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FREE HOMEOMORPHISMS OF TWO DIMENSIONAL MANIFOLDS

JONG-JIN PARK

ABSTRACT. M. Brown [2] posed an open question on the class of free homeomorphisms as follows: *if f is a free homeomorphism of two manifold and k is a positive integer then is f^k free?* In this paper we show that the answer of the open question is true.

1. Introduction

The class of free homeomorphisms has been introduced and studied by M. Brown in [1, 2], and then other of mathematicians developed the theory of free homeomorphisms.

In particular, E. Slaminka [5] proved a generalization of the Brouwer translation theorem using the concept of free homeomorphisms as follows:

THEOREM 1. *Let h be a free homeomorphism of the two sphere S^2 with finite fixed point set F. Then each point $p \in S^2 \setminus F$ lies in the image of an embedding $\phi_p : (\mathbb{R}^2, 0) \to (S^2 \setminus F,\ p)$ such that*

1) *$h\phi_p = \phi\tau$, where $\tau(z) = z + 1$ is the canonical translation of the plane,*
2) *the image of a vertical line under ϕ_p is closed in $S^2 \setminus F$.*

Moreover G. Lucien [3] extended the Brouwer plane translation theorem using the notion of free homeomorphisms as follows:

THEOREM 2. *Let h be a free homeomorphism of the two sphere S^2 with a finite fixed point set F. Then each point $p \in S^2 \setminus F$ lies in the image of an embedding $\phi_p : (\mathbb{R}^2, 0) \to (S^2 \setminus F,\ p)$ such that*

1) *$h\phi_p = \phi_p\tau$, where $\tau(x, y) = (x + 1,\ y)$,*

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2) *on each line* $x \times \mathbb{R}$, ϕ_p *restricts to a proper embedding, i.e.,* $\phi_p(x \times \mathbb{R})$ *is closed in* $S \setminus F$.

The purpose of this paper is to give an affirmative answer to the following open question posed by M. Brown in [3]: *Suppose f is a free homeomorphism of two-manifold M and k is a positive integer. Is f^k free homeomorphism?*

For this purpose, we let M be a connected two dimensional manifold. For any subset N of M, we denote ∂N by the boundary of N and $intN$ by the interior of N.

A homeomorphism f of M is called *free homeomorphism* provided that whenever D is a disk in M and $f(D) \cap D = \phi$ then $f^p(D) \cap f^q(D) = \phi$ whenever p, q are distinct integers.

We denote the *fixed point set* of f by $F = Fix(f)$. If $x \in M$ then the *orbit* of x is the set $\{f^i(x) | i \in \mathbb{Z}\}$.

f is said to be a *locally free homeomorphism* of M provided that for each $x \in M \setminus F$, there exists a disk D_x containing x in its interior(relative M) such that $f^p(D_x) \cap f^q(D_x) = \phi$ whenever p, q are distinct integers. It is clear that if a homeomorphism f on M is free then it is locally free. But the converse does not hold in general.

Let f be a homeomorphism of M. An arc($\cong$ [0,1]) $\alpha = \widehat{pq}$ is called a *translation arc* for f if

$$f(p) = q \text{ and } f(\alpha \setminus \{q\}) \cap (\alpha \setminus \{q\}) = \phi.$$

A homeomorphism f is said to have the *translation arc property* provided that whenever $\alpha = \widehat{pq}$ is a translation arc for f

$$f^n(\alpha \setminus \{q\}) \cap (\alpha \setminus \{q\}) = \phi$$

for all $n > 1$. The set $L_\alpha = \cup_{n=-\infty}^{\infty} f^n(\alpha)$ is homeomorphic to the real line $\mathbb{R}^1$ by Lemma 2.3 in the following section.

We call L_α a *translation line* for f generated by the translaton arc α. Brouwer introduced the notion of the translation arc and proved that a fixed point free orientation preserving homeomorphism of the plane has the translation arc property.

For a sequence $\{A_n\}$ of subsets of M, we define the set $\limsup A_n$ [4] by

$$\limsup A_n = \{x \in M | \text{ for each neighorhood } U \text{ of } x, \; U \text{ intersects infinitely many of the sets } A_n\}.$$

2. Main theorem

MAIN THEOREM. *If f is a free homeomorphism of M then f^k is also a free homeomorphism on M for any $k \in \mathbb{N}$.*

REMARKS. The inverse of the above main theorem does not hold in general. In fact, let $f : S^2 \longrightarrow S^2$ be a homeomorphism defined by

$$\begin{aligned} &f(\sqrt{1-z^2}\cos\theta,\ \sqrt{1-z^2}\sin\theta,\ z) \\ =&(\sqrt{1-z^2}\cos(\theta+\pi/2),\ \sqrt{1-z^2}\sin(\theta+\pi/2),\ z). \end{aligned}$$

Then it is clear that f is not free, but f^4 is free.

To prove our main theorem we need several lemmas.

LEMMA 2.1. ([4], Lemma 3.1) *Let f be a free homeomorphism of M. Then if C is a continuum and $C \cap f(C) = \phi$ then $f^p(C) \cap f^q(C) = \phi$ whenever $p \neq q$.*

LEMMA 2.2. ([4], Lemma 4.1) *Let f be a homeomorphism of M. If x and $f(x)$ are in the same component of $M \setminus F$ then there is a translation* arc *from x to $f(x)$.*

LEMMA 2.3. ([4], Lemma 4.7) *Let f be a free homeomorphism of M and let L be a translation line for f. Then L is a homeomorphic to the real line $\mathbb{R}^1$.*

LEMMA 2.4. ([4], Lemma 4.8) *Let f be a free homeomorphism of M and let L be a translation line for f. Then f does not (locally) interchange the two sides of L.*

LEMMA 2.5. *Let f be a locally free homeomorphism of M. If N is a compact subset of M and F is the fixed point set of M, then for each x in M,* $\limsup f^n(x) \cap (N \setminus F_N) = \phi$ *where $F_N = F \cap N$.*

PROOF. Let U be an open subset of F_N in M. Then$N \setminus U$ is a compact subset of M. By the compactness of $N \setminus U$, there exists a finite collection $\{D_i\}$ of disks covering $N \setminus U$ such that for each i,

$$f^p(D_i) \cap f^q(D_i) = \phi,$$

whenever p, q are distinct integers, since f is a locally free homeomorphism. Let $x \in M$. Then the orbit of x can intersect each D_i at most once; i.e.,

$$f^n(x) \in M \setminus (N \setminus U)$$

for all but a finite number of values of n. □

COROLLARY 2.6. *For each translation arc $\alpha = \widehat{pq}$ in M,*

$$\limsup\ f^n(\alpha) \cap (N \setminus F_N) = \phi.$$

Now we introduce the concept of the h-disk which allows certain holes, and then we extend some properties of the disk to those of h-disk using the method of M. Brown [3]. We say that a subset G of M is a generalized disk in M if it is a homeomorphic to a subset of D which contains $intD$, where $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$.

DEFINITION 2.7. A subset H of M is called an n-disk, $n \in \mathbb{N}$, in M if there are generalized (n+1)-disks $G_1, \cdots, G_{n+1}$ such that

1) $G_i \subset intG_1$ for all $i = 2, \cdots, n+1$
2) $G_i \cap G_j = \phi$ if $i, j > 1$ and $i \neq j$
3) $H \cong \overline{G_1} \setminus (\cup_{i=2}^{n+1} G_i)$

We denote $\overline{\partial} H$ by the boundary of the disk G_1. We say that a subset H of M is h-disk in M if it is an n-disk in M for some $n \in \mathbb{N}$.

LEMMA 2.8. *Let f be a homeomorphism of M and H a h-disk in M. If there exists $t \in \mathbb{N}$ such that $f^t(H) \subset H \setminus \overline{\partial} H$ then f is not free.*

PROOF. We suppose that $t > 1$ and $f(H) \cap H = \phi$. Take $x \in \overline{\partial} H$. Let α be an arc from x to $f^t(x)$ in the set $int(H \setminus f^t(H)) \cup \{x, f^t(x)\}$. Then $f(\alpha) \cap \alpha = \phi$ and $\alpha \cap f^t(\alpha)$ contains the point $f^t(x)$. Hence f is not free by Lemma 2.1.

Suppose $f(H) \cap H \neq \phi$. Take $x \in \overline{\partial} H \setminus f^{-1}(f(H) \cap H)$ and let α be an arc from x to $f^t(x)$ in the set $int(H \setminus f^t(H)) \cup \{x, f^t(x)\}$ satisfying

$$\alpha \cap [f^{-1}(f(H) \cap H)] = \phi \text{ or } \alpha \cap (f(H) \cap H) = \phi.$$

Then $f(\alpha) \cap \alpha = \phi$ and $f^t(\alpha) \cap \alpha$ contains the point $f^t(x)$. Hence f is not free by Lemma 2.1.

Now we suppose that $t = 1$. Then it is proved by Lemma 5.2 in [2]. □

If we apply the same techniques by M. Brown in [2], we obtain the following Lemmas: 2.9, 2.10 and 2.11. So we will omit the proof.

LEMMA 2.9. *Let f be a locally free homeomorphism of M and H a h-disk in M. If there exists a natural number $t \in \mathbb{N}$ such that $f^t(H) \subset H$ and f has no fixed point on $\overline{\partial} H$, then for each $x \in \overline{\partial} H$, there is an* *unique integer* $N \geq 1$ *such that*

$$f^{it}(x) \in \begin{cases} \overline{\partial} H, & 1 \leq i < N \\ H \backslash \overline{\partial} H, & i \geq N. \end{cases}$$

LEMMA 2.10. *Let* f *be a locally free homeomorphism of* M *and* H *a* h*-disk in* M. *If there exists a natural number* $t \in \mathbb{N}$ *such that* $f^t(H) \subset H$ *and* f *has no fixed point on* $\overline{\partial} H$ *then there is a smallest number* $N \geq 1$ *such that* $f^{Nt}(H) \subset H \backslash \overline{\partial} H$.

LEMMA 2.11. *Let* f *be a locally free homeomorphism of* M *and* H *a* h*-disk in* M. *If there exists a natural number* $t \in \mathbb{N}$ *such that* $f^t(H) \subset H$ *and* f *has no fixed point on* $\overline{\partial} H$ *then there exists an* h*-disk* E *in* M *such that* $f^t(E) \subset E \backslash \overline{\partial} E$.

Now to prove our main theorem, we denote an arc β from x to y in $X(\subset M)$ by $\beta = \widehat{xy}(X)$.

PROOF OF THE MAIN THEOREM. Suppose $k \geq 2$, and let D be a disk in M. If $f(D) \cap D = \phi$ then the proof is clear. So we suppose

$$(1)\ f^k(D) \cap D = \phi \text{ and } f^i(D) \cap D \neq \phi$$

for all $1 \leq i \leq k$. To derive a contradiction, we suppose that f^k is not free. Then there exist positive integer p, q with $p < q$ satisfying

$$(2)\ f^{kp}(D) \cap f^{kq}(D) \neq \phi$$

Let $r = kq \setminus kp$. Then we can see that $r > k$. In fact, if $r = k$ then $q = p + 1$. Take a point z in the set $f^{kp}(D) \cap f^{kq}(D)$. Then we have

$$f^{-k}(z) \in f^{kp}(D) \quad \text{and} \quad f^{-kp}(f^{-k}(z)), f^{-kp}(z) \in D.$$

Hence we get

$$f^k(f^{-kp}(f^{-k}(z))) \in D \quad \text{and so} \quad f^k(D) \cap D \neq \phi.$$

This is a contradiction such that $r > k$.

Since $f^{kp}(D) \cap f^{kq}(D) \neq \phi$ by (2), we take a point z in $f^{kp}(D) \cap f^{kq}(D)$. Let $y = f^{-r}(z)$. Then the set $f^{kp}(D)$ contains the points y, z. If we apply Lemma 2.2 then we can choose an arc α from y to $f(y)$ in M. Put

$$L = \cup_{n=0}^{\infty} f^n(\alpha).$$

Then the set $[f^{kp}(D) \cap L]$ contain the point y, z. Choose an arc γ from y to z in $f^{kp}(D)$. Then we have the following cases.

First we have two cases:

$$\gamma \cap L = \{y,\ z\} \text{ or } \gamma \cap L \neq \{y,\ z\}.$$

If $\gamma \cap L \neq \{y,\ z\}$, then we have two cases:

$$\gamma \cap L \text{ is a finite set or } \gamma \cap L \text{ is an infinite set.}$$

If $\gamma \cap L$ is a finite, then we have two cases:

$$\gamma \cap [\,\cup_{n=0}^{r-1} f^n(\alpha)\,] = \{y,\ z\} \text{ or } \gamma \cap [\cup_{n=0}^{r-1} f^n(\alpha)\,] \neq \{y,\ z\}.$$

If $\gamma \cap [\cup_{n=0}^{r-1} f^n(\alpha)\,] \neq \{y,\ z\}$, then we have two cases:

$$\gamma \cap [\cup_{n=r}^{\infty} f^n(\alpha)] = z \text{ or } \gamma \cap [\cup_{n=r}^{\infty} f^n(\alpha)] \neq z.$$

For each case, we will show that f is not free. Then the contradiction completes the proof of our theorem.

Case 1:

- $\gamma \cap L = \{y,\ z\}$.

Let E be bounded by $\gamma \cup [\cup_{n=0}^{r-1} f^n(\alpha)]$. Then E is an h-disk and

$$\cup_{n=r}^{\infty} f^n(\alpha) \subset E \text{ or } \cup_{n=r}^{\infty} f^n(\alpha) \cap E = z$$

(see Figure 1 or 2), since L is homeomorphic to a ray of a real line $\mathbb{R}^1$(see Lemma 2.3).

Let $\cup_{n=r}^{\infty} f^n(\alpha) \subset E$ (see Figure 1). Then since

$$f^{kp}(D) \cap f^{k(p+1)}(D) = \phi \text{ and } r > k,$$

there exists some i, $1 \leq i < r$, such that $f^i(\gamma) \cap \gamma = \phi$, and so by Lemma 2.4, $f^i(E) \subset E$. But since f has no fixed point on $\overline{\partial} E$ by Lemma 2.11 and 2.8, f is not a free homeomorphism.

Let $\cup_{n=r}^{\infty} f^n(\alpha) \cap E = z$ (see Figure 2). Then by Lemma 2.4, for all i, $1 \leq i \leq r-1$,

$$f^i(\gamma) \cap \gamma \neq \phi$$

since $f^i(y) \in E$ and $f^i(z) \notin E$. But since $k \leq r-1$,

$$f^{kp}(D) \cap f^{k(p+1)}(D) \neq \phi$$

and f is not a free homeomorphism.

Case 2:

- $\gamma \cap L \neq \{y,\ z\}$,
- $\gamma \cap L$ is a finite set, and
- $\gamma \cap [\,\cup_{n=0}^{r-1} f^n(\alpha)\,] = \{y,\ z\}$.

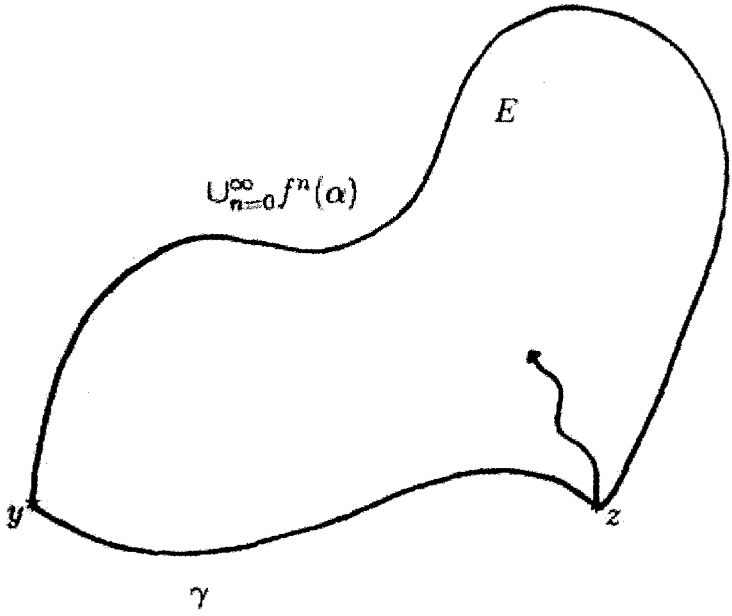

FIGURE 1

Then there exists a point t in γ such that

$$\widehat{yt}(\gamma) \cap \cup_{n=0}^{\infty} f^n(\alpha) = \{y,\ t\} \text{ and so } t \neq z.$$

And there exists j, $r-1 < j < \infty$, such that

$$t \in \{f^j(\alpha) \setminus f^{j+1}(y)\}.$$

Let E' be bounded by $\widehat{yt}(\gamma) \cup \widehat{yt}(L)$. Then E' is an h-disk and

$$f^{j+1}(y) \in E' \text{ or } f^{j+1}(y) \notin E'.$$

By the proof of Case 1, f is not a free homeomorphism.

Case 3:

- $\gamma \cap L \neq \{y,\ z\}$,
- $\gamma \cap L$ is a finite set, and
- $\gamma \cap [\cup_{n=0}^{r-1} f^n(\alpha)] \neq \{y,\ z\}$.

Let $y = t_0, t_1, \cdots, t_n = z$ be points in $\gamma \cap \widehat{yz}(L)$ such that

$$\gamma \cap \widehat{yt_i}(L) = \{\, y, t_1, \cdots, t_i \,\},\ i = 1, \cdots, n$$

(see Figure 3). If there are t_s, t_m in $\{\, y,\ t_1, \cdots, t_{n-1},\ z \,\}$ such that

$$\widehat{t_s t_m}(\gamma) \cap \{y,\ t_1, \cdots, t_{n-1},\ z\} = \{t_s,\ t_m\}$$

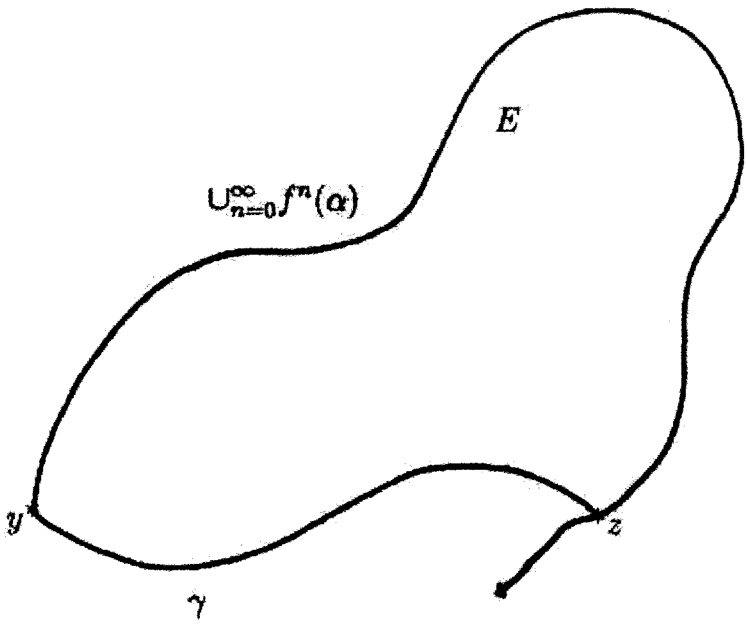

FIGURE 2

and $\widehat{t_s t_m}(L)$ contain more than $k+1$ points of the orbit of y, then we consider an h-disk E'' bounded by $\widehat{t_s t_m}(\gamma) \cup \widehat{t_s t_m}(L)$. Then by the proof of Case 1,

$$f^k(\widehat{t_s t_m}(\gamma)) \cap \widehat{t_s t_m}(\gamma) \neq \phi.$$

Hence

$$f^{kp}(D) \cap f^{k(p+1)}(D) \neq \phi$$

and f is not a free homeomorphism.

If there exists a t_i in $[\{y, t_1, \cdots, t_{n-1}, z\} \cap \widehat{yf^{-k}}(z)(L)]$ such that $f^k(t_i) \in \gamma$, then also

$$f^k(\gamma) \cap \gamma \neq \phi$$

and it is a contradiction.

Now we assume that for each pair t_s, t_m in $\{y, t_1, \cdots, t_{n-1}, z\}$ satisfying

$$\widehat{t_s t_m}(\gamma) \cap \{y, t_1, \cdots, t_{n-1}, z\} = \{t_s, t_m\},$$

$\widehat{t_s t_m}(L)$ contain less than k points of the orbit of y and

$$\text{for each } t_i \in [\{y, t_1, \cdots, t_{n-1}, z\} \cap \widehat{yf^{-k}}(z)(L)],\ f^k(t_i) \notin \gamma.$$

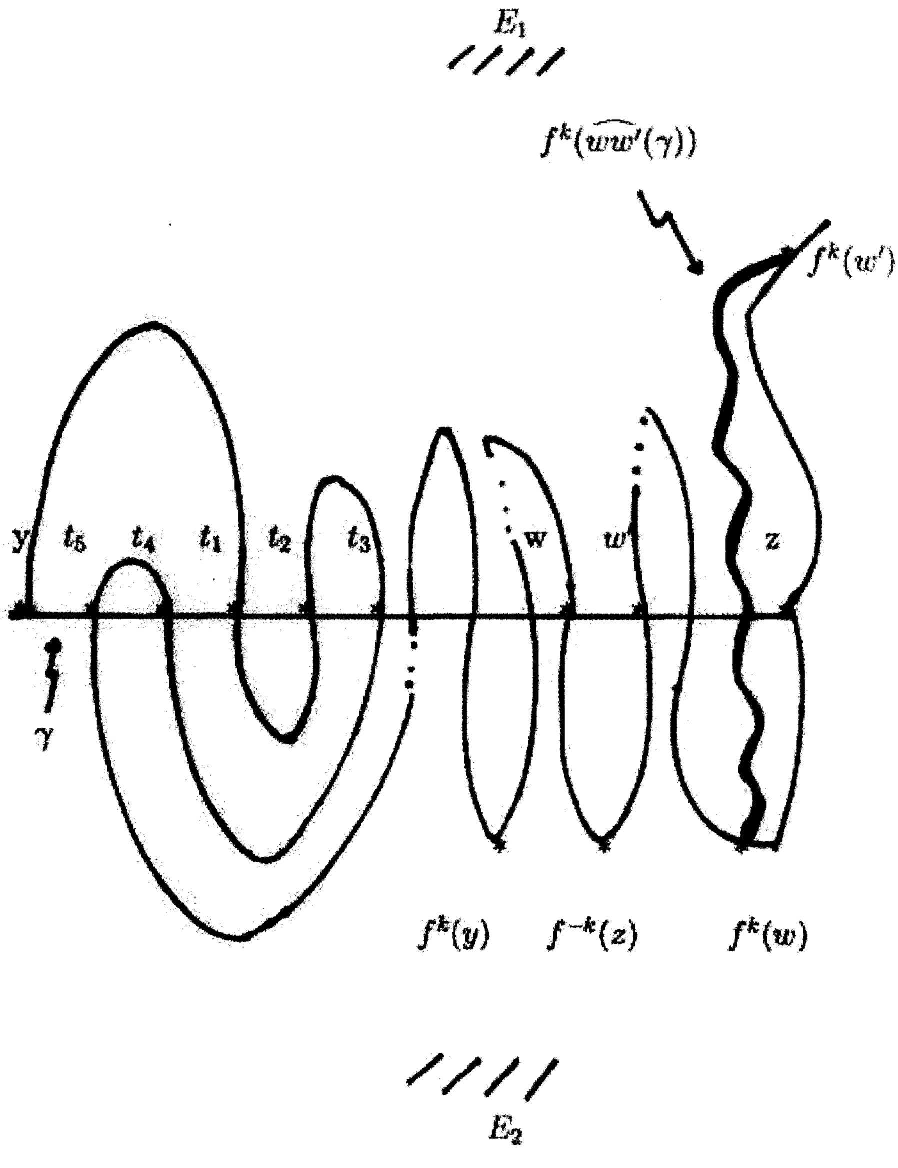

FIGURE 3

Let E_1 be (locally) one side of γ such that $\widehat{yt_1}(L) \subset E_1 \cup \gamma$ and let E_2 be (locally) the other side of γ (see Figure 3).

If

$$f^k(\{y, t_1, \cdots, t_{n-1}, z\}) \cap \widehat{yz}(L) \cap E_l \neq \phi, \; l = 1, 2,$$

then

$$\gamma \cap f^k(\gamma) \neq \phi$$

and also

$$f^{kp}(D) \cap f^{k(p+1)}(D) \neq \phi.$$

Case 4:

- $\gamma \cap L \neq \{y,\ z\}$,
- $\gamma \cap L$ is a finite set,
- $\gamma \cap [\cup_{n=0}^{r-1} f^n(\alpha)] \neq \{y,\ z\}$, and
- $\gamma \cap [\cup_{n=r}^{\infty} f^n(\alpha)] = z$.

We use the notations and their meanings of the assumption of the proof of Case 3. Let

$$f^k(\{y,\ t_1,\ \cdots,\ t_{n-1},\ z\}) \cap \widehat{yz}(L) \cap E_1 = \phi.$$

If $f^{-k}(z) \in E_2$, then there exist two points $w,\ w'$ in $\{y,\ t_1,\ \cdots,\ t_{n-1},\ z\}$ such that

$$f^{-k}(z) \in \widehat{ww'}(L),\ \widehat{ww'}(L) \cap \gamma = \{w, w'\} \text{ and } w' \notin \widehat{yw}(\gamma).$$

Since

$$f^k(w) \in E_2 \text{ and } f^k(f^{-k}(z)) \in \gamma,$$

by Lemma 2.4,

$$f^k(\widehat{ww'}(\gamma)) \cap \gamma \neq \phi \text{ (see Figure 3).}$$

Now let $f^{-k}(z) \in E_1$. Then we consider some points of the orbit of $f^k(y)$ in $\widehat{yf^{-k}(z)}(L) \cap E_1$ as $\{f^{2k}(y), \cdots, f^{-k}(z)\}$. We can choose a point $f^{nk}(y)$ in $[\{f^{2k}(y),\ \cdots, f^{-k}(z)\} \cap E_1]$ such that

$$\{f^k(y),\ \cdots,\ f^{k(n-1)}(y)\} \cap E_1 = \phi,$$

since $f^{-k}(z) \in E_1$. Then since $f^k(y) \in E_2$, there exist two points $w,\ w'$ such that

$$f^{k(n-1)}(y) \in \widehat{ww'}(L),\ \widehat{ww'}(L) \cap \gamma = \{w,\ w'\} \text{ and } w' \notin \widehat{yw}(\gamma).$$

Since $f^{nk}(y) \in E_1$ and $f^k(w),\ f^k(w')$ in E_2, by Lemma 2.4,

$$f^k(\widehat{ww'}(\gamma)) \cap \gamma \neq \phi.$$

Let

$$f^k(\{y,\ t_1,\ \cdots,\ t_{n-1},\ z\}) \cap \widehat{yz}(L) \cap E_2 = \phi.$$

By the same techniques as above, we can easily check that f is not free.

Case 5:

- $\gamma \cap L \neq \{y,\ z\}$,
- $\gamma \cap L$ is a finite set,

- $\gamma \cap [\cup_{n=0}^{r-1} f^n(\alpha)] \neq \{y, z\}$, and
- $\gamma \cap [\cup_{n=r}^{\infty} f^n(\alpha)] \neq z$.

We can prove the case by the similar methods of the proof of Case 4.

Case 6:

- $\gamma \cap L$ is an infinite set.

Then γ contains a point of $\limsup f^n(\alpha)$. By Corollary 2.6, γ contains a fixed point and so

$$f^{kp}(D) \cap f^{k(p+1)}(D) \neq \phi.$$

Therefore f is not a free homeomorphism for all the cases. This completes the proof. □

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Department of Mathematics
Chonbuk National University
Chonbuk 561-756, Korea
E-mail: jjpark46@moak.chonbuk.ac.kr