insert Eqs. (3) and (4) into Eq. (5), the sensitivities of eigenvalues become

$$\frac{\partial\lambda_i}{\partial P_{kj}}=\phi_i^T[K_{ej}]_0\phi_i \quad (6)$$

$$\frac{\partial\lambda_i}{\partial P_{mj}}=\phi_i^T(-\lambda_i[M_{ej}]_0)\phi_i \quad (7)$$

Using the FE analysis (Petyt, 1989), the element mass and stiffness matrices were formed and the eigenvalues and eigenvectors were calculated for a cantilever beam with length 270 mm, width 35 mm, thickness 1.5 mm, Young's modulus 175×10^9 N/m^2, and density 7850 kg/m^3. The beam is composed of five beam elements with equal length and is shown in Fig. 1. The sensitivities of eigenvalues to the element correction factors in Eqs. (6) and (7) were calculated and are listed in Table 1. Examining the table, it can be found that the sensitivity to the stiffness correction factor of one element is almost equal and opposite to the sensitivity to the mass correction factor of the element in symmetric position. As Fig. 1 shows, elements 1 and 5, and 2 and 4 are in symmetric positions. The above observation can be expressed in equations as follows.

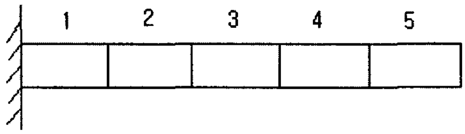

Fig. 1 Cantilever beam composed of five elements

Table 1 Sensitivity of the eigenvalues to the element correction factors for a cantilever beam (units : rad^2/s^2)

Element i	1	2	3	4	5
$\frac{\partial\lambda_1}{\partial P_{ki}}$	5.8359e3	2.7688e3	0.9421e3	0.1704e3	0.0064e3
$\frac{\partial\lambda_1}{\partial P_{mi}}$	−0.0066e3	−0.1704e3	−0.9421e3	−2.7687e3	−5.8358e3
$\frac{\partial\lambda_2}{\partial P_{ki}}$	1.0742e5	0.3845e5	1.4500e5	0.8502e5	0.0636e5
$\frac{\partial\lambda_2}{\partial P_{mi}}$	−0.0645e5	−0.8496e5	−1.4499e5	−0.3852e5	−1.0734e5
$\frac{\partial\lambda_3}{\partial P_{ki}}$	0.5653e6	0.8227e6	0.2218e6	1.1607e6	0.2450e6
$\frac{\partial\lambda_3}{\partial P_{mi}}$	−0.2446e6	−1.1566e6	−0.2231e6	−0.8243e6	−0.5669e6
$\frac{\partial\lambda_4}{\partial P_{ki}}$	2.3904e6	1.8709e6	3.1323e6	2.3059e6	2.0686e6
$\frac{\partial\lambda_4}{\partial P_{mi}}$	−1.9972e6	−2.3236e6	−3.1128e6	−1.8690e6	−2.4656e6

$$\frac{\partial \lambda_i}{\partial P_{k1}} = -\frac{\partial \lambda_i}{\partial P_{m5}} \tag{8}$$

$$\frac{\partial \lambda_i}{\partial P_{k2}} = -\frac{\partial \lambda_i}{\partial P_{m4}} \tag{9}$$

In general, these relations can be written as

$$\frac{\partial \lambda_i}{\partial P_{kj}} = -\frac{\partial \lambda_i}{\partial P_{m(N+1-j)}} \tag{10}$$

The relationship (10) holds true exactly for lower modes, and there are slight differences between the two sensitivities for higher modes. For example, the maximum difference for the fourth mode was 3.45%.

The relationship means that to increase stiffness in one element has the same effects on the eigenvalues as to decrease mass by the same proportion in the symmetrically positioned element. For the above cantilever beam, the eigenvalues were calculated for the case with the stiffness of the second element increased by 20%, that is, $P_{k2}=0.2$. Next, the eigenvalues were calculated with the mass of the fourth element decreased by 20%, that is, $P_{m4}=-0.2$. These eigenvalues are compared in Table 2 and agree very well.

Table 2 Eigenvalues of the cantilever beam for the two cases (units: rad^2/s^2)

Case 1 (P_{k2}=0.2)	Case 2 (P_{m4}=-0.2)
0.0161e3	0.0162e3
0.0993e3	0.0994e3
0.2833e3	0.2848e3
0.5538e3	0.5555e3
0.9207e3	0.9239e3

The above relationship may be useful in deciding the amount of modification in the element mass and/or stiffness matrices so that the eigenvalues of the FE models agree with the measured ones. Here it should be noted that varying the thickness of an element does not result in stiffness variation only, since the mass of the element also varies.

2.3 Sensitivity of the eigenvalues for beams with other boundary conditions

The sensitivities of eigenvalues to changes in the element correction factors were calculated for a simply supported beam with the same material properties and dimensions as the previous cantilever beam. This beam is also composed of 5 elements. Examining the calculated sensitivities in Table 3, it can be found that the same relationship as for a cantilever beam holds true. Moreover, because of the symmetry of the simply supported beam, the following relationship also holds

$$\frac{\partial \lambda_i}{\partial P_{kj}} = -\frac{\partial \lambda_i}{\partial P_{mj}} \tag{11}$$

Table 3 Sensitivity of the eigenvalues to the element correction factors for a simply supported beam (units: rad^2/s^2)

Element i	1	2	3	4	5
$\frac{\partial \lambda_1}{\partial P_{ki}}$	0.3727e4	1.9757e4	2.9664e4	1.9757e4	0.3727e4
$\frac{\partial \lambda_1}{\partial P_{mi}}$	−0.3727e4	−1.9757e4	−2.9664e4	−1.9757e4	−0.3727e4
$\frac{\partial \lambda_2}{\partial P_{ki}}$	1.8852e5	3.9642e5	0.6002e5	3.9642e5	1.8852e5
$\frac{\partial \lambda_2}{\partial P_{mi}}$	−1.8856e5	−3.9630e5	−0.6018e5	−3.9630e5	−1.8856e5
$\frac{\partial \lambda_3}{\partial P_{ki}}$	1.4536e6	0.7565e6	1.8845e6	0.7565e6	1.4536e6
$\frac{\partial \lambda_3}{\partial P_{mi}}$	−1.4518e6	−0.7613e6	−1.8786e6	−0.7613e6	−1.4518e6
$\frac{\partial \lambda_4}{\partial P_{ki}}$	4.6566e6	3.8950e6	3.4243e6	3.8950e6	4.6566e6
$\frac{\partial \lambda_4}{\partial P_{mi}}$	−4.6178e6	−3.9098e6	−3.4723e6	−3.9098e6	−4.6178e6

Table 4 Sensitivity of the eigenvalues to the element correction factors for a clamped - simply supported beam (units : rad^2/s^2)

Element i	1	2	3	4	5
$\frac{\partial \lambda_1}{\partial P_{ki}}$	6.3173e4	0.6633e4	4.8904e4	5.5600e4	1.2761e4
$\frac{\partial \lambda_1}{\partial P_{mi}}$	−0.1746e4	−2.8388e4	−7.5234e4	−6.7227e4	−1.4477e4
$\frac{\partial \lambda_2}{\partial P_{ki}}$	3.8962e5	5.2240e5	1.8318e5	5.2571e5	3.5302e5
$\frac{\partial \lambda_2}{\partial P_{mi}}$	−1.2006e5	−7.7120e5	−2.1846e5	−5.1363e5	−3.5059e5
$\frac{\partial \lambda_3}{\partial P_{ki}}$	1.7069e6	1.6369e6	2.2876e6	1.0230e6	2.0775e6
$\frac{\partial \lambda_3}{\partial P_{mi}}$	−1.3053e6	−2.0880e6	−2.2331e6	−1.0320e6	−2.0733e6
$\frac{\partial \lambda_4}{\partial P_{ki}}$	5.9950e6	4.7015e6	4.9469e6	5.2059e6	5.4471e6
$\frac{\partial \lambda_4}{\partial P_{mi}}$	−5.6830e6	−5.0665e6	−4.9549e6	−5.1999e6	−5.3921e6

Explaining in words, the sensitivity of eigenvalues to the stiffness correction factor in one element is equal and opposite to the sensitivity to the mass correction factor in the same element.

The sensitivities of eigenvalues were calculated for beams with other types of boundary conditions : a clamped-simply supported beam and a clamped-clamped beam. However, for these beams the previous relationship (10) does not hold. Table 4 lists the calculated sensitivities for a clamped-simply supported beam.

3. Proof of the Sensitivity Relationship

From the definition of eigenvalues and eigenvectors, we have the following relation

$$[K]\phi_i=\lambda_i[M]\phi_i \tag{12}$$

Pre-multiplying ϕ_i^T on both sides of the above equation,

$$\phi_i^T[K]\phi_i=\phi_i^T(\lambda_i[M])\phi_i \tag{13}$$

Expressing the mass and stiffness matrices as sums of the element matrices,

$$\begin{aligned}&\phi_i^T([K_{e1}]+[K_{e2}]+\cdots+[K_{eN}])\phi_i\\&=\phi_i^T(\lambda_i[M_{e1}]+\lambda_i[M_{e2}]+\cdots+\lambda_i[M_{eN}])\phi_i\end{aligned} \tag{14}$$

Here the subscripts 0 of the element matrices are omitted for brevity. Substituting Eqs. (6) and (7) into the above equation, we obtain

$$\sum_{j=1}^{N}\frac{\partial \lambda_i}{\partial P_{kj}}=-\sum_{j=1}^{N}\frac{\partial \lambda_i}{\partial P_{mj}} \tag{15}$$

Therefore, the sum of the sensitivities to P_{kj} is equal and opposite to the sum of the sensitivities to P_{mj}. However, it does not explain the equality between the corresponding terms of the sums, Eq. (10).

To prove the relation described by Eq. (10), a simply supported beam was considered. When a beam with length L and flexural rigidity EI is composed of N elements, the stiffness matrix of the j-th element from one end, $[K_{ej}]$, is of size 2N×2N. The elements of the matrix have the values represented in the following equation for $(2j-2)$th to $(2j+1)$th rows and columns, and zeros for the remainings.

$$[K_{ej}]=\frac{EI}{l^3}\begin{bmatrix}12 & 6l & -12 & 6l\\ 6l & 4l^2 & -6l & 2l^2\\ -12 & -6l & 12 & -6l\\ 6l & 2l^2 & -6l & 4l^2\end{bmatrix} \tag{16}$$

where l is the length of one element and equal to $l=L/N$.

The analytical eigenvectors of a simply supported beam are known to be $\sin\beta_i x$. Calculating translational displacements and rotational displacements at all the nodes from the expression and its differentiation $\beta_i \cos \beta_i x$, respectively, the eigenvector of mode i, ϕ_i, is obtained as follows.

$$\phi_i^T = [\beta_i \cdots \sin(j-1)\beta_i l \;\; \beta_i \cos(j-1)\beta_i l \;\; \sin j\beta_i l \;\; \beta_i \cos j\beta_i l \cdots] \tag{17}$$

where $\beta_i = i\pi/L$

Using the expressions for $[K_{ej}]$ and ϕ_i, the sensitivity of eigenvalue to the stiffness correction factor in Eq. (6) was calculated and is expressed in the following equation with $\alpha = \beta_i l$.

$$\begin{aligned}\phi_i^T[K_{ej}]\phi_i = \frac{EI}{l^3}[&12\sin^2(j-1)\alpha + 12\alpha\sin(j-1)\alpha\cos(j-1)\alpha \\ &-24\sin(j-1)\alpha\sin j\alpha + 12\alpha\sin(j-1)\alpha\cos j\alpha \\ &+4\alpha^2\cos^2(j-1)\alpha - 12\alpha\cos(j-1)\alpha\sin j\alpha \\ &+4\alpha^2\cos(j-1)\alpha\cos j\alpha + 12\sin^2 j\alpha \\ &-12\alpha\sin j\alpha\cos j\alpha + 4\alpha^2\cos^2 j\alpha]\end{aligned} \tag{18}$$

Using the trigonometric relations

$$\sin(j-1)\alpha = \sin j\alpha\cos\alpha - \cos j\alpha\sin\alpha \tag{19}$$

$$\cos(j-1)\alpha = \cos j\alpha\cos\alpha + \sin j\alpha\sin\alpha \tag{20}$$

and expanding $\sin\alpha$ and $\cos\alpha$ into Taylor series,

$$\sin\alpha = \alpha - \frac{\alpha^3}{6} + \frac{\alpha^5}{120} - \frac{\alpha^7}{5040}\cdots \tag{21}$$

$$\cos\alpha = 1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{24} - \frac{\alpha^6}{720} + \frac{\alpha^8}{40320}\cdots \tag{22}$$

Eq. (18) is expressed including upto α^8 terms as follows.

$$\begin{aligned}\phi_i^T[K_{ej}]\phi_i = \frac{EI}{l^3}\Big[&\Big(\alpha^4 - \frac{1}{3}\alpha^6 + \frac{47}{720}\alpha^8\Big)\sin^2 j\alpha \\ &+\Big(-\alpha^5 + \frac{1}{3}\alpha^7\Big)\sin j\alpha\cos j\alpha \\ &+\Big(\frac{1}{3}\alpha^6 - \frac{1}{30}\alpha^8\Big)\cos^2 j\alpha\Big]\end{aligned} \tag{23}$$

The mass matrix of the $(N+1-j)$th element which is in the symmetric position to the j-th element, $[M_{e(N+1-j)}]$ is of size $2N \times 2N$. The elements of the matrix have the values represented in the following equation for $(2N-2j)$th to $(2N-2j+3)$th rows and columns, and zeros for the remainings.

$$[M_{e(N+1-j)}] = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \tag{24}$$

where ρ and A represent the density of the material and the cross sectional area, respectively.

The analytical eigenvalues of the beam are known to be

$$\lambda_i = \omega_i^2 = (\beta_i L)^4 \frac{EI}{\rho A L^4} \tag{25}$$

Following a similar procedure to the one used in calculating the sensitivity of eigenvalue to the stiffness correction factor in Eq. (6), the sensitivity of eigenvalue to the mass correction factor in Eq. (7) becomes

$$\begin{aligned}\phi_i^T(\lambda_i[M_{e(N+1-j)}])\phi_i = \frac{EI}{l^3}\Big[&\Big(\alpha^4 - \frac{1}{3}\alpha^6 + \frac{161}{2520}\alpha^8\Big)\sin^2 j\alpha \\ &+\Big(-\alpha^5 + \frac{1}{3}\alpha^7\Big)\sin j\alpha\cos j\alpha \\ &+\Big(\frac{1}{3}\alpha^6 - \frac{1}{15}\alpha^8\Big)\cos^2 j\alpha\Big]\end{aligned} \tag{26}$$

Eq. (23) represents the sensitivity of eigenvalue to the stiffness correction factor, and Eq. (26) the opposite of the sensitivity of eigenvalue to the mass correction factor in the symmetric position. Comparing the two equations, they are identical to each other for lower terms of α and differ from the α^8 term. Consequently, the two sensitivities are almost equal and opposite to each other for small values of α. Therefore, the relation in Eq. (10) has been proven for simply supported beams. Since $\alpha = \beta_i l$ and β_i is small for lower modes, the value of α becomes small for lower modes. This fact agrees with the observation that the two sensitivities are almost equal for lower modes. Also, as the number of elements increases, the element length l decreases and the value of α becomes small. For beams with other boundary conditions, a similar procedure can be followed to prove the sensitivity relations which were observed previously.

4. Conclusions

Some characteristics of the eigenvalue sensitivities for beams have been found in the paper. For cantilever beams and simply supported beams, the sensitivities of the eigenvalues to the stiffness correction factor of one element are equal and opposite to the sensitivities to the mass correction factor of the symmetrically positioned element. The relationship means that to increase stiffness in one element has the same effects on the eigenvalues as to decrease mass by the same proportion in the symmetrically positioned element. The relationship has been proven analytically for simply supported beams, for which the eigenvalues and eigenvectors are expressed in simple forms. For beams with other boundary conditions, however, the relationship does not hold.

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