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WEIGHTED CONTINUITY OF MULTILINEAR MARCINKIEWICZ OPERATORS FOR THE EXTREME CASES OF p

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ABSTRACT. In this paper, we prove the weighted continuity of multilinear Marcinkiewicz operators for the extreme cases of p.

1. Introduction and results

Suppose that S^{n-1} is the unit sphere of $R^n (n \geq 2)$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let Ω be homogeneous of degree zero and satisfy the following two conditions:

(i) $\Omega(x)$ is continuous on S^{n-1} and satisfies the Lip_γ condition on $S^{n-1}(0 < \gamma \leq 1)$, i.e.

$$|\Omega(x') - \Omega(y')| \leq M|x' - y'|^\gamma, \quad x', y' \in S^{n-1};$$

(ii) $\int_{S^{n-1}} \Omega(x')dx' = 0$.

Let m be a positive integer and A be a function on R^n. We denote that $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x - y| < t\}$ and the characteristic of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The multilinear Marcinkiewicz operator is defined by

$$\mu_S^A(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$F_t^A(f)(x,y) = \int_{|y-z|\leq t} \frac{\Omega(y-z)}{|y-z|^{n-1}} \frac{R_{m+1}(A;x,z)}{|x-z|^m} f(z)dz$$

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and

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha|\le m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha.$$

We denote that

$$F_t(f)(y) = \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz.$$

We also define that

$$\mu_S(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2},$$

which is the Marcinkiewicz integral operator (see [10]).

Let H be the Hilbert space

$$H = \left\{ h : ||h|| = \left(\int\int_{R_+^{n+1}} |h(t)|^2 dydt/t^{n+3} \right)^{1/2} < \infty \right\},$$

then for each fixed $x \in R^n$, $F_t^A(f)(x,y)$ may be viewed as a mapping from $(0,+\infty)$ to H, and it is clear that

$$\mu_S^A(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t^A(f)(x,y)\right|\right|,$$

$$\mu_S(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t(f)(y)\right|\right|.$$

We also consider the variant of μ_S^A, which is defined by

$$\tilde{\mu}_S^A(f)(x) = \left(\int\int_{\Gamma(x)} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2},$$

where

$$\tilde{F}_t^A(f)(x,y) = \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} \frac{Q_{m+1}(A;x,z)}{|x-z|^m} f(z)dz$$

and

$$Q_{m+1}(A;x,z) = R_m(A;x,z) - \sum_{|\alpha|=m} \frac{1}{\alpha!} D^\alpha A(x)(x-z)^\alpha.$$

Note that when $m=0$, μ_S^A is just the commutator of Marcinkiewicz integral operator (see [15], [18]). It is well known that multilinear operators, as the extension of commutators, are of great interest in harmonic analysis and have been widely studied by many authors (see

[3-6], [8], [9], [13], [14]). In [12], the endpoint boundedness properties of the commutators generated by the Calderon-Zygmund operator and BMO functions are obtained. The main purpose of this paper is to discuss the weighted continuity properties of the multilinear Marcinkiewicz operators for the extreme cases of p. Throughout this paper, B will denote a ball of R^n. For a ball B and any locally integral function f on R^n, we denote that $f(B) = \int_B f(x)dx$, $f_B = |B|^{-1}\int_B f(x)dx$ and $f^{\#}(x) = \sup_{x\in B}|B|^{-1}\int_B |f(y) - f_B|dy$. Moreover, for a weight functions $w \in A_1$(see [11]), f is said to belong to $BMO(w)$ if $f^{\#} \in L^\infty(w)$ and define $||f||_{BMO(w)} = ||f^{\#}||_{L^\infty(w)}$, if $w = 1$, we denote that $BMO(R^n) = BMO(w)$. Also, we give the concepts of atom and weighted H^1 space. A function a is called a $H^1(w)$ atom if there exists a ball B such that a is supported on B, $||a||_{L^\infty(w)} \le w(B)^{-1}$ and $\int_{R^n} a(x)dx = 0$. It is well known that, for $w \in A_1$, the weighted Hardy space $H^1(w)$ has the atomic decomposition characterization(see [1]).

We shall prove the following theorems in Section 3.

THEOREM 1. *Let $D^\alpha A \in BMO(R^n)$ for $|\alpha| = m$ and $w \in A_1$. Then μ_S^A maps $L^\infty(w)$ continuously into $BMO(w)$.*

THEOREM 2. *Let $D^\alpha A \in BMO(R^n)$ for $|\alpha| = m$ and $w \in A_1$. Then $\tilde{\mu}_S^A$ maps $H^1(w)$ continuously into $L^1(w)$.*

THEOREM 3. *Let $D^\alpha A \in BMO(R^n)$ for $|\alpha| = m$ and $w \in A_1$. Then μ_S^A maps $H^1(w)$ continuously into weak $L^1(w)$.*

THEOREM 4. *Let $D^\alpha A \in BMO(R^n)$ for $|\alpha| = m$ and $w \in A_1$.*

(i) If for any $H^1(w)$-atom a supported on certain cube Q and $u \in 3Q \setminus 2Q$, there is

$$\int_{(4Q)^c}\left\|\chi_{\Gamma(x)}\sum_{|\alpha|=m}\frac{1}{\alpha!}\frac{(x-u)^\alpha}{|x-u|^m}\frac{\Omega(y-u)}{|y-u|^{n-1}}\chi_{\Gamma(y)}(u,t)\right.$$
$$\left.\times\int_Q D^\alpha A(z)a(z)dz\right\|w(x)dx$$
$$\le C,$$

then μ_S^A is bounded from $H^1(w)$ to $L^1(w)$;

(ii) *If for any cube Q and $u \in 3Q \setminus 2Q$, there is*

$$\frac{1}{w(Q)}\int_Q \left|\left|\chi_{\Gamma(x)}\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x)-(D^\alpha A)_Q)\right.\right.$$
$$\left.\left.\times\int_{(4Q)^c}\frac{(u-z)^\alpha}{|u-z|^m}\frac{\Omega(y-z)\chi_{\Gamma(y)}(z,t)}{|y-z|^{n-1}}f(z)dz\right|\right|w(x)dx$$
$$\leq C||f||_{L^\infty(w)},$$

then $\tilde{\mu}_S^A$ is bounded from $L^\infty(w)$ to $BMO(w)$.

2. Some lemmas

We begin with some preliminary lemmas.

LEMMA 1. (see [6]) *Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A;x,y)| \leq C|x-y|^m \sum_{|\alpha|=m}\left(\frac{1}{|\tilde{B}(x,y)|}\int_{\tilde{B}(x,y)}|D^\alpha A(z)|^q dz\right)^{1/q},$$

where $\tilde{B}(x,y)$ is the ball centered at x and having radius $5\sqrt{n}|x-y|$.

LEMMA 2. *Let $w \in A_1$, $1 < p < \infty$, $1 < r \leq \infty$, $1/q = 1/p + 1/r$ and $D^\alpha A \in BMO(R^n)$ for $|\alpha| = m$. Then μ_S^A is bound from $L^p(w)$ to $L^q(w)$, that is*

$$||\mu_S^A(f)||_{L^q(w)} \leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^p(w)}.$$

PROOF. Note that $|x-z| \leq 2t$, $|y-z| \geq |x-z| - t \geq |x-z| - 3t$ when $|x-y| \leq t$, $|y-z| \leq t$. By Minkowski inequality, we have

$$\begin{aligned}\mu_S^A(f)(x) &\leq \int_{R^n}\left[\int\int_{|x-y|\leq t}\left(\frac{|\Omega(y-z)||R_{m+1}(A;x,z)||f(z)|}{|y-z|^{n-1}|x-z|^m}\right)^2\right.\\ &\qquad\left.\times\chi_{\Gamma(z)}(y,t)\frac{dydt}{t^{n+3}}\right]^{1/2}dz\\ &\leq C\int_{R^n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^m}\end{aligned}$$

$$\times \left[\int \int_{|x-y|\le t} \frac{\chi_{\Gamma(z)}(y,t) t^{-n-3}}{(|x-z|-3t)^{2n-2}} dy dt \right]^{1/2} dz$$

$$\le C \int_{R_n} \frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+3/2}}$$

$$\times \left[\int_{|x-z|/2}^{\infty} \frac{dt}{(|x-z|-3t)^{2n-2}} \right]^{1/2} dz$$

$$\le C \int_{R_n} \frac{|R_{m+1}(A;x,z)|}{|x-z|^{m+n}} |f(z)| dz,$$

thus, the lemma follows from [8], [9]. □

3. Proofs of theorems

PROOF OF THEOREM 1. It is only to prove that there exists a constant C_B such that

$$\frac{1}{w(B)} \int_B |\mu_S^A(f)(x) - C_B| w(x) dx \le C ||f||_{L^\infty(w)}$$

holds for any ball B. Fix a ball $B = B(x_0, l)$. Let $\tilde{B} = 5\sqrt{n}B$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{\tilde{B}} x^\alpha$, then $R_m(A;x,y) = R_m(\tilde{A};x,y)$ and $D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A)_{\tilde{B}}$ for $|\alpha| = m$. We write, for $f_1 = f\chi_{\tilde{B}}$ and $f_2 = f\chi_{R^n \setminus \tilde{B}}$,

$$F_t^A(f)(x) = F_t^A(f_1)(x) + F_t^A(f_2)(x),$$

then

$$\begin{aligned} & \frac{1}{w(B)} \int_B |\mu_S^A(f)(x) - \mu_S^A(f_2)(x_0)| w(x) dx \\ = & \frac{1}{w(B)} \int_B \Big| ||\chi_{\Gamma(x)} F_t^A(f)(x,y)|| - ||\chi_{\Gamma(x)} F_t^A(f_2)(x_0,y)|| \Big| \, w(x) dx \\ \le & \frac{1}{w(B)} \int_B \mu_S^A(f_1)(x) w(x) dx \\ & + \frac{1}{w(B)} \int_B ||\chi_{\Gamma(x)} F_t^A(f_2)(x,y) - \chi_{\Gamma(x)} F_t^A(f_2)(x_0,y)|| w(x) dx \\ := & I + II. \end{aligned}$$

Now, let us estimate I and II. First, by the L^∞ boundedness of μ_S^A(Lemma 2), we gain

$$I \le ||\mu_S^A(f_1)||_{L^\infty(w)} \le C||f||_{L^\infty(w)}.$$

To estimate II, we write

$$\begin{aligned}
&\chi_{\Gamma(x)}F_t^A(f_2)(x,y) - \chi_{\Gamma(x)}F_t^A(f_2)(x_0,y)\\
= &\int_{|y-z|\le t}\left[\frac{1}{|x-z|^m}-\frac{1}{|x_0-z|^m}\right]\frac{\chi_{\Gamma(x)}\Omega(y-z)R_m(\tilde{A};x,z)f_2(z)}{|y-z|^{n-1}}dz\\
&+\int_{|y-z|\le t}\frac{\chi_{\Gamma(x)}\Omega(y-z)f_2(z)}{|y-z|^{n-1}|x_0-z|^m}[R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)]dz\\
&+\int_{|y-z|\le t}(\chi_{\Gamma(x)}-\chi_{\Gamma(x_0)})\frac{\Omega(y-z)R_m(\tilde{A};x_0,z)f_2(z)}{|y-z|^{n-1}|x_0-z|^m}dz\\
&-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{|y-z|\le t}\left[\frac{\chi_{\Gamma(x)}(x-z)^\alpha}{|x-z|^m}-\frac{\chi_{\Gamma(x_0)}(x_0-z)^\alpha}{|x_0-z|^m}\right]\\
&\times\frac{\Omega(y-z)D^\alpha\tilde{A}(z)f_2(z)}{|y-z|^{n-1}}dz\\
:= &II_1^t(x)+II_2^t(x)+II_3^t(x)+II_4^t(x).
\end{aligned}$$

Note that $|x-z|\sim|x_0-z|$ for $x\in\tilde{B}$ and $z\in R^n\setminus\tilde{B}$, and by the similar method to the proof of Lemma 2 and by Lemma 1, we have

$$\begin{aligned}
&\frac{1}{w(B)}\int_B||II_1^t(x)||w(x)dx\\
\le &\frac{C}{w(B)}\int_B\left(\int_{R^n\setminus\tilde{B}}\frac{|x-x_0||f(z)|}{|x-z|^{n+m+1}}|R_m(\tilde{A};x,z)|dz\right)w(x)dx\\
\le &\frac{C}{w(B)}\int_B\left(\sum_{k=0}^\infty\int_{2^{k+1}\tilde{B}\setminus 2^k\tilde{B}}\frac{|x-x_0||f(z)|}{|x-z|^{n+m+1}}|R_m(\tilde{A};x,z)|dz\right)w(x)dx\\
\le &C\sum_{k=1}^\infty\frac{kl(2^kl)^m}{(2^kl)^{n+m+1}}\sum_{|\alpha|=m}||D^\alpha A||_{BMO}(\int_{2^{k+1}\tilde{B}}|f(z)|dz)\\
\le &C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)}\sum_{k=1}^\infty k2^{-k}\\
\le &C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)};
\end{aligned}$$

For $II_2^t(x)$, by the formula (see [6]):

$$\begin{aligned} &R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)\\ =\ &R_m(\tilde{A};x,x_0)\\ &+\sum_{0<|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};x_0,z)(x-x_0)^\beta \end{aligned}$$

and Lemma 1, we get

$$\begin{aligned} &|R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)|\\ \le\ &C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}(|x-x_0|^m\\ &+\sum_{0<|\beta|<m}|x_0-z|^{m-|\beta|}|x-x_0|^{|\beta|}), \end{aligned}$$

thus, for $x\in B$,

$$\begin{aligned} &||II_2^t(x)||\\ \le\ &C\int_{R^n}\frac{|f_2(z)|}{|x-z|^{m+n}}|R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)|dz\\ \le\ &C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\\ &\times\int_{R^n}\frac{|x-x_0|^m+\sum_{0<|\beta|<m}|x_0-z|^{m-|\beta|}|x-x_0|^{|\beta|}}{|x_0-z|^{m+n}}|f_2(z)|dz\\ \le\ &C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=0}^{\infty}\frac{kl^m}{(2^kl)^{m+n}}\int_{2^{k+1}\tilde{B}}|f(z)|dz\\ \le\ &C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)}\sum_{k=1}^{\infty}k2^{-km}\\ \le\ &C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)}; \end{aligned}$$

For $II_3^t(x)$, note that $|x+y-z|\sim|x_0+y-z|$ for $x\in\tilde{B}$ and $z\in R^n\setminus\tilde{B}$, we obtain from the similar method to the estimate of II_1,

$$||II_3^t(x)||\ \le\ C\int_{R^n}\left(\int_{R_+^{n+1}}\left[\frac{|f_2(z)||\Omega(y-z)|\chi_{\Gamma(z)}(y,t)|R_m(\tilde{A};x_0,z)|}{|y-z|^{n-1}|x_0-z|^m}\right.\right.$$

$$\times(\chi_{\Gamma(x)}(y,t)-\chi_{\Gamma(x_0)}(y,t))\Bigg]^2\frac{dydt}{t^{n+3}}\Bigg)^{1/2}dz$$

$$\leq C\int_{R^n}\frac{|f_2(z)||R_m(\tilde{A};x_0,z)|}{|x_0-z|^m}\Bigg|\int\int_{|x-y|\leq t}\frac{t^{-n-3}\chi_{\Gamma(z)}(y,t)}{|y-z|^{2n-2}}dydt$$

$$-\int\int_{|x_0-y|\leq t}\frac{t^{-n-3}\chi_{\Gamma(z)}(y,t)}{|y-z|^{2n-2}}dydt\Bigg|^{1/2}dz$$

$$\leq C\int_{R^n}\frac{|f_2(z)||R_m(\tilde{A};x_0,z)|}{|x_0-z|^m}\Bigg(\int\int_{|y|\leq t,|x+y-z|\leq t}\Bigg|\frac{1}{|x+y-z|^{2n-2}}$$

$$-\frac{1}{|x_0+y-z|^{2n-2}}\Bigg|\frac{dydt}{t^{n+3}}\Bigg)^{1/2}dz$$

$$\leq C\int_{R^n}\frac{|f_2(z)||R_m(\tilde{A};x_0,z)|}{|x_0-z|^m}$$

$$\times\Bigg(\int\int_{|y|\leq t,|x+y-z|\leq t}\frac{|x-x_0|}{|x+y-z|^{2n+2}}t^{-n}dydt\Bigg)^{1/2}dz$$

$$\leq C\int_{R^n}\frac{|f_2(z)||x-x_0|^{1/2}|R_m(\tilde{A};x_0,z)|}{|x_0-z|^{m+n+1/2}}dz$$

$$\leq C\sum_{k=0}^{\infty}\frac{kl^{1/2}(2^kl)^m}{(2^kl)^{n+m+1/2}}\sum_{|\alpha|=m}||D^\alpha A||_{BMO}(\int_{2^{k+1}\tilde{B}}|f(z)|dz)$$

$$\leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)}\sum_{k=0}^{\infty}k2^{-k/2}$$

$$\leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^\infty(w)};$$

For $II_4^t(x)$, by the similar method to the estimates of $II_1^t(x)$ and $II_3^t(x)$, we have

$$||II_4^t(x)||\leq C\int_{R^n\backslash\tilde{B}}\left[\frac{|x-x_0|}{|x-z|^{n+1}}+\frac{|x-x_0|^{1/2}}{|x-z|^{n+1/2}}\right]\times\sum_{|\alpha|=m}|D^\alpha\tilde{A}(z)||f(z)|dz$$

$$\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} ||f||_{L^\infty(w)} \sum_{k=0}^{\infty} k(2^{-k} + 2^{-k/2})$$
$$\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} ||f||_{L^\infty(w)}.$$

Combining these estimates, we completes the proof of Theorem 1. □

PROOF OF THEOREM 2. It suffices to show that there exists a constant $C > 0$ such that for every H^1-atom a(that is that a satisfies: supp$a \subset B = B(x_0, r)$, $||a||_{L^\infty(w)} \leq w(B)^{-1}$ and $\int_{R^n} a(y)dy = 0$(see [1])), the following holds:

$$||\tilde{\mu}_S^A(a)||_{L^1(w)} \leq C.$$

We write

$$\int_{R^n} \tilde{\mu}_S^A(a)(x)w(x)dx = \left[\int_{|x-x_0|\leq 2r} + \int_{|x-x_0|>2r}\right] \tilde{\mu}_S^A(a)(x)w(x)dx$$
$$:= J + JJ.$$

For J, by the following equality

$$Q_{m+1}(A; x, y) = R_{m+1}(A; x, y) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-y)^\alpha (D^\alpha A(x) - D^\alpha A(y)),$$

we have, by the similar method to the proof of Lemma 2,

$$\tilde{\mu}_S^A(a)(x) \leq \mu_S^A(a)(x) + C \sum_{|\alpha|=m} \int_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^n} |a(y)|dy,$$

thus, $\tilde{\mu}_\delta^A$ is L^∞-bounded by Lemma 2 and [2]. We see that

$$J \leq C||\tilde{\mu}_S^A(a)||_{L^\infty(w)} w(2B) \leq C||a||_{L^\infty(w)} w(B) \leq C.$$

To obtain the estimate of JJ, we denote that

$$\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{2B} x^\alpha,$$

then $Q_m(A; x, y) = Q_m(\tilde{A}; x, y)$. We write, by the vanishing moment of a and $Q_{m+1}(A; x, y) = R_m(A; x, y) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-y)^\alpha D^\alpha A(x)$, for

$x \in (2B)^c$,

$$\begin{aligned}
\tilde{F}_t^A(a)(x,y) &= \int_{|y-z|\le t} \frac{\Omega(y-z)R_m(\tilde{A};x,z)}{|y-z|^{n-1}|x-z|^m} a(z)dz \\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{|y-z|\le t} \frac{\Omega(y-z)D^\alpha \tilde{A}(x)(x-z)^\alpha}{|y-z|^{n-1}|x-z|^m} a(z)dz \\
&= \int_{R^n} \left[\frac{\chi_{\Gamma(y)}(z,t)\Omega(y-z)R_m(\tilde{A};x,z)}{|y-z|^{n-1}|x-z|^m} \right. \\
&\quad \left. - \frac{\chi_{\Gamma(y)}(x_0,t)\Omega(y-x_0)R_m(\tilde{A};x,x_0)}{|y-x_0|^{n-1}|x-x_0|^m} \right] a(z)dz \\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \left[\frac{\chi_{\Gamma(y)}(z,t)\Omega(y-z)(x-z)^\alpha}{|y-z|^{n-1}|x-z|^m} \right. \\
&\quad \left. - \frac{\chi_{\Gamma(y)}(x_0,t)\Omega(y-x_0)(x-x_0)^\alpha}{|y-x_0|^{n-1}|x-x_0|^m} \right] D^\alpha \tilde{A}(x)a(z)dz,
\end{aligned}$$

thus, by the similar method to the proof of II in Theorem 1, we obtain

$$\begin{aligned}
||\tilde{F}_t^A(a)(x,y)|| \le C\frac{|B|^{1+1/n}}{w(B)} \Big(\sum_{|\alpha|=m} ||D^\alpha A||_{BMO} |x-x_0|^{-n-1} \\
+ |x-x_0|^{-n-1} |D^\alpha \tilde{A}(x)| \Big),
\end{aligned}$$

note that if $w \in A_1$, then $\frac{w(B_2)}{|B_2|}\frac{|B_1|}{w(B_1)} \le C$ for all balls B_1, B_2 with $B_1 \subset B_2$. Thus, by Hölder' inequality and the reverse of Hölder' inequality for $w \in A_1$ and some $p > 1$ with $1/p + 1/p' = 1$, we obtain

$$\begin{aligned}
JJ &\le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} \sum_{k=1}^\infty 2^{-k} \left(\frac{|B|}{w(B)} \frac{w(2^{k+1}B)}{|2^{k+1}B|} \right) \\
&\quad + C \sum_{|\alpha|=m} \sum_{k=1}^\infty 2^{-k} \frac{|B|}{w(B)} \left(\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} w(x)^p dx \right)^{1/p} \\
&\quad \times \left(\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |D^\alpha \tilde{A}(x)|^{p'} dx \right)^{1/p'}
\end{aligned}$$

$$\begin{aligned} &\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \sum_{k=1}^{\infty} k2^{-k} \left(\frac{w(2^{k+1}B)}{|2^{k+1}B|} \frac{|B|}{w(B)} \right) \\ &\le C, \end{aligned}$$

which together with the estimate for J yields the desired result. This finishes the proof of Theorem 2. □

PROOF OF THEOREM 3. By the equality

$$R_{m+1}(A;x,y) = Q_{m+1}(A;x,y) + \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-y)^\alpha (D^\alpha A(x) - D^\alpha A(y))$$

and the similarity to the proof of Lemma 2, we get

$$\mu_S^A(f)(x) \le \tilde{\mu}_S^A(f)(x) + C \sum_{|\alpha|=m} \int_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^n} |f(y)| dy.$$

By Theorem 1, 2 and [2], we obtain

$$\begin{aligned} & w(\{x \in R^n : \mu_S^A(f)(x) > \lambda\}) \\ \le\ & w(\{x \in R^n : \tilde{\mu}_S^A(f)(x) > \lambda/2\}) \\ & + w\left(\left\{ x \in R^n : \sum_{|\alpha|=m} \int_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^n} |f(y)| dy > C\lambda \right\} \right) \\ \le\ & C\|f\|_{H^1(w)}/\lambda. \end{aligned}$$

This completes the proof of Theorem 3. □

PROOF OF THEOREM 4. **(i).** It suffices to show that there exists a constant $C > 0$ such that for every $H^1(w)$-atom a with $\text{supp} a \subset Q = Q(x_0, d)$, there is

$$\|\mu_S^A(a)\|_{L^1(w)} \le C.$$

Let $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}} x^\alpha$. We write, by the vanishing moment of a and for $u \in 3Q \setminus 2Q$,

$$\begin{aligned} & F_t^A(a)(x,y) \\ =\ & \chi_{4Q}(x) F_t^A(a)(x,y) \end{aligned}$$

$$+\chi_{(4Q)^c}(x)\int_{R^n}\left[\frac{R_m(\tilde{A};x,z)}{|x-z|^m}\frac{\Omega(y-z)}{|y-z|^{n-1}}\chi_{\Gamma(y)}(z,t)\right.$$
$$\left.-\frac{R_m(\tilde{A};x,u)}{|x-u|^m}\frac{\Omega(y-u)}{|y-u|^{n-1}}\chi_{\Gamma(y)}(u,t)\right]a(z)dz$$
$$-\chi_{(4Q)^c}(x)\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\left[\frac{(x-z)^\alpha}{|x-z|^m}\frac{\Omega(y-z)}{|y-z|^{n-1}}\chi_{\Gamma(y)}(z,t)\right.$$
$$\left.-\frac{(x-u)^\alpha}{|x-u|^m}\frac{\Omega(y-u)}{|y-u|^{n-1}}\chi_{\Gamma(y)}(u,t)\right]D^\alpha\tilde{A}(z)a(z)dz-\chi_{(4Q)^c}(x)$$
$$\times\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\frac{(x-u)^\alpha}{|x-u|^m}\frac{\Omega(y-u)}{|y-u|^{n-1}}\chi_{\Gamma(y)}(u,t)D^\alpha\tilde{A}(z)a(z)dz,$$

then

$$\begin{aligned}
&\mu_S^A(a)(x)\\
=\;&\left\|\chi_{\Gamma(x)}F_t^A(a)(x,y)\right\|\\
\le\;&\chi_{4Q}(x)\left\|\chi_{\Gamma(x)}F_t^A(a)(x,y)\right\|\\
&+\chi_{(4Q)^c}(x)\left\|\chi_{\Gamma(x)}\int_{R^n}\left[\frac{R_m(\tilde{A};x,z)}{|x-z|^m}\frac{\Omega(y-z)\chi_{\Gamma(y)}(z,t)}{|y-z|^{n-1}}\right.\right.\\
&\left.\left.-\frac{R_m(\tilde{A};x,u)}{|x-u|^m}\frac{\Omega(y-u)\chi_{\Gamma(y)}(u,t)}{|y-u|^{n-1}}\right]a(z)dz\right\|\\
&+\chi_{(4Q)^c}(x)\left\|\sum_{|\alpha|=m}\frac{\chi_{\Gamma(x)}}{\alpha!}\int_{R^n}\left[\frac{(x-z)^\alpha}{|x-z|^m}\frac{\Omega(y-z)\chi_{\Gamma(y)}(z,t)}{|y-z|^{n-1}}\right.\right.\\
&\left.\left.-\frac{(x-u)^\alpha}{|x-u|^m}\frac{\Omega(y-u)\chi_{\Gamma(y)}(u,t)}{|y-u|^{n-1}}\right]D^\alpha\tilde{A}(z)a(z)dz\right\|\\
&+\chi_{(4Q)^c}(x)\left\|\chi_{\Gamma(x)}\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\frac{(x-u)^\alpha}{|x-u|^m}\frac{\Omega(y-u)}{|y-u|^{n-1}}\chi_{\Gamma(y)}(u,t)\right.\\
&\left.\times D^\alpha\tilde{A}(z)a(z)dy\right\|\\
=\;&K_1(x)+K_2(x,u)+K_3(x,u)+K_4(x,u).
\end{aligned}$$

By the $L^p(w)$-boundedness of μ_S^A, we get

$$\int_{R^n} K_1(x)w(x)dx = \int_{4Q} \mu_S^A(a)(x)w(x)dx \le C||a||_{L^\infty(w)}w(Q) \le C;$$

For $K_2(x,u)$, we write

$$\begin{aligned}
&\frac{R_m(\tilde{A};x,z)}{|x-z|^m}\frac{\Omega(y-z)}{|y-z|^{n-1}}\chi_{\Gamma(y)}(z,t)\\
&-\frac{R_m(\tilde{A};x,u)}{|x-u|^m}\frac{\Omega(y-u)}{|y-u|^{n-1}}\chi_{\Gamma(y)}(u,t)\\
=\ &(\chi_{\Gamma(y)}(z,t)-\chi_{\Gamma(y)}(u,t))\frac{\Omega(y-z)R_m(\tilde{A};x,z)}{|y-z|^{n-1}|x-z|^m}\\
&+\left[\frac{\Omega(y-z)}{|y-z|^{n-1}}-\frac{\Omega(y-u)}{|y-u|^{n-1}}\right]\frac{R_m(\tilde{A};x,z)}{|x-z|^m}\chi_{\Gamma(y)}(u,t)\\
&+\frac{\Omega(y-u)\chi_{\Gamma(y)}(u,t)}{|y-u|^{n-1}}\left(\frac{R_m(\tilde{A};x,z)}{|x-z|^m}-\frac{R_m(\tilde{A};x,u)}{|x-u|^m}\right).
\end{aligned}$$

By the following inequality (see [18]):

$$\left|\frac{\Omega(y-z)}{|y-z|^{n-1}}-\frac{\Omega(y-u)}{|y-u|^{n-1}}\right| \le C\left(\frac{|z-u|}{|y-z|^n}+\frac{|z-u|^\gamma}{|y-z|^{n-1+\gamma}}\right)$$

and note that

$$\begin{aligned}
&\left\|\chi_{\Gamma(x)}\int_{R^n}(\chi_{\Gamma(y)}(z,t)-\chi_{\Gamma(y)}(u,t))\frac{\Omega(y-z)R_m(\tilde{A};x,z)}{|y-z|^{n-1}|x-z|^m}a(z)dz\right\|\\
\le\ & C\int_{R^n}\frac{|a(z)||R_m(\tilde{A};x,z)|}{|x-z|^m}\\
&\times\left(\int\int_{R_+^{n+1}}\frac{\chi_{\Gamma(x)}(y,t)|\chi_{\Gamma(y)}(z,t)-\chi_{\Gamma(y)}(u,t)|^2}{|y-z|^{2n-2}}\frac{dydt}{t^{n+3}}\right)^{1/2}dz\\
\le\ & C\int_{R^n}\frac{|a(z)||R_m(\tilde{A};x,z)|}{|x-z|^m}\\
&\times\left|\int\int_{\Gamma(x),\ \Gamma(z)}\frac{t^{-n-3}dydt}{|y-z|^{2n-2}}-\int\int_{\Gamma(x),\ \Gamma(u)}\frac{t^{-n-3}dydt}{|y-z|^{2n-2}}\right|^{1/2}dz
\end{aligned}$$

$$\leq C\int_{R^n}\frac{|a(z)||R_m(\tilde{A};x,z)|}{|x-z|^m}\left(\int\int_{|y|\leq t,|x+y-z|\leq t}\left|\frac{1}{|x+y-z|^{2n-2}}\right.\right.$$

$$\left.\left.-\frac{1}{|x+y-u|^{2n-2}}\right|\frac{dydt}{t^{n+3}}\right)^{1/2}dz$$

$$\leq C\int_{R^n}\frac{|a(z)||R_m(\tilde{A};x,z)|}{|x-z|^m}$$

$$\times\left(\int\int_{|y|\leq t,|x+y-z|\leq t}\frac{|u-z|t^{-n-3}dydt}{|x+y-z|^{2n-1}}\right)^{1/2}dz$$

$$\leq C\int_{R^n}\frac{|a(z)||R_m(\tilde{A};x,z)|}{|x-z|^m}\frac{|u-z|^{1/2}}{|x-z|^{n+1/2}}dz,$$

by the similarity of the proof of Theorem 1, we obtain

$$\int_{R^n}K_2(x,u)w(x)dx$$

$$\leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=2}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}$$

$$\times\int_Q k\left(\frac{|u-z|}{|x-z|^{n+1}}+\frac{|u-z|^{1/2}}{|x-z|^{n+1/2}}+\frac{|u-z|^\gamma}{|x-z|^{n+\gamma}}\right)|a(z)|dzw(x)dx$$

$$\leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=2}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}k$$

$$\times\left(\frac{d}{(2^kd)^{n+1}}+\frac{d^{1/2}}{(2^kd)^{n+1/2}}+\frac{d^\gamma}{(2^kd)^{n+\gamma}}\right)||a||_{L^\infty(w)}|Q|w(x)dx$$

$$\leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=2}^{\infty}k(2^{-k}+2^{-k/2}+2^{-\gamma k})\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}\frac{|Q|}{w(Q)}$$

$$\leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=2}^{\infty}k(2^{-k}+2^{-k/2}+2^{-\gamma k})\leq C;$$

Similarly, we get

$$\int_{R^n}K_3(x,u)w(x)dx$$

$$\le C\sum_{|\alpha|=m}\sum_{k=2}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\int_Q\left(\frac{|u-z|}{|x-z|^{n+1}}+\frac{|u-z|^{1/2}}{|x-z|^{n+1/2}}\right.$$
$$\left.+\frac{|u-z|^{\gamma}}{|x-z|^{n+\gamma}}\right)|D^\alpha\tilde{A}(z)||a(z)|dzw(x)dx$$
$$\le C\sum_{|\alpha|=m}\sum_{k=2}^{\infty}\left(\frac{d}{(2^kd)^{n+1}}+\frac{d^{1/2}}{(2^kd)^{n+1/2}}+\frac{d^{\gamma}}{(2^kd)^{n+\gamma}}\right)$$
$$\times\left(\frac{1}{|Q|}\int_Q|D^\alpha\tilde{A}(y)|dy\right)||a||_{L^\infty(w)}|Q|w(2^{k+1}Q)$$
$$\le C.$$

Thus, by using the condition of $K_4(x,u)$, we obtain

$$\int_{R^n}\mu_S^A(a)(x)w(x)dx\le C.$$

(ii). For any cube $Q=Q(x_0,d)$, let

$$\tilde{A}(x)=A(x)-\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}}x^\alpha.$$

We write, for $f=f\chi_{4Q}+f\chi_{(4Q)^c}=f_1+f_2$ and $u\in 3Q\setminus 2Q$,

$$\tilde{F}_t^A(f)(x,y)$$
$$=\tilde{F}_t^A(f_1)(x,y)+\int_{|y-z|\le t}\frac{R_m(\tilde{A};x,z)}{|x-z|^m}\frac{\Omega(y-z)}{|y-z|^{n-1}}f_2(z)dz$$
$$-\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x)-(D^\alpha A)_Q)$$
$$\times\int_{|y-z|\le t}\left[\frac{(x-z)^\alpha\Omega(y-z)}{|x-z|^m|y-z|^{n-1}}-\frac{(u-z)^\alpha\Omega(y-z)}{|u-z|^m|y-z|^{n-1}}\right]f_2(z)dz$$
$$-\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x)-(D^\alpha A)_Q)$$
$$\times\int_{|y-z|\le t}\frac{(u-z)^\alpha}{|u-z|^m}\frac{\Omega(y-z)}{|y-z|^{n-1}}f_2(z)dz,$$

then

$$\begin{aligned}
&\left|\tilde{\mu}_S^A(f)(x) - \mu_S\left(\frac{R_m(\tilde{A};x_0,\cdot)}{|x_0-\cdot|^m}f_2\right)(x_0)\right| \\
= &\left|\left\|\chi_{\Gamma(x)}\tilde{F}_t^A(f)(x,y)\right\| - \left\|\chi_{\Gamma(x_0)}F_t\left(\frac{R_m(\tilde{A};x_0,\cdot)}{|x_0-\cdot|^m}f_2\right)(y)\right\|\right| \\
\le &\left\|\chi_{\Gamma(x)}\tilde{F}_t^A(f)(x,y) - \chi_{\Gamma(x_0)}F_t\left(\frac{R_m(\tilde{A};x_0,\cdot)}{|x_0-\cdot|^m}f_2\right)(y)\right\| \\
\le &\left\|\chi_{\Gamma(x)}(y,t)\tilde{F}_t^A(f_1)(x,y)\right\| \\
&+\left\|\chi_{\Gamma(x)}(y,t)\int_{|y-z|\le t}\left[\frac{R_m(\tilde{A};x,z)\Omega(y-z)}{|x-z|^m|y-z|^{n-1}}\right.\right. \\
&\left.\left.-\chi_{\Gamma(x_0)}(y,t)\int_{|y-z|\le t}\frac{R_m(\tilde{A};x_0,z)\Omega(y-z)}{|x_0-z|^m|y-z|^{n-1}}\right]f_2(z)dz\right\| \\
&+\left\|\chi_{\Gamma(x)}(y,t)\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x)\right. \\
&-(D^\alpha A)_Q)\int_{|y-z|\le t}\left[\frac{\Omega(y-z)(x-z)^\alpha}{|y-z|^{n-1}|x-z|^m}\right. \\
&\left.\left.-\frac{\Omega(y-z)(u-z)^\alpha}{|y-z|^{n-1}|u-z|^m}\right]f_2(z)dz\right\| \\
&+\left\|\chi_{\Gamma(x)}(y,t)\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x)\right. \\
&\left.-(D^\alpha A)_Q)\int_{|y-z|\le t}\frac{\Omega(y-z)(u-z)^\alpha}{|y-z|^{n-1}|u-z|^m}f_2(z)dz\right\| \\
= &L_1(x)+L_2(x)+L_3(x,u)+L_4(x,u).
\end{aligned}$$

By the $L^p(w)$-boundedness of $\tilde{\mu}_S^A$, we get

$$\frac{1}{w(Q)}\int_Q L_1(x)w(x)dx \le C||f||_{L^\infty(w)};$$

For $L_2(x)$, we write

$$\begin{aligned}
&\chi_{\Gamma(x)}(y,t)\frac{R_m(\tilde{A};x,z)\Omega(y-z)}{|x-z|^m|y-z|^{n-1}} \\
&-\chi_{\Gamma(x_0)}(y,t)\frac{R_m(\tilde{A};x_0,z)\Omega(y-z)}{|x_0-z|^m|y-z|^{n-1}} \\
= \ &\chi_{\Gamma(x)}(y,t)\left[\frac{R_m(\tilde{A};x,z)}{|x-z|^m}-\frac{R_m(\tilde{A};x_0,z)}{|x_0-z|^m}\right]\frac{\Omega(y-z)}{|y-z|^{n-1}} \\
&+(\chi_{\Gamma(x)}(y,t)-\chi_{\Gamma(x_0)}(y,t))\frac{R_m(\tilde{A};x_0,z)}{|x_0-z|^m}\frac{\Omega(y-z)}{|y-z|^{n-1}},
\end{aligned}$$

then, by similarity to the proof of Lemma 2 and $K_2(x,u)$, we obtain

$$\begin{aligned}
&\frac{1}{w(Q)}\int_Q L_2(x)w(x)dx \\
\leq \ &C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=2}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}k \\
&\times\left(\frac{|x-x_0|}{|x-y|^{n+1}}+\frac{|x-x_0|^{1/2}}{|x-y|^{n+1/2}}+\frac{|x-x_0|^\gamma}{|x-y|^{n+\gamma}}\right)|f(y)|dy \\
\leq \ &C||f||_{L^\infty(w)};
\end{aligned}$$

Similarly, we get

$$\frac{1}{w(Q)}\int_Q L_3(x,u)w(x)dx \leq C||f||_{L^\infty(w)}.$$

Thus, by using the condition of $L_4(x,u)$, we obtain

$$\begin{aligned}
&\frac{1}{w(Q)}\int_Q\left|\tilde{\mu}_S^A(f)(x)-\mu_S\left(\frac{R_m(\tilde{A};x_0,\cdot)}{|x_0-\cdot|^m}f_2\right)(x_0)\right|w(x)dx \\
&\leq C||f||_{L^\infty(w)}.
\end{aligned}$$

This completes the proof of Theorem 4. □

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