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DIMENSION OF DEFORMED SELF-SIMILAR SETS

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ABSTRACT. We generalize S. Ikeda's results for perturbed cantor sets showing how we get the dimensions for deformed self-similar sets.

1. Introduction

In [5] we define deformed self-similar sets. The construction of these sets go as follows. For more details (see [4], [5]).

Put $X = [0,1]$. Fix $m \geq 2$, write $S_k = \{1, 2, \cdots, m\}^k$ and $S^* = \cup_{k=1}^{\infty} S_k$. Consider a sequence of similarities $\{\phi_\sigma : X \to X | \sigma \in S^*\}$. Suppose that each ϕ_σ has a contraction ratio r_σ, that is, $|\phi_\sigma(x) - \phi_\sigma(y)| = r_\sigma |x - y|$ for any $x, y \in [0,1]$, where $|\cdot|$ is the Euclidean norm. We further assume there exists $0 < \alpha, \beta < 1$ such that $\alpha < r_\sigma < \beta$ for any $\sigma \in S^*$ and $\phi_{i_1 i_2 \cdots i_{k-1} i_k}(X) \cap \phi_{i_1 i_2 \cdots i_{k-1} i_{k'}}(X) = \emptyset$ if $i_k \neq i'_k$. For brevity, we write

$$\Phi_\sigma \equiv \phi_{i_1} \circ \phi_{i_1 i_2} \circ \cdots \circ \phi_{i_1 i_2 \cdots i_k}$$
$$R_\sigma \equiv r_{i_1} r_{i_1 i_2} \cdots r_{i_1 i_2 \cdots i_k}$$

for any $\sigma = i_1 i_2 \cdots i_k \in S_k$ and write $|\sigma| = k$ if $\sigma \in S_k$. Then we obtain an unique compact set K,

$$K = \bigcap_{k=1}^{\infty} \bigcup_{|\sigma|=k} \Phi_\sigma(X).$$

We call this set K a deformed self-similar set on $[0,1]$. We note that K is a generalized Cantor set since K is a loosely self-similar set ([2]) if we take $r_{\sigma j} = r_j$ for all σ and that K is a perturbed Cantor set ([3]) if we append the following condition to the sequence of similarities

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$\{\phi_\sigma : X \to X | \sigma \in S^*\}$; for $\sigma = i_1 i_2 \cdots i_n$ and $\tau = j_1 j_2 \cdots j_n$, there exists $\delta > 0$ such that

$$\mathrm{dist}(\Phi_\sigma(X), \Phi_\tau(X)) \geq \delta \max(r_\sigma(X), r_\tau(X))$$

if $i_k = j_k (k = 1, 2, \cdots, n-1)$ and $i_n \neq j_n$.

Now we show how to calculate the dimensions of the deformed self-similar sets. We note that it works for perturbed cantor sets.

2. Hausdorff dimension of a deformed self-similar set

We begin to recall the definition of the Hausdorff measure and dimension ([1]). In this paper, we write $|A|$ for the diameter of A. Let E be a bounded subset of $\mathbf{R}$ and $s \geq 0$.

$$H^s(E) = \lim_{\delta \to 0} H^s_\delta(E)$$

where

$$H^s_\delta(E) = \inf\{\sum_{i=1}^{\infty} |U_i|^s : E \subset \cup_i U_i, |U_i| \leq \delta\}.$$

The Hausdorff dimension of E is defined by

$$\dim_H E = \sup\{s \geq 0 : H^s(E) = \infty\} = \inf\{s \geq 0 : H^s(E) = 0\}.$$

In [5], we define a new metric outer measure M^s ;

$$M^s(E) = \lim_{n \to \infty} M^s_n(E)$$

where

$$M^s_n(E) = \inf\{\sum |\Phi_\sigma(X)|^s : E \subset \cup_\sigma \Phi_\sigma(X), |\sigma| \geq n\}.$$

The following lemma tells us that we may use M^s-measure to get the Hausdorff dimension of deformed self-similar sets.

Lemma 1. [5] *Let K be a deformed self-similar set. Then*

$$\dim_H K = \sup\{s \geq 0 : M^s(K) = \infty\} = \inf\{s \geq 0 : M^s(K) = 0\}.$$

For simplicity, we denote $\Phi_\sigma(x)$ for $\Phi_\sigma(X)$ containing x.

Theorem 2. *Let K be a deformed self-similar set. Suppose that there exists a finite Borel measure ν with supp $(\nu) \subset K$ and $d \in (0,1)$ such that if for any $x \in K$*

$$\lim_{k \to \infty} \sup_{|\sigma| \geq k} \frac{\nu(\Phi_\sigma(x))}{|\Phi_\sigma(x)|^s} = 0 \quad \textit{for any } s < d.$$

Then

$$d \leq \dim_H K.$$

PROOF. Since $\lim_{k\to\infty} \sup_{|\sigma|\geq k} \frac{\nu(\Phi_\sigma(x))}{|\Phi_\sigma(x)|^s} = 0$ for any $s < d$ and for any $x \in K$, there exists n such that $\nu(\Phi_\sigma(x)) < |\Phi_\sigma(x)|^s$ for $|\sigma| \geq n$ and $x \in K$. For any $\rho > 0$, let

$$K_\rho = \{x \in K : |\Phi_\sigma(x)| \geq \rho \text{ or } \nu(\Phi_\sigma(x)) <| \Phi_\sigma(x) |^s \text{ for any } \Phi_\sigma(x)\}.$$

Then K_ρ increases to K as $\rho \to 0$ and there exists $\rho > 0$ such that $\nu(K_\rho) > 0$ since $\nu(K) > 0$.

For any $\varepsilon > 0$ and some $n \in \mathbf{N}$ with $\beta^n \leq \rho$, there exists a cover $\{\Phi_\sigma(x)|x \in K_\rho\}$ of K_ρ such that $|\sigma| \geq n$ and $\sum |\Phi_\sigma(x)|^s \leq M_n^s(K_\rho) + \varepsilon$ by definition of M_n^s. Hence for given $\varepsilon > 0$,

$$\begin{aligned} M_n^s(K) + \varepsilon &\geq M_n^s(K_\rho) + \varepsilon \\ &\geq \sum |\Phi_\sigma(x)|^s \\ &> \sum \nu(\Phi_\sigma(x)) \\ &\geq \nu(\bigcup \Phi_\sigma(x)) \\ &> \nu(K_\rho). \end{aligned}$$

Since ε is arbitrary, $M_n^s(K) \geq \nu(K_\rho) > 0$. Therefore $M^s(K) > 0$ for any $s < d$. This implies $d \leq \dim_H K$ by Lemma 1. □

THEOREM 3. *Let K be a deformed self-similar set. Suppose that there exists a finite Borel measure ν with supp $(\nu) \subset K$ and $d \in (0,1)$ such that if for any $x \in K$*

$$\lim_{k\to\infty} \sup_{|\sigma|\geq k} \frac{\nu(\Phi_\sigma(x))}{|\Phi_\sigma(x)|^s} = \infty \quad \text{for any } s > d.$$

Then

$$\dim_H K \leq d.$$

PROOF. Since $\lim_{k\to\infty} \sup_{|\sigma|\geq k} \frac{\nu(\Phi_\sigma(x))}{|\Phi_\sigma(x)|^s} = \infty$ for any $s > d$ and for $x \in K$, we have that $|\Phi_\sigma(x)|^s < \nu(\Phi_\sigma(x))$ for infinitely many σ and any $x \in K$. For any $\rho > 0$, let

$$\mathcal{C}_\rho = \{\Phi_\sigma(x) : x \in K, |\Phi_\sigma(x)| \leq \rho \text{ and } |\Phi_\sigma(x)|^s < \nu(\Phi_\sigma(x))\}.$$

Then $\mathcal{C}_\rho$ becomes a Vitali covering of K, and then by the Vitali covering theorem, there exists a countable disjoint sequence $\{\Phi_{\sigma_i}(x_i)\}$ from $\mathcal{C}_\rho$ such that $M^s(K \setminus \cup_{i=1}^\infty \Phi_{\sigma_i}(x_i)) = 0$ since $M^s(K) < \infty$ (see [5]).

We see $M^s(E) = \lim M_n^s(E)$, so for any $\varepsilon > 0$, we take n_o such that $M^s(E) - \varepsilon \leq M_{n_o}^s(E)$. And put $\rho = \alpha^{n_o}$. Then for some countable disjoint sequence $\{\Phi_{\sigma_i}(x_i)\}$ from $\mathcal{C}_\rho$ with $M^s(K \setminus \cup_{i=1}^{\infty} \Phi_{\sigma_i}(x_i)) = 0$ we get

$$\begin{aligned}
M^s(K) - \varepsilon &\leq M_{n_o}^s(K) \\
&\leq M_{n_o}^s(K \cap \bigcup \Phi_{\sigma_i}(x_i)) + M_{n_o}^s(K \setminus \bigcup_{i=1}^{\infty} \Phi_{\sigma_i}(x_i)) \\
&\leq M_{n_o}^s(\bigcup \Phi_{\sigma_i}(x_i)) + M^s(K \setminus \bigcup_{i=1}^{\infty} \Phi_{\sigma_i}(x_i)) \\
&\leq \sum | \Phi_{\sigma_i}(x_i) |^s \\
&< \sum \nu(\Phi_{\sigma_i}(x_i)) \\
&= \nu(\bigcup \Phi_{\sigma_i}(x_i)) \\
&\leq \nu(\mathbf{R}).
\end{aligned}$$

Since ε is arbitrary, we have $M^s(K) \leq \nu(\mathbf{R}) < \infty$ for any $s > d$. Thus $\dim_H K \leq d$ by Lemma 1. □

3. Packing dimension of a deformed self-similar set

Let's begin with recalling the packing classes for a bounded E of $\mathbf{R}$ ([7]). Let $\mathcal{C}_0$ stand for the class of all countable families of disjoint balls $\{B_i(x_i)\}$ with their centers $x_i \in E$ and $\mathcal{C}_1$ stand for the class of all countable families of disjoint open balls $\{B_i\}$ with $\overline{B_i} \cap \overline{E} \neq \emptyset$.

For each i, $\{I_{i_n}\}$ is called $\mathcal{C}_i$-type δ-packing of E if $\{I_{i_n}\} \in \mathcal{C}_i$ with $|I_{i_n}| < \delta$, for arbitrary i_n. For $s \geq 0$, put

$$P_i^s(E) = \limsup_{\delta \to 0} \{\sum |I_{i_n}|^s : \{I_{i_n}\} \text{ is a } \mathcal{C}_i\text{-type } \delta\text{-packing of } E\}, i = 0, 1$$

and

$$p_i^s(E) = \inf\{\sum P_i^s(E_n) : E_n \text{ is bounded and } E = \cup E_n\}, i = 0, 1.$$

Then $p_i^s(E)$ is a metric measure. Usually we call $p_0^s(E)$ the s-dimensional packing measure of E. For any bounded $E \subset \mathbf{R}$,

$$\mathcal{C}_i\text{-Dim}E \equiv \sup\{s > 0 : p_i^s(E) = \infty\} = \inf\{s > 0 : p_i^s(E) = 0\}.$$

It is well known that

$$\text{Dim}E \equiv \mathcal{C}_0\text{-Dim}E = \mathcal{C}_1\text{-Dim}E$$

where Dim$E$ is the packing dimension of E.

Before we go to our main result, let's put an easy but useful property for packing measure.

LEMMA 4. [7] *Let E be any subset of a bounded set K. Then*

$$p_i^s(E) = \inf\{\lim_{n\to\infty} P_i^s(E_n) : E_n \uparrow E\}, i = 0, 1.$$

In [4], we used an auxiliary packing measure of K to get the packing dimension of K. This goes as follows; $\{\Phi_\sigma(x)\}$ is called $\mathcal{C}_2$-type δ-packing of $E \subset K$ if it satisfies

(1) $\Phi_\sigma(x) \cap \Phi_{\sigma'}(x') = \emptyset$, for any $\sigma \neq \sigma'$ and $x, x' \in E$

(2) $|\sigma| \geq n$, for $n \geq \dfrac{\log \delta}{\log \beta}$

(3) $\Phi_\sigma(x) \cap \overline{E} \neq \emptyset$.

For $s \geq 0$, put

$$P_2^s(E) = \lim_{\delta\to 0} \sup\{\sum |\Phi_\sigma(x)|^s : \{\Phi_\sigma(x)\} \text{ is a } \mathcal{C}_2\text{-type } \delta\text{-packing of } E\}$$

and

$$p_2^s(E) = \inf\{\sum P_2^s(E_n) : E_n \text{ is a bounded and } E = \cup E_n\}.$$

The following lemma says that we may use the above p_2^s measure to get the packing dimension of deformed self-similar sets.

LEMMA 5. [4] *Let K be a deformed self-similar set. Then for $D \subset K$,*

$$p_2^s(D) \leq p_1^s(D)$$

and

$$p_0^s(D) \leq \left(\frac{2}{\alpha}\right)^s p_2^s(D)$$

where α is the number as in the introduction. In particular, we have

$$\text{Dim}K = \sup\{s > 0 : p_2^s(K) = \infty\}.$$

Now we calculate the packing dimension of K by using above lemmas. The idea is paralleling with those of Hausdorff dimension.

THEOREM 6. *Let K be a deformed self-similar set. Suppose that there exists a finite Borel measure ν with supp $(\nu) \subset K$ and $d \in (0,1)$ such that if for any $x \in K$*

$$\lim_{k\to\infty} \inf_{|\sigma|\geq k} \frac{\nu(\Phi_\sigma(x))}{|\Phi_\sigma(x)|^s} = 0 \quad \textit{for any } s < d.$$

Then

$$d \leq \text{Dim } K.$$

PROOF. For $\varepsilon > 0$ there exists a sequence $\{K_n\}$ such that K_n increases to K and $\lim_{n\to\infty} P_1^s(K_n) < p_1^s(K) + \varepsilon$ by Lemma 4. We note that there exists n_o such that $\nu(K_{n_o}) > 0$ since $\nu(K) > 0$.

Since $\lim_{k\to\infty} \inf_{|\sigma|\geq k} \frac{\nu(\Phi_\sigma(x))}{|\Phi_\sigma(x)|^s} = 0$ for any $s < d$ and for any $x \in K$, it is true that $\nu(\Phi_\sigma(x)) < |\Phi_\sigma(x)|^s$ for infinitely many σ and $x \in K$. For any $\rho > 0$, let

$$\mathcal{C}_\rho = \{\Phi_\sigma(x) : x \in K_{n_o}, |\Phi_\sigma(x)| < \rho \text{ and } \nu(\Phi_\sigma(x)) < |\Phi_\sigma(x)|^s\}.$$

Then $\mathcal{C}_\rho$ becomes a Vitali covering of K_{n_o} and by Vitali covering theorem there exists a countable disjoint sequence $\{\Phi_{\sigma_i}(x_i)\}$ from $\mathcal{C}_\rho$ such that $\nu(K_{n_o} \setminus \cup_{i=1}^\infty \Phi_{\sigma_i}(x_i)) = 0$. Here $\{\Phi_{\sigma_i}(x_i)\}$ is C_2-type ρ-packing of K_{n_o}.

Therefore

$$\begin{aligned} 0 < \nu(K_{n_o}) &= \nu(K_{n_o} \cap \bigcup_{i=1}^{\infty} \Phi_{\sigma_i}(x_i)) + \nu(K_{n_o} \setminus \bigcup_{i=1}^{\infty} \Phi_{\sigma_i}(x_i)) \\ &\leq \nu(\bigcup \Phi_{\sigma_i}(x_i)) \\ &= \sum \nu(\Phi_{\sigma_i}(x_i)) \\ &< \sum | \Phi_{\sigma_i}(x_i) |^s \\ &\leq P_2^s(K_{n_o}). \end{aligned}$$

Hence we get $0 < \nu(K_{n_o}) < P_2^s(K_{n_o}) \leq \lim_{n\to\infty} P_2^s(K_n) \leq \lim_{n\to\infty} P_1^s(K_n) < p_1^s(K) + \varepsilon$ by Lemma 5. We conclude that $p_1^s(K) > 0$ for any $s < d$ and Dim $K \geq d$. □

THEOREM 7. *Let K be a deformed self-similar set. Suppose that there exists a finite Borel measure ν with supp $(\nu) \subset K$ and $d \in (0,1)$ such that if for any $x \in K$*

$$\lim_{k\to\infty} \inf_{|\sigma|\geq k} \frac{\nu(\Phi_\sigma(x))}{|\Phi_\sigma(x)|^s} = \infty \quad \text{for any } s > d.$$

Then

$$\text{Dim } K \leq d.$$

PROOF. Since $\lim_{k\to\infty} \inf_{|\sigma|\geq k} \frac{\nu(\Phi_\sigma(x))}{|\Phi_\sigma(x)|^s} = \infty$ for any $s < d$ and any $x \in K$, there exists n such that $\nu(\Phi_\sigma(x)) > |\Phi_\sigma(x)|^s$ for any $|\sigma| \geq n$. Now for $\rho > 0$, let

$$K_\rho = \{x \in K : |\Phi_\sigma(x)| \geq \rho \text{ or } | \Phi_\sigma(x) |^s < \nu(\Phi_\sigma(x)) \text{ for any } \Phi_\sigma(x)\}.$$

Then K_ρ increases to K as $\rho \to 0$. Hence for any $\delta > 0$

$$\begin{aligned} P_2^s(K_\rho) &\leq \sup\{\sum|\Phi_\sigma(x)|^s : \{\Phi_\sigma(x)\} \text{ is a } \mathcal{C}_2\text{-type } \delta\text{-packing of } K_\rho\} \\ &\leq \sup\{\sum \nu(\Phi_\sigma(x)) : \{\Phi_\sigma(x)\} \text{ is a } \mathcal{C}_2\text{-type } \delta\text{-packing of } K_\rho\} \\ &= \sup\{\nu(\bigcup \Phi_\sigma(x)) : \{\Phi_\sigma(x)\} \text{ is a } \mathcal{C}_2\text{-type } \delta\text{-packing of } K_\rho\} \\ &< \nu(\mathbf{R}) \end{aligned}$$

Therefore we get $p_0^s(K) \leq \lim_{\rho\to 0} P_0^s(K_\rho) \leq (\frac{2}{\alpha})^s \lim_{\rho\to 0} P_2^s(K_\rho) \leq (\frac{2}{\alpha})^s \nu(\mathbf{R}) < \infty$ for any $s > d$ by Lemma 4 and Lemma 5. This implies that $\mathrm{Dim}(K) \leq d$. □

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