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ON THE SOLUTION AND STABILITY OF THE QUADRATIC TYPE FUNCTIONAL EQUATIONS

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ABSTRACT. In this paper we prove the solution and stability of the quadratic type functional equations with two variables

$$4f(\frac{x-2y}{2}) + f(x) + 2f(y) = 8f(\frac{x-y}{2}) + f(2y)$$

and

$$f(x-2y) + f(x) + 2f(y) = 2f(x-y) + f(2y).$$

1. Introduction

In 1940 S. M. Ulam ([6]) gave a wide range talk before the mathematic club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms :

Let X be a group and let Y be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : X \to Y$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta$$

for all $x, y \in X$, then there exists a homomorphism $H : X \to Y$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in X$?

The case of approximately additive mappings was solved by D. H. Hyers ([2]) under the assumptions that X and Y are Banach spaces.

In 1978 Th. M. Rassias ([3]) generalized the result of Hyers as follows : Let $f : X \to Y$ be a mapping between Banach spaces and let $0 \le p < 1$ be fixed. If f satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \theta(\|x\|^p + \|y\|^p)$$

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for some $\theta \geq 0$ and all $x, y \in X$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\|A(x) - f(x)\| \leq \frac{2\theta}{2-2^p}\|x\|^p$$

for all $x \in X$. If, in addition, $f(tx)$ is continuous in t for each fixed $x \in X$, then A is linear.

In 2001 J. H. Bae and I. S. Chang ([1]) investigated the Hyers-Ulam-Rassias stability of a quadratic functional equation

$$f(x+y+z) + f(x-y) + f(y-z) + f(x-z) = 3f(x) + 3f(y) + 3f(z)$$

and the stability of the above functional equation on bounded domains.

In 2002 Y. W. Lee ([4]) showed the solution and stability of a quadratic Jensen type functional equation with three variables

$$\begin{aligned} & 9f(\frac{x+y+z}{3}) + f(x) + f(y) + f(z) \\ = \; & 4[f(\frac{x+y}{2}) + f(\frac{y+z}{2}) + f(\frac{z+x}{2})]. \end{aligned}$$

That is, let X and Y be real linear spaces. A function $f : X \to Y$ satisfies the above functional equation for all $x, y, z \in X$ if and only if there exist an element $B \in Y$, an additive function $A : X \to Y$, and a quadratic function $Q : X \to Y$ such that

$$f(x) = Q(x) + A(x) + B$$

for all $x \in X$. And he proved the stability of the above functional equation.

The following functional equation with three variables

$$\begin{aligned} & 9f(\frac{x+y+z}{3}) + 9f(\frac{x-y+z}{3}) \\ & +9f(\frac{x+y-z}{3}) + 9f(\frac{-x+y+z}{3}) \\ = \; & 4f(x) + 4f(y) + 4f(z) \end{aligned}$$

has a solution presented as the sum of an additive function and a quadratic function ([5]).

The quadratic function $f(x) = x^2$ is a solution of the following functional equations

$$f(x+y) + f(x-y) = 2f(x) + 2f(y),$$

$$f(x-y-z) + f(x) + f(y) + f(z) = f(x-y) + f(y+z) + f(z-x),$$

$$f(x+y+z)+f(x-y+z)+f(x+y-z)+f(-x+y+z)$$
$$=4f(x)+4f(y)+4f(z)$$

and

$$f(x+y+z)+f(x-y)+f(y-z)+f(z-x)=3f(x)+3f(y)+3f(z).$$

So it is natural that each equation is called a quadratic functional equation. In particular every solution of the quadratic functional equation

$$f(x-y-z)+f(x)+f(y)+f(z)=f(x-y)+f(y+z)+f(z-x)$$

is called a quadratic function.

In this paper we investigate the stability of the quadratic type functional equations

$$4f(\frac{x-2y}{2})+f(x)+2f(y)=8f(\frac{x-y}{2})+f(2y)$$

and

$$f(x-2y)+f(x)+2f(y)=2f(x-y)+f(2y).$$

2. Solutions of the quadratic functional equation

Throughout this section X and Y will be real linear spaces. Given a function $f: X \to Y$, consider the following functional equations ;

$$4f(\frac{x-2y}{2})+f(x)+2f(y)=8f(\frac{x-y}{2})+f(2y) \tag{1}$$

and

$$f(x-2y)+f(x)+2f(y)=2f(x-y)+f(2y). \tag{2}$$

We prove that the solutions of the functional equations (1) and (2) are $Q(x)$ and $Q(x)+A(x)$, respectively, where $Q(x)$ is quadratic and $A(x)$ is additive. And then we also prove the stability of equations (1) and (2).

THEOREM 2.1. *Let X and Y be real vector spaces. A function f : $X \to Y$ satisfies* (1) *for all $x, y \in X$ if and only if f is a quadratic function.*

PROOF. (Necessity). By putting $x=y=0$ in (1), we have $f(0)=0$. By letting $y=0$ in (1), we get $4f(\frac{x}{2})=f(x)$ and so $f(2x)=4f(x)$ for all $x \in X$. Thus f satisfies the functional equation

$$4f(\frac{x-2y}{2})+f(x)+2f(y)=8f(\frac{x-y}{2})+f(2y)$$

for all $x, y \in X$. Therefore we have

$$f(x-2y)+f(x)+2f(y)=2\cdot 4f(\frac{x-y}{2})+4f(y)$$

for all $x, y \in X$. Hence we get

$$f(x-2y)+f(x)=2f(x-y)+2f(y)$$

for all $x, y \in X$.

Letting $x-y=u$ and $y=v$, we have

$$f(u-v)+f(u+v)=2f(u)+2f(v)$$

for all $u, v \in X$. Thus f is a quadratic function.

(Sufficiency). Let f be a quadratic function. That is, we know that the functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

holds for all $x, y \in X$.

From the above functional equation we get

$$f(0)=0, f(-x)=f(x), f(2x)=4f(x)$$

and $f(\frac{x}{2})=\frac{1}{4}f(x)$ for all $x \in X$.

Thus we obtain

$$\begin{aligned} & 4f(\frac{x-2y}{2})+f(x)+2f(y)-8f(\frac{x-y}{2})-f(2y) \\ = \;& f(x-2y)+f(x)+2f(y)-2f(x-y)-4f(y) \\ = \;& f(x-2y)-2f(x-y)+f(x)-2f(y) \\ = \;& 2f(x-y)+2f(y)-2f(x-y)-2f(y) \\ = \;& 0. \end{aligned}$$

for all $x, y \in X$. □

THEOREM 2.2. *Let X and Y be real linear spaces. A function $f : X \to Y$ satisfies (2) for all $x, y \in X$ if and only if there exist a quadratic function $Q : X \to Y$ and an additive function $A : X \to Y$ such that*

$$f(x)=Q(x)+A(x)$$

for all $x, y \in X$.

PROOF. (Necessity). By putting $x, y = 0$ in (2), we see that $f(0)=0$. Let $Q(x)=\frac{1}{2}\{f(x)+f(-x)\}$ and $A(x)=\frac{1}{2}\{f(x)-f(-x)\}$.

Then we get $Q(0)=0$, $Q(-x)=Q(x)$, $A(0)=0$, and $A(-x)=-A(x)$ for all $x \in X$.

Since f satisfies the functional equation (2), we have the equations as follows ;

$$f(x) = Q(x) + A(x) \tag{3}$$

$$Q(x-2y) + Q(x) + 2Q(y) = 2Q(x-y) + Q(2y) \tag{4}$$

and

$$A(x-2y) + A(x) + 2A(y) = 2A(x-y) + A(2y) \tag{5}$$

for all $x, y \in X$.

We claim that Q is quadratic and A is additive.

Replacing y by x in (4), we have $4Q(x) = Q(2x)$ for all $x \in X$.

From $4Q(x) = Q(2x)$ and (4), we get

$$Q(x-2y) + Q(x) = 2Q(x-y) + 2Q(y) \tag{6}$$

for all $x, y \in X$.

Replacing $x - y$ by u and y by v in (6), we have

$$Q(u+v) + Q(u-v) = 2Q(u) + 2Q(v)$$

for all $u, v \in X$. Thus $Q(x)$ is a quadratic function.

Also replacing y by x in (5), we have $2A(x) = A(2x)$ for all $x \in X$. Rewriting (5), we get

$$A(x-2y) + A(x) = 2A(x-y)$$

for all $x, y \in X$.

Replacing x by $2x$ in above functional equation, we have

$$A(x-y) + A(x) = A(2x-y)$$

for all $x, y \in X$.

Putting $x - y = u$ and $x = v$, we have

$$A(u) + A(v) = A(u+v)$$

for all $u, v \in X$.

Thus A is an additive function.

(Sufficiency). That is obvious. □

3. Stability of the quadratic type functional equation

In this section we prove the stability of the functional equations

$$4f(\frac{x-2y}{2})+f(x)+2f(y)=8f(\frac{x-y}{2})+f(2y) \tag{7}$$

and

$$f(x-2y)+f(x)+2f(y)=2f(x-y)+f(2y)$$

for all $x, y \in X$.

Throughout this section X and Y will be a real normed linear space and a real Banach space respectively.

Let $\varphi : X \times X \longrightarrow [0,\infty)$ be a mapping such that

$$\Phi(x,y) := \sum_{i=1}^{\infty} \frac{1}{2^{2i}}\varphi(2^i x, 2^i y) < \infty$$

for all $x, y \in X$.

THEOREM 3.1. *If a function $f : X \to Y$ satisfies*

$$\|4f(\frac{x-2y}{2})+f(x)+2f(y)-8f(\frac{x-y}{2})-f(2y)\| \leq \varphi(x,y)$$

for all $x, y \in X$, then there exists a unique quadratic function $Q : X \to Y$ such that

$$\|f(x)-Q(x)\| \leq \Phi(x,0)+\frac{1}{3}\|f(0)\|$$

for all $x \in X$.

PROOF. Putting $y = 0$ in (7), we have

$$\|f(x)-2^2 f(\frac{x}{2})\| \leq \varphi(x,0)+\|f(0)\|$$

for all $x \in X$.

By dividing by 2^{2n-1} and replacing x by $2^{n-1}x$, we get

$$\|\frac{f(2^n x)}{2^{2n}}-\frac{f(2^{n-1}x)}{2^{2n-1}}\| \leq \frac{\varphi(2^n x,0)}{2^{2n}}+\frac{\|f(0)\|}{2^{2n}}$$

for all $x \in X$.

Induction argument implies

$$\begin{aligned} \|\frac{f(2^n x)}{2^{2n}}-f(x)\| &\leq \sum_{i=1}^{n}\frac{\varphi(2^i x,0)}{2^{2i}}+\sum_{i=1}^{n}\frac{\|f(0)\|}{2^{2i}} \\ &\leq \Phi(x,0)+\frac{1}{3}(1-\frac{1}{2^{2n}})\|f(0)\| \end{aligned} \tag{8}$$

for all $x \in X$ and $n \in N$.

Hence we obtain

$$\|\frac{f(2^n x)}{2^{2n}} - \frac{f(2^k x)}{2^{2k}}\| \leq \sum_{i=k+1}^{n} \frac{\varphi(2^i x, 0)}{2^{2i}} + \sum_{i=k+1}^{n} \frac{\|f(0)\|}{2^{2i}}$$

for all $x \in X$ and $n, k \in N$ with $n > k$.

This shows that the sequence $\{\frac{f(2^n x)}{2^{2n}}\}$ is a Cauchy sequence for all $x \in X$ and thus converges. Therefore we can define a function $Q : X \longrightarrow Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^{2n}}$$

for all $x \in X$.

By (7) we have

$$\begin{aligned}
& \|4Q(\frac{x-2y}{2}) + Q(x) + 2Q(y) - 8Q(\frac{x-y}{2}) - Q(2y)\| \\
= & \lim_{n\to\infty} \frac{1}{2^{2n}} \|4f(\frac{2^n(x-2y)}{2}) + f(2^n x) + 2f(2^n y) \\
& -8f(\frac{2^n(x-y)}{2} - f(2^{n+1}y)\| \\
\leq & \lim_{n\to\infty} \frac{1}{2^{2n}} \varphi(2^n x, 2^n y) \\
= & \ 0
\end{aligned}$$

for all $x, y \in X$.

Hence we get

$$4Q(\frac{x-2y}{2}) + Q(x) + 2Q(y) = 8Q(\frac{x-y}{2}) + Q(2y)$$

for all $x, y \in X$.

By Theorem 2.1, Q is a quadratic function. By the inequality in (8), we get

$$\begin{aligned}
\|f(x) - Q(x)\| &= \lim_{n\to\infty} \|f(x) - \frac{f(2^n x)}{2^{2n}}\| \\
&\leq \Phi(x, 0) + \frac{1}{3}\|f(0)\|
\end{aligned}$$

for all $x \in X$. Now we prove the uniqueness.

Let $B : X \longrightarrow Y$ be an another quadratic function such that

$$\|f(x) - B(x)\| \leq \Phi(x, 0) + \frac{1}{3}\|f(0)\|$$

for all $x \in X$. By Theorem 2.1, B satisfies the equation

$$4B(\frac{x-2y}{2}) + B(x) + 2B(y) = 8B(\frac{x-y}{2}) + B(2y)$$

for all $x, y \in X$.

Replacing y by 0, and x by $2x$ and dividing 2^2, we have

$$\frac{B(2x)}{2^2} = B(x) = \cdots\cdots = \frac{B(2^n x)}{2^{2n}}$$

for all $x \in X$. Hence we have

$$\begin{aligned} \|Q(x) - B(x)\| &\leq \|\frac{Q(2^n x)}{2^{2n}} - \frac{f(2^n x)}{2^{2n}}\| + \|\frac{f(2^n x)}{2^{2n}} - \frac{B(2^n x)}{2^{2n}}\| \\ &\leq \frac{2}{2^{2n}}\Phi(2^n x, 0) + \frac{\frac{2}{3}}{2^{2n}}\|f(0)\| \\ &= \frac{2}{2^{2n}}\sum_{i=1}^{\infty}\frac{1}{2^{2i}}\varphi(2^{i+n}x, 0) + \frac{\frac{2}{3}}{2^{2n}}\|f(0)\| \\ &= 2\sum_{i=n+1}^{\infty}\frac{1}{2^{2i}}\varphi(2^i x, 0) + \frac{\frac{2}{3}}{2^{2n}}\|f(0)\| \\ &\longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \end{aligned}$$

Thus we get $Q(x) = B(x)$ for all $x \in X$.
Hence we complete the proof. □

COROLLARY 3.1. *If the function $f : X \to Y$ satisfies*

$$\|4f(\frac{x-2y}{2}) + f(x) + 2f(y) - 8f(\frac{x-y}{2}) - f(2y)\| \leq \delta$$

for every $x, y \in X$ and for some $\delta > 0$, then there exists a unique quadratic function $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{\delta}{3} + \frac{1}{3}\|f(0)\|$$

for all $x \in X$.

PROOF. Let $\varphi(x, y) = \delta$ for all $x, y \in X$ in (7). Then we get

$$\Phi(x, 0) = \sum_{i=1}^{\infty}\frac{1}{2^{2i}}\delta = \frac{\delta}{3}$$

for all $x \in X$. By Theorem 3.1, we complete the proof. □

COROLLARY 3.2. *If the function $f : X \to Y$ satisfies*

$$\|4f(\frac{x-2y}{2}) + f(x) + 2f(y) - 8f(\frac{x-y}{2}) - f(2y)\| \le \|x\|^p + \|y\|^p$$

for all $x, y \in X$ and $0 < p < 2$, then there exists a unique quadratic function $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{2^{2-p}-1}\|x\| + \frac{1}{3}\|f(0)\|$$

for all $x \in X$.

PROOF. Let $\varphi(x, y) = \|x\|^p + \|y\|^p$ for all $x, y \in X$. Then we get

$$\Phi(x) = \sum_{i=1}^{\infty} \frac{1}{2^{2i}}\|2^i x\|^p = \sum_{i=1}^{\infty} \frac{1}{2^{(2-p)i}}\|x\| = \frac{1}{2^{2-p}-1}\|x\|$$

for all $x \in X$. By Theorem 3.1 we complete the proof. □

Let $\Psi : X \times X \longrightarrow [0, \infty)$ be a function satisfying one of the condition $(a), (b)$ and one of the condition $(c), (d)$;

$$\Phi_1(x, y) := \sum_{i=0}^{\infty} \frac{1}{4^i}\psi(2^i x, 2^i y) < \infty \qquad (a)$$

$$\Phi_2(x, y) := \sum_{i=0}^{\infty} 4^i \psi(\frac{x}{2^i}, \frac{y}{2^i}) < \infty \qquad (b)$$

$$\Psi_1(x, y) := \sum_{i=0}^{\infty} \frac{1}{2^i}\psi(2^i x, 2^i y) < \infty \qquad (c)$$

$$\Psi_2(x, y) := \sum_{i=0}^{\infty} 2^i \psi(\frac{x}{2^i}, \frac{y}{2^i}) < \infty \qquad (d)$$

for all $x, y \in X$.

One of the condition $(a), (b)$ will be needed to derive a quadratic function and one of the condition $(c), (d)$ will be needed to derive an additive function in the following theorem.

THEOREM 3.2. *If the function $f : X \to Y$ satisfies $f(0) = 0$ and*

$$(9) \qquad \|f(x-2y) + f(x) + 2f(y) - 2f(x-y) - f(2y)\| \le \varphi(x, y)$$

for all $x, y \in X$, then there exist a unique additive function $A : X \to Y$ satisfying the equation (2) such that

$$\|f(x) - Q(x) - A(x)\| \le \varepsilon_i(x) + \delta_j(x),$$

$$\|\frac{f(x)+f(-x)}{2} - Q(x)\| \le \varepsilon_i(x),$$

and

$$\|\frac{f(x)-f(-x)}{2} - A(x)\| \le \delta_j(x)$$

for all $x \in X$ *and for* $i = 1$ *or* 2,$j = 1$ *or* 2, *where* $\varepsilon_i(x) = \frac{1}{8}[\Phi_i(x,x) + \Phi_i(-x,-x)]$ *and* $\delta_j(x) = \frac{1}{4}[\Psi_j(x,x) + \Psi_j(-x,-x)]$.

In particular the functions Q *and* A *are given by*

$$\begin{aligned} Q(x) &= \lim_{n\to\infty} \frac{f(2^n x) + f(-2^n x)}{2\cdot 4^n} && \text{if } (a) \text{ holds.}\\ Q(x) &= \lim_{n\to\infty} \frac{4^n[f(\frac{x}{2^n}) + f(-\frac{x}{2^n})]}{2} && \text{if } (b) \text{ holds.}\\ A(x) &= \lim_{n\to\infty} \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} && \text{if } (c) \text{ holds.}\\ A(x) &= \lim_{n\to\infty} \frac{2^n[f(\frac{x}{2^n}) - f(-\frac{x}{2^n})]}{2} && \text{if } (d) \text{ holds.} \end{aligned}$$

PROOF. Let $f_1 : X \to Y$ be a function defined by $f_1(x) = \frac{1}{2}[f(x) + f(-x)]$ for all $x \in X$. Then we get $f_1(0) = 0$ and $f_1(x) = f_1(-x)$, and

$$\begin{aligned} &\|f_1(x-2y) + f_1(x) + 2f_1(y) - 2f_1(x-y) - f_1(2y)\| \\ (10) \qquad &\le \frac{1}{2}[\psi(x,y) + \psi(-x,-y)] \end{aligned}$$

for all $x, y \in X$. Putting $y = x$ in (10), we have

$$(11) \qquad \|4f_1(x) - f_1(2x)\| \le \frac{1}{2}[\psi(x,x) + \psi(-x,-x)]$$

for all $x \in X$. Dividing both sided by 4 in (11) we get

$$\|f_1(x) - \frac{f_1(2x)}{4}\| \le \frac{1}{8}[\psi(x,x) + \psi(-x,-x)]$$

for all $x \in X$.

Case I. Assume that ψ satisfies the condition (a). Replacing x by $2^{n-1}x$ and dividing by 4^{n-1} in (11), we obtain

$$\begin{aligned} (12) \qquad &\|\frac{f_1(2^{n-1}x)}{4^{n-1}} - \frac{f_1(2^n x)}{4^n}\| \\ &\le \frac{1}{8}\cdot\frac{1}{4^{n-1}}[\psi(2^{n-1}x, 2^{n-1}x) + \psi(-2^{n-1}x, -2^{n-1}x)] \end{aligned}$$

for all $x \in X$ and for all $n \in N$.

An induction argument implies easily that

$$(13)\qquad \|f_1(x) - \frac{f_1(2^n x)}{4^n}\| \le \frac{1}{8}\sum_{i=0}^{n-1}\frac{1}{4^i}[\psi(2^i x, 2^i x) + \psi(-2^i x, -2^i x)]$$

for all $x \in X$ and for all $n \in N$.

Hence we obtain that

$$\begin{aligned} \|\frac{f_1(2^n x)}{4^n} - \frac{f_1(2^m x)}{4^m}\| &\le \frac{1}{4^m}\|\frac{f_1(2^{n-m}\cdot 2^m x)}{4^{n-m}} - f_1(2^m x)\| \\ (14)\qquad &\le \frac{1}{8}\sum_{i=m}^{n-1}\frac{1}{4^i}[\psi(2^i x, 2^i x) + \psi(-2^i x, -2^i x)] \end{aligned}$$

for all $x \in X$ and for all $n, m \in N$ with $n > m$. Since the right hand side of (14) tends to zero as $n \to \infty$, the sequence $\{\frac{f_1(2^n x)}{4^n}\}$ is a Cauchy sequence for all $x \in X$ and thus converges by the completeness of Y. Therefore we can define a function $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{f_1(2^n x)}{4^n}$$

for all $x \in X$. Note that $Q(0) = 0$ and $Q(x) = Q(-x)$ for all $x \in X$.

Replacing x, y in (10) by $2^n x, 2^n y$ respectively and dividing both sides by 4^n, and then taking the limit in the resulting inequality, we have

$$(15)\qquad Q(x - 2y) + Q(x) + 2Q(y) - 2Q(x - y) - Q(2y) = 0.$$

By Theorem 2.2, Q is a quadratic function.

Taking the limit in (12) as $n \to \infty$, we obtain

$$(16)\qquad \|f_1(x) - Q(x)\| \le \frac{1}{8}[\Phi_1(x, x) + \Phi_1(-x, -x)]$$

for all $x \in X$.

To prove the uniqueness let $Q'(x)$ be another quadratic function satisfying (16). Then we get $Q'(0) = 0, Q'(2^n x) = 4^n Q'(x)$ and $Q'(-x) = Q'(x)$ for all $x \in X$. Thus we have

$$\begin{aligned} \|Q(x) - Q'(x)\| &\le \|\frac{Q(2^n x)}{4^n} - \frac{f_1(2^n x)}{4^n}\| + \|\frac{f_1(2^n x)}{4^n} - \frac{Q'(2^n x)}{4^n}\| \\ &\le \frac{1}{4^n}\{\|Q(2^n x) - f_1(2^n x)\| + \|f_1(2^n x) - Q'(2^n x)\|\} \\ &\le \frac{1}{4}\cdot\frac{1}{4^n}\{\Phi_1(2^n x, 2^n x) + \Phi_1(-2^n x, -2^n x)\} \\ &= \frac{1}{4}[\sum_{i=n+1}^{\infty}\frac{1}{4^i}\psi(2^i x, 2^i x) + \sum_{i=n+1}^{\infty}\frac{1}{4^i}\psi(-2^i x, -2^i x)] \end{aligned}$$

for all $x \in X$.

Taking the limit as $n \to \infty$, we conclude that $Q(x) = Q'(x)$ for all $x \in X$.

Case II. Assume that ψ satisfies the condition (b). Replacing x by $\frac{x}{2}$ in (11) we get

$$\|4f_1(\frac{x}{2}) - f_1(x)\| \le \frac{1}{2}[\psi(\frac{x}{2}, \frac{x}{2}) + \psi(-\frac{x}{2}, -\frac{x}{2})]$$

for all $x \in X$.

Replacing x by $\frac{x}{2^{n-1}}$ and multiplying by 4^{n-1} in the above inequality we obtain

$$\begin{aligned} &\|4^n f_1(\frac{x}{2^n}) - 4^{n-1} f_1(\frac{x}{2^{n-1}})\| \\ \le\ & \frac{1}{2} \cdot 4^{n-1}[\psi(\frac{x}{2^n}, \frac{x}{2^n}) + \psi(-\frac{x}{2^n}, -\frac{x}{2^n})] \end{aligned}$$

for all $x \in X$ and for all $n \in N$.

An induction argument implies

$$\|4^n f_1(\frac{x}{2^n}) - f_1(x)\| \le \frac{1}{8}\sum_{i=1}^{n} 4^i[\psi(\frac{x}{2^i}, \frac{x}{2^i}) + \psi(-\frac{x}{2^i}, -\frac{x}{2^i})] \tag{17}$$

for all $x \in X$ and for all $n \in N$.

Hence we get

$$\begin{aligned} &\|4^n f_1(\frac{x}{2^n}) - 4^m f_1(\frac{x}{2^m})\| \\ \le\ & \frac{1}{8} \sum_{i=m+1}^{n} 4^i[\psi(\frac{x}{2^i}, \frac{x}{2^i}) + \psi(-\frac{x}{2^i}, -\frac{x}{2^i})] \end{aligned}$$

for all $x \in X$ and for all $n, m \in N$ with $n > m$.

This shows that the sequence $\{4^n f_1(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$ and thus converges. Therefore we can define a function $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 4^n f_1(\frac{x}{2^n})$$

for all $x \in X$. By (9) we have

$$Q(x - 2y) + Q(x) + 2Q(y) - 2Q(x - y) - Q(2y) = 0$$

for all $x, y \in X$ and thus Q is a quadratic.

Taking the limit in (17) as $n \to \infty$, we obtain

$$\|Q(x) - f_1(x)\| \le \frac{1}{8}[\Phi_2(x, x) + \Phi_2(-x, -x)] \tag{18}$$

for all $x \in X$.

Using the similar argument to that of Case I, we easily have the uniqueness of Q satisfying (18).

Now let $f_2 : X \to Y$ be a function defined by $f_2 := \frac{1}{2}[f(x) - f(-x)]$ for all $x \in X$. Then we get $f_2(0) = 0, f_2(-x) = -f_2(x)$, and the relation (9) can be written by

$$\begin{aligned} &\|f_2(x-2y) + f_2(x) + 2f_2(y) - 2f_2(x-y) - f_2(2y)\| \\ (19) \qquad \leq\ & \frac{1}{2}[\psi(x,y) + \psi(-x,-y)] \end{aligned}$$

for all $x, y \in X$.

Putting $y = x$ in (19), we have

$$(20) \qquad \|f_2(2x) - 2f_2(x)\| \leq \frac{1}{2}[\psi(x,y) + \psi(-x,-y)]$$

for all $x \in X$. Dividing the above inequality by 2 we have

$$(21) \qquad \|\frac{f_2(2x)}{2} - f_2(x)\| \leq \frac{1}{4}[\psi(x,x) + \psi(-x,-x)]$$

for all $x \in X$.

Case III. Assume that ψ satisfies the condition (c). Replacing x by $2^{n-1}x$ in (21) and dividing by 2^{n-1} we obtain

$$\begin{aligned} &\|\frac{f_2(2^n x)}{2^n} - \frac{f_2(2^{n-1}x)}{2^{n-1}}\| \\ \leq\ & \frac{1}{2^{n+1}}[\psi(2^{n-1}x, x^{n-1}x) + \psi(-2^{n-1}x, -2^{n-1}x)] \end{aligned}$$

for all $x \in X$ and for all $n \in N$.

It follows by an induction argument that

$$(22) \qquad \|\frac{f(2^n x)}{2^n}) - f_2(x)\| \leq \frac{1}{4}\sum_{i=0}^{n-1}[\frac{1}{2^i}\psi(2^i x, 2^i x) + \frac{1}{2^i}\psi(-2^i x, -2^i x)]$$

for all $x \in X$ and for all $n \in N$.

Hence we get

$$\|\frac{f_2(2^n x)}{2^n} - \frac{f_2(2^m x)}{2^m}\| \leq \frac{1}{4}\sum_{i=m}^{n-1}[\frac{1}{2^i}\psi(2^i x, 2^i x) + \frac{1}{2^i}\psi(-2^i x, -2^i x)]$$

for all $x \in X$ and for all $n, m \in N$ with $n > m$. This shows that the sequence $\{\frac{f_2(2^n x)}{2^n})\}$ is a Cauchy sequence for all $x \in X$ and thus converges in Y. Therefore we can define a function $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{f_2(2^n x)}{2^n}$$

for all $x \in X$. Note that $A(0) = 0$ and $A(-x) = -A(x)$ for all $x \in X$.

By (19) we get

$$A(x-2y) + A(x) + 2A(y) - 2A(x-y) - A(2y) = 0$$

for all $x, y \in X$ and thus A is an additive as in the proof of Theorem 2.2.

Taking the limit in (22) as $n \to \infty$, we obtain

$$\|f_2(x) - A(x)\| \le \frac{1}{4}[\Psi_1(x,x) + \Psi_1(-x,-x)] \tag{23}$$

for all $x \in X$.

If is another function satisfying the inequality (23), then we get $A'(0) = 0, A'(-x) = A'(x)$ and $A'(2^n x) = 2^n A'(x)$ for all $x \in X$. Thus we obtain that by (23)

$$\begin{aligned} \|A(x) - A'(x)\| &\le \frac{1}{2^n}(\|A(2^n x) - f_2(2^n x)\| + \|f_2(2^n x) - A'(2^n x)\|) \\ &\le \frac{\Psi_1(2^n x, 2^n x) + \Psi_1(-2^n x, -2^n x)}{2^{n+1}} \\ &= \frac{1}{2}\sum_{i=n+1}^{\infty} \frac{1}{2^i}[\psi(2^i x, 2^i x) + \psi(-2^i x, -2^i x)] \end{aligned}$$

for all $x \in X$. Taking the limit as $n \to \infty$, we can conclude that we obtain $A(x) = A'(x)$ for all $x \in X$.

Case IV. Assume that ψ satisfies the condition (d). Replacing x by $\frac{x}{2}$ in (20) we get

$$\|f_2(x) - 2f_2(\frac{x}{2})\| \le \frac{1}{2}[\psi(\frac{x}{2}, \frac{x}{2}) + \psi(-\frac{x}{2}, -\frac{x}{2})] \tag{24}$$

for all $x \in X$.

Replacing x by $\frac{x}{2^{n-1}}$ in (24) and multiplying by 2^{n-1} we obtain

$$\begin{aligned} &\|2^{n-1} f_2(\frac{x}{2^{n-1}}) - 2^n f_2(\frac{x}{2^n})\| \\ \le\ & 2^{n-2}[\psi(\frac{x}{2^n}, \frac{x}{2^n}) + \psi(-\frac{x}{2^n}, -\frac{x}{2^n})] \end{aligned}$$

for all $x \in X$ and for all $n \in N$.

An induction argument implies

$$\|f_2(x) - 2^n f_2(\frac{x}{2^n})\| \le \frac{1}{4}\sum_{i=1}^{n} 2^i [\psi(\frac{x}{2^i}, \frac{x}{2^i}) + \psi(-\frac{x}{2^i}, -\frac{x}{2^i})] \tag{25}$$

for all $x \in X$ and for all $n \in N$.

Hence we get

$$\begin{aligned} &\|2^n f_2(\frac{x}{2^n}) - 2^m f_2(\frac{x}{2^m})\| \\ \leq\ & \frac{1}{4}\sum_{i=m+1}^{n} 2^i[\psi(\frac{x}{2^i},\frac{x}{2^i}) + \psi(-\frac{x}{2^i},-\frac{x}{2^i})] \end{aligned}$$

for all $x \in X$ and for all $n, m \in N$ with $n > m$. This implies that the sequence $\{2^n f_2(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$ and thus converges. Therefore we can define a function $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f_2(\frac{x}{2^n})$$

for all $x \in X$. Note that $A(0) = 0, A(-x) = -A(x)$ for all $x \in X$ and thus A is an additive. Taking the limit in (25) as $n \to \infty$, we obtain

$$\|f_2(x) - A(x)\| \leq \frac{1}{4}[\Psi_2(x,x) + \Psi_2(-x,-x)] \tag{26}$$

for all $x \in X$.

Similarly we have easily that A is a unique additive mapping subject to (26). We complete the proof. □

COROLLARY 3.3. *Let $p \neq 1, p \neq 2$ and $\epsilon \geq 0$ be real numbers. Assume that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\begin{aligned} &\|f(x-2y) + f(x) + 2f(y) - 2f(x-y) - f(2y)\| \\ \leq\ & \epsilon(\|x\|^p + \|y\|^p) \end{aligned}$$

for all $x, y \in X (x, y \in X \setminus \{0\}$ if $p < 0)$. Then there exist a unique quadratic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ such that

$$\|f(x) - Q(x) - A(x)\| \leq 2\epsilon\|x\|^p(\frac{1}{|4-2^p|} + \frac{1}{|2-2^p|}),$$

$$\|\frac{f(x)+f(-x)}{2} - Q(x)\| \leq 2\epsilon\|x\|^p\frac{1}{|4-2^p|},$$

and

$$\|\frac{f(x)-f(-x)}{2} - A(x)\| \leq 2\epsilon\|x\|^p\frac{1}{|2-2^p|}$$

for all $x \in X (x \in X \setminus \{0\}$ if $p < 0)$.

PROOF. Let $\varphi(x,y) := \epsilon(\|x\|^p + \|y\|^p)$ for all $x, y \in X(x \in X \setminus \{0\}$ if $p < 0)$. If $p < 2$, then we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{4^n}\psi(2^n x, 2^n y) &= \sum_{n=0}^{\infty} \frac{2^{np}\epsilon(\|x\|^p + \|y\|^p)}{4^n} \\ &= \frac{4\epsilon(\|x\|^p + \|y\|^p)}{4 - 2^p} \end{aligned}$$

for all $x, y \in X(x \in X \setminus \{0\}$ if $p < 0)$.

If $p > 2$, then we have

$$\begin{aligned} \sum_{n=1}^{\infty} 4^n \psi(\frac{x}{2^n}, \frac{y}{2^n}) &= \sum_{n=1}^{\infty} \frac{4^n\epsilon(\|x\|^p + \|y\|^p)}{2^{np}} \\ &= \frac{4\epsilon(\|x\|^p + \|y\|^p)}{2^p - 4} \end{aligned}$$

for all $x, y \in X$

If $p < 1$, then we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{2^n}\psi(2^n x, 2^n y) &= \sum_{n=0}^{\infty} \frac{2^{np}\epsilon(\|x\|^p + \|y\|^p)}{2^n} \\ &= \frac{2\epsilon(\|x\|^p + \|y\|^p)}{2 - 2^p} \end{aligned}$$

for all $x, y \in X(x \in X \setminus \{0\}$ if $p < 0)$.

If $p > 1$, then we have

$$\begin{aligned} \sum_{n=1}^{\infty} 2^n \psi(\frac{x}{2^n}, \frac{y}{2^n}) &= \sum_{n=1}^{\infty} \frac{2^n\epsilon(\|x\|^p + \|y\|^p)}{2^{np}} \\ &= \frac{2\epsilon(\|x\|^p + \|y\|^p)}{2^p - 2} \end{aligned}$$

for all $x, y \in X$. □

Thus applying Corollary 3.3 for the there case $p < 1, 1 < p < 2$ and $p > 2$, we easily obtain the following result.

COROLLARY 3.4. *Assume that for some $\delta > 0$, a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|f(x - 2y) + f(x) + 2f(y) - 2f(x - y) - f(2y)\| \le \delta$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ such that

$$\|f(x) - Q(x) - A(x)\| \le \frac{4}{3}\delta,$$

$$\|\frac{f(x)+f(-x)}{2} - Q(x)\| \leq \frac{1}{3}\delta,$$

and $\|\frac{f(x)-f(-x)}{2} - A(x)\| \leq \delta$ *for all* $x \in X$.

PROOF. Putting $\varphi(x,y) = \delta$, we immediately get the result. □

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