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# ON INFINITE CLASSES OF GENUS TWO 1-BRIDGE KNOTS

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ABSTRACT. We study a family of 2-bridge knots with 2-tangles in the 3-sphere admitting a genus two 1-bridge splitting. We also observe a geometric relation between $(g-1,1)$-splitting and $(g,0)$-splitting for $g=2,3$. Moreover we construct a family of closed orientable 3-manifolds which are $n$-fold cyclic coverings of the 3-sphere branched over those 2-bridge knots.

## 1. Preliminaries and definitions

By a handlebody, we mean a bounded connected oriented 3-manifold $V$ which admits mutually disjoint proper disks such that they split $V$ into solid tori. Let $M$ be a closed connected oriented 3-manifold. A Heegaard handlebody of $M$ is a handlebody $V$ in $M$ such that $V' = Cl(M-V)$, the closure of $M-V$, is also a handlebody. The surface $H = V \cap V'$ and $V \cup_H V'$ are called a Heegaard surface(or Heegaard diagram) and a Heegaard splitting of $M$, respectively. Every closed connected oriented 3-manifold $M$ admits a Heegaard splitting.

DEFINITION 1.1. A properly embedded arc $t$ in a handlebody $V$ is called trivial if it is boundary parallel, namely, there is a disk $C$ embedded in $V$ such that $t \subset \partial C$ and $C \cap \partial V = Cl(\partial C - t)$. This disk $C$ is a cancelling disk of $t$.

DEFINITION 1.2. We call $K$ a $(g,1)$-knot in $M$ if $M$ is a union of two handlebodies $W_1$ and $W_2$ of genus $g$ glued along their boundary handlebodies $\partial W_1$ and $\partial W_2$ and if $K$ intersects each handlebody $W_i$ in a trivial arc $t_i$ for $i=1,2$. The splitting $(W_1,t_1)\cup_H(W_2,t_2)$ is called a $(g,1)$-splitting of $(M,K)$, where $H = W_1\cap W_2 = \partial W_1 = \partial W_2$ and $K = t_1 \cup_H t_1$.

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We note that a $(g,1)$-knot $K = t_1 \cup_F t_2$ is a singular core and such knots are important in the light of some results and conjectures in Dehn surgery on knots (see [1], [2], [3], [8], [9], and [10]).

Note that all 2-bridge knots, torus knots and 3-bridge knots with tunnel number one admits a $(1,1)$-splitting.

DEFINITION 1.3. Let $M$ be the union of two handlebodies $U_1$ and $U_2$ of genus $g$, and let $K$ be a core in $U_1$. Then the splitting $(M,K) = (U_1,K) \cup_H (U_2,\emptyset)$ is called a $(g,0)$-splitting, where $H = \partial U_1 = \partial U_2$. When this is the case, we call $K$ a $(g,0)$-knot.

Let $K$ be a $(2,1)$-knot in $M$ and $H$ be a Heegaard splitting surface of $M$. Then we have three types of $H$ as following: (i) The connected sum of each closed surface for given two different $(1,1)$-splittings, (ii) the stabilized closed surface of the closed surface for given a $(1,1)$-splitting, or (iii) remainder case except (i) and (ii). In case of (i), a $(2,1)$-knot $K$ is the connected sum of two $(1,1)$-knots; of (ii) the knot $K$ is a $(1,1)$-knot; of (iii) the knot $K$ is a prime $(2,1)$-knot. In view of (ii), it was shown in [7] that for each $g \geq 3$, every genus $g$ Heegaard splitting of the exterior $E(K)$ of a non-trivial two bridge knot $K$ is reducible, and that in [6], every $(2,0)$-splitting for a 2-bridge knot $K$ is meridianally stabilized. If a 2-bridge knot $K$ is non-trivial, a $(2,0)$-splitting $(\mathbb{S}^3,K) = (W_1,K) \cup_H (W_2,\emptyset)$ is a weakly $K$-reducible if and only if $H$ is meridianally stabilized.

In sections 2 and 3, we study a $(2,1)$-splitting as the connected sum of two $(1,1)$-splittings, and in the special case of $M = \mathbb{S}^3$ a family of certain $(2,1)$-splittings which is stabilized by a $(1,1)$-splitting and destabilized by a $(3,0)$-splitting. The former represents the connected sum of two $(1,1)$-knots in $\mathbb{S}^3$, and the latter represents the $(1,1)$-knot in $\mathbb{S}^3$. A graph $\Gamma$ in a compact orientable 3-manifold $M$ is called a Heegaard graph if $Cl(M - N(\Gamma))$ is a handlebody, where $N(\cdot)$ denotes a regular neighborhood. In case $\Gamma$ is a singular arc, it is called an unknotting tunnel or a Heegaard string. From meridianally stabilized $(3,0)$-splitting, we show that one arc of Heegaard graph is the tunnel of a 2-bridge knot $K$ induced by certain family of $(2,1)$-splitting.

In section 4, we will construct an infinite family of closed orientable 3-manifolds which are the $n$-fold cyclic branched covering spaces of $\mathbb{S}^3$, and we show that its orbifold space with a branched set admits a $(2,1)$-decomposition. Moreover, we calculate the homology group of the covering spaces.

## 2. Certain classes of genus two 1-bridge knots

THEOREM 2.1. *Suppose that $(N, J)$ and $(R, S)$ admit $(1,1)$-splittings and that $N$ and $R$ are either 3-spheres or lens spaces. Then the connected sum $J\sharp S$ of two $(1,1)$-knots $J$ and $S$ is a $(2,1)$-knot.*

PROOF. Let $(X_1, j_1) \cup_H (X_2, j_2)$ and $(Y_1, s_1) \cup_{H'} (Y_2, s_2)$ be $(1,1)$-splittings of $(N, J)$ and $(R, S)$ respectively. Let $P_1$ and $P_2$ be points in $J$ and $S$ respectively, and $(B_1^3, B_1^1)$ and $(B_2^3, B_2^1)$ be regular neighborhoods of $P_1$ and $P_2$, which are trivial ball pair, in $(N, J)$ and $(R, S)$ respectively. Note that, for any two given knots, the knot type of the connected sum is uniquely determined. From the fact of $K = J\sharp S$, there is an orientation-reversing homeomorphism $\phi : (\partial B_2^3, \partial B_2^1) \to (\partial B_1^3, \partial B_1^1)$. Therefore we obtain the connected sum $(N - intB_1^3, J - intB_1^1) \cup_\phi (R - intB_1^3, S - intB_1^1) = (N\sharp R, K)$ where *int* means interior. By the uniqueness of connected sum, there are small disks $D_1, D_2$(resp. $U_1, U_2$) of points of $\partial j_i$(resp. $\partial s_i$) on $H$(resp. $H'$) for $i = 1, 2$ such that $D_1 \cup D_2 = \partial B_1^3$ and $U_1 \cup U_2 = \partial B_2^3$. Then for $i = 1, 2$, $(X_i, j_i)\sharp(Y_i, s_i) = (W_i, K_i)$ is a genus two handlebody such that $j_i \sharp s_i = K_i$ is a trivial arc and $H'' = H\sharp H' = \partial W_i$. Thus $(N\sharp R, K) = (W_1, K_1) \cup_{H''} (W_2, K_2)$ is a $(2,1)$-splitting. □

We now give a definite example for the connected sum of two $(1,1)$-knots in $\mathbb{S}^3$ in the following.

EXAMPLE 2.2. We show that the connected sum of two $(1,1)$-splittings representing a trefoil knot is a $(2,1)$-splitting representing the connected sum of two treifoil knots(see Figure 1). We note that each in Figure 1.a is called the (1,1)-splitting of a trefoil knot $K$ in $\mathbb{S}^3$, and is denoted by $(\mathbb{S}^3, K) = (V_1 \cup_H V_2, K_1 \cup K_2)$ in general, where the trivial arc $K_1$ in $V_1$ is properly embedded so that $\partial K_1$ is located in the interior of outermost regions wrapping by a simple closed curve $l$, and the trivial arc $K_2$ is properly embedded in its dual $V_2$ satisfying that $K_2 \cap \partial V_2 = \partial K_2$ is to be located in the interior of the same outmost regions as $V_2$. Since $V_1 \cup_H V_2 = \mathbb{S}^3$, we could bring $H$ into the standard one such that $|m \cap l| = 1$ by a (special) geometric $T$-transformations, i.e., wave reducing moves on $H$, where $m$ is a meridian curve. Indeed, these moves can be realized by cut and glue moves in Heegaard diagram $H$. Under these moves, $K_1$ is also transformed to $K_1'$ in the standard Heegaard splitting of $\mathbb{S}^3$. To eliminate 1-handle with 2-handle, we may attach 2-handle along a simple closed curve $l$ on standard Heegaard splitting. Finally, projecting a singular core of 2-handle onto the standard

Heegaard diagram, we may obtain the desired $(1,1)$-knot, in particular, a treifoil knot in our example.

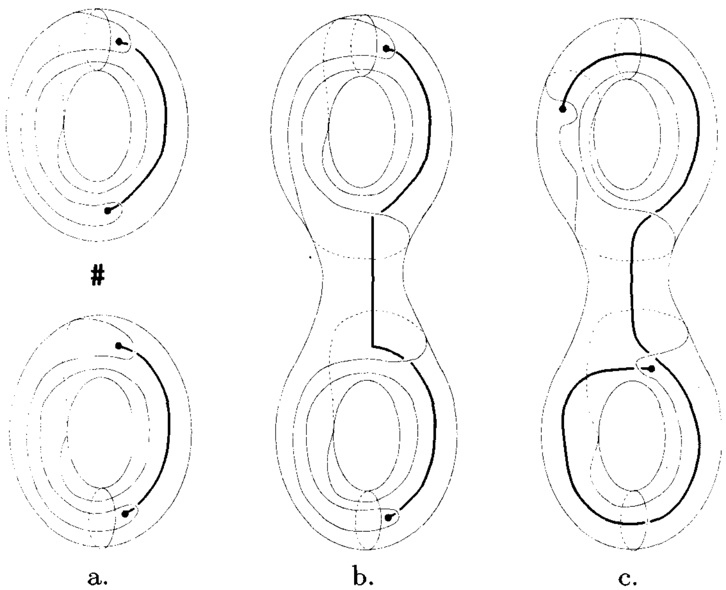


FIGURE 1. (2,1)-splitting of two (1,1)-splittings by the connected sum

Figure 1.b is the result of connected sum of two $(1,1)$-decompositions and Figure 1.c is the result of transformation of the original splitting into a standard one. By the same method, attaching a 2-handle along each simple closed curve, we may obtain the connected sum of two treifoil knots.

We have an interest in prime knots admitting a $(2,1)$-splitting including all tunnel number one knots, where tunnel number one knot means that, given a pair $(M, K)$, $M$ is the union of genus two handlebodies $W_1 \cup_{\partial W_1} W_2$ and $K$ is a core in $W_1$, and call this splitting $(M, K) = (W_1, K) \cup_{\partial W_1} (W_2, \emptyset)$ a $(2,0)$ -splitting.

In the next, we make a study of $(2,1)$-knots such that

$$(M, K) = (V_1, t_1) \cup_{H_1} (V_2, t_2) = (W_1, k_1) \cup_{H_2} (W_2, k_2)$$

and a study of $n$-fold branched covering with a branched set.

For integers $m, s \geq 3$ and not both even, we call the diagram in Figure 2 the $(m, s)$-diagram.

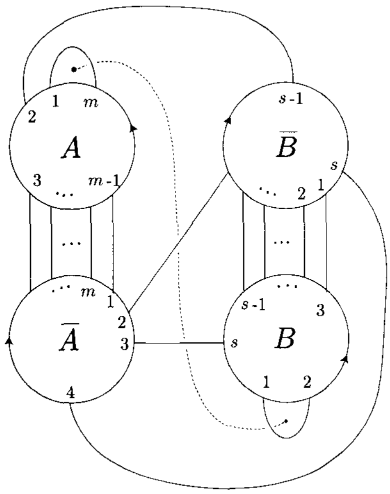
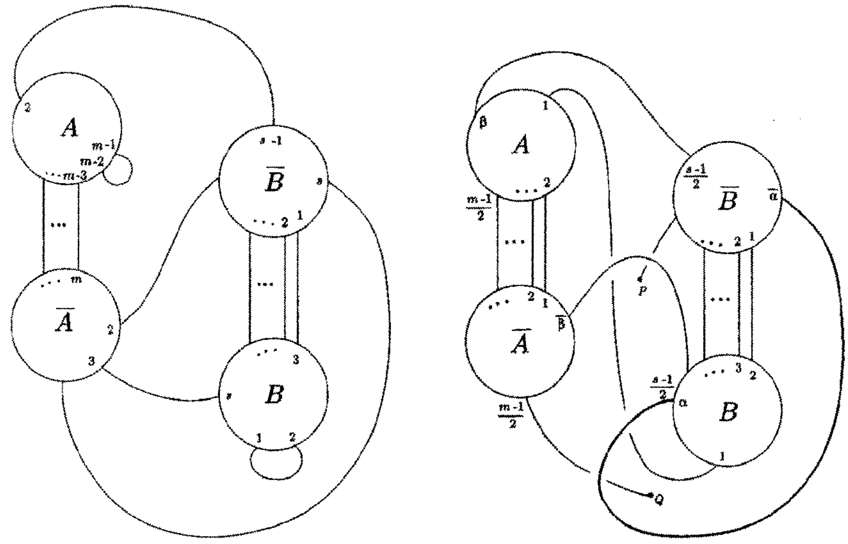



FIGURE 2. The $(m, s)$-diagram

We now consider an infinite family of $(2,1)$-splittings induced by $(m,s)$-diagrams. In $(2,1)$-splittings induced by $(m,s)$-diagrams, two oriented meridian disks $\{A, \overline{A}\}$ and $\{B, \overline{B}\}$ are 1-handles, and edges connecting these two meridian disks are 2-handles. Regions which are bounded by meridian disks and edges are 3-handles. The dotted line connecting two points in 3-handles with bigon boundaries(or outmost region) is a trivial arc in a genus two handlebody.

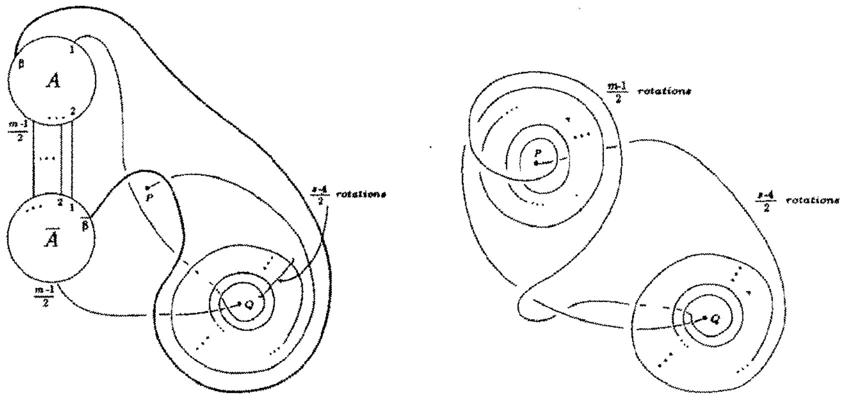
THEOREM 2.3. *Let $(M, K)$ admit a $(2,1)$-splitting induced by a $(m, s)$ -diagram, where $m, s \geq 3$ and not both even. Then $M = \mathbb{S}^3$ and $K$ is the 2-bridge knot $(m-1) + \frac{1}{s-1}$.*

PROOF. Let $(M, K)$ admit a $(2,1)$-splitting with a Heegaard diagram $H$ induced by a $(m,s)$-diagram(see Figure 2). To show that $M$ is the 3-sphere, we move the Heegaard diagram $H$ to a standard Heegaard diagram of $\mathbb{S}^3$ by simple isotopic moves. We have two types of standard Heegaard diagrams depending on $m$ and $s$. Let $m$ and $s$ be odd and even respectively, and we consider two points 1 and $m$ on meridian disk $A$. By cancelling a 3-handle by 2-handle along $\{1, m\}$, we have a diagram as shown in Figure 3.a. We continue this procedures $\frac{m-3}{2}$ times more on $A$ and $\frac{s-1}{2}$ times on $B$. We then get a reduced Heegaard splitting of the 3-sphere. Indeed we see the cancelling procedures with the singular curve from Figure 3 to Figure 4. By connecting two points $P$ and $Q$, we have the 2-bridge knot $(m-1)+\frac{1}{s-1}$. For even integers $m, s$, we can apply the same argument.




a. b.

FIGURE 3




a. b.

FIGURE 4

□

COROLLARY 2.4. *The 2-bridge knots* $\pm(m-1) \pm (\frac{1}{s-1})$ *or symmetrically* $\pm(s-1) \pm (\frac{1}{m-1})$ *admit a* $(2,1)$*-splitting and this knots are amphicherial.*

PROOF. We observe that a trivial arc has some orientation along each hole of $(2,1)$-splitting induced by a standard $(m,s)$-diagram. Define the orientation of a trivial arc along each hole of a standard diagram by $+$ having the same orientation, and $-$ otherwise. This orientation determines the $+$ or $-$ sign of crossings, and so $\pm(m-1) \pm (\frac{1}{s-1})$ or symmerically $\pm(s-1) \pm (\frac{1}{m-1})$ is obtained by applying same process as Theorem 2.3. □

## 3. On the splittings related to a $(m,s)$-diagram

Let $M$ be a closed orientable 3-manifold and let the Heegaard surface $H$ of $M$ split $M$ into two handlebodies $V_1$ and $V_2$. The surface $H$ is said to be weakly $K$-reducible if $V_i$ contains a meridian disc $D_i$ and $V_j$ contains a cancelling disc $D_j$ of $t_j$ such that int $D_i$ intersects $t_i$ transversely in one point and $\partial D_i \cap \partial D_j = \emptyset$ for $i=1, j=2$ or $i=2, j=1$.

THEOREM 3.1. *Let* $S = (W_1, t_1) \cup_H (W_2, t_2)$ *be the* $(2,1)$*-splitting of* $(\mathbb{S}^3, K)$*, where* $H$ *is the genus two Heegaard surface represented by a* $(m,s)$*-diagram. Then the following properties hold;*
(i) $S$ *is weakly* $K$*-reducible.*
(ii) $S$ *admits a* $(2,0)$*-splitting.*
(iii) $S$ *admits a meridianally stabilized* $(3,0)$*-splitting.*

PROOF. (i) Suppose that $(\mathbb{S}^3, K)$ admit genus two 1-bridge splitting induced by a $(m,s)$-diagram, where $m, s \geq 3$ and not both even. Then there are genus two handlebodies $W_1$ and $W_2$ such that $W_1 \cup_H W_2 = \mathbb{S}^3$, $K \cap W_1 = K_1$ and $K \cap W_2 = K_2$. See Figure 2 for a Heegaard splitting in view of $W_1$. It consists of two disjoint simple closed curves and a boundary parallel trivial arc. The dotted line bounds a meridian disc $D_j$ in $\partial W_2$ intersecting with $K$ in one point. One of simple closed curves, say thick line $D_i$ in Figure 5, is a cancelling disc such that $\partial D_i \cap \partial D_j = \emptyset$ for $i=1, j=2$. Thus the splitting is weakly $K$-reducible.

(ii) By definition of weakly $K$-reducibility, $W_1$ contains a meridian disc $D_1$ and $W_2$ contains a cancelling disc $D_2$ of $K_2$ such that int $D_2$ intersects $K_2$ transversely in one point and $\partial D_2 \cap \partial D_1 = \emptyset$. We push the $K_2$ out of $W_2$ into $W_1$ fixing two points $K \cap \partial W_1$ along $D_2 \cap \partial W_2$. Thus we obtain the desired conclusion(see Figure 6).

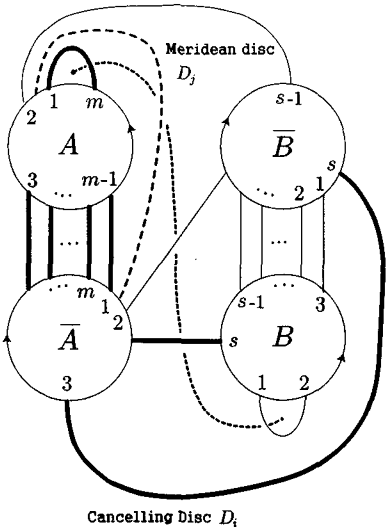



FIGURE 5. Weakly $K$-reducible

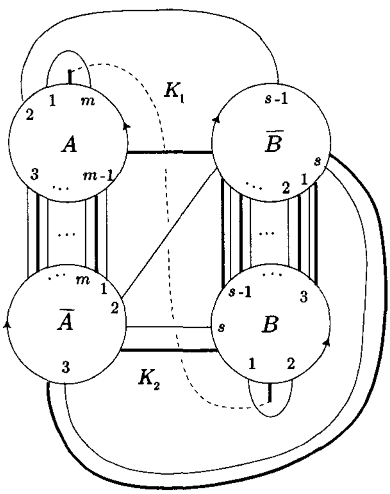



FIGURE 6. (2, 0)-splitting

(iii) Let $H$ be a genus two Heegaard diagram represented by a $(m, s)$-diagram, and $K$ a (2, 1)-knot. Then there are handlebodies $W_1$ and $W_2$ such that $H_2 = \partial W_1 \cap \partial W_2$ and $K \cap W_i = K_i$ is a 1-string trivial arc in $W_i$ $(i = 1, 2)$. We consider the regular neighborhood $N(K_i)$ of the arc $K_i$, say, $N(K_2)$. Note that $Cl(W_2 - N(K_2))$ is a genus three handlebody, denote it by $U_2$, and that $W_1 \cup N(K_2)$ is a genus three handlebody, denote it by $U_1$. By handle sliding on outermost region, we obtain the genus three Heegaard diagram $H'$(see Figure 7). Reversing this process and dragging the relator discs along, we may go back to the Heegaard diagram $H$. Thus we obtain that $(\mathbb{S}^3, K) = (W_1, K_1) \cup_H (W_2, K_2) = (U_1, K) \cup_{H'} (W_2, \emptyset)$. These splittings are equivalent up to meridianally stabilization.

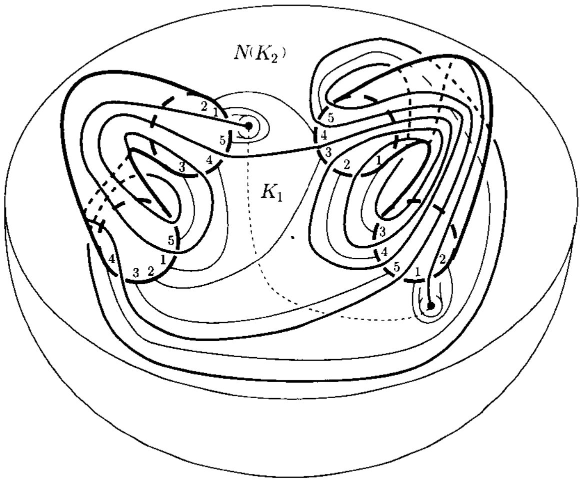



FIGURE 7. (3, 0)-splitting

□

COROLLARY 3.2. *Let $S$ be as in Theorem 3.1 and $T = (\mathbb{S}^3, K) = (W_1, K) \cup_H (W_2, \emptyset)$ be a $(2,0)$-splitting induced by $S$. Then the following hold;*

(i) *$T$ is the meridianally stabilized splitting from a $(1,1)$-splitting of $K$.*

(ii) *$T$ is weakly $K$-reducible.*

PROOF. (i) We just note that every $(2,0)$-splitting for a 2-bridge knot is meridianally stabilized([6]).

(ii) It follows from a result of [4]. □

To every Heegaard diagram, there is an associated Heegaard graph in a canonical way. To define it fix a mid-point $x$ for every meridian disk $D$ and every 0-handle. Then let $\Gamma$ be the union of all arcs $x \times I$ and the cones of their end-points over the mid-points in the 0-handles. Denote by $\Gamma$ the result graph, which is called a Heegaard graph.

THEOREM 3.3. *One arc of a Heegaard graph from a meridianally stabilized $(3,0)$-splitting is the tunnel of 2-bridge knot $K$.*

PROOF. Let $(\mathbb{S}^3, K) = (U_1, K) \cup_{H'} (U_2, \emptyset)$ be a $(3,0)$-splitting. Now, attaching 0-handle to $H'$, intersection of 1-handle and 2-handle, and connecting cone point of 0-handle with each center point of meridian disks, then we obtain graph of (1-handle $\cap$ 2-handle) $\cup$ 0-handle. And again we attach the trivial 1-handles, which is connected the same number of each meridian, to each meridian disk of (1-handle $\cap$ 2-handle) $\cup$ 0-handle. Finally, if we connect each central point of meridian, we get the Heegaard graph which consists of three closed curves, called Heegaard string. Note that one of curves, obtained by handle sliding, is the knot $K$ in $\mathbb{S}^3$ and others are unknotting tunnels for $K$. □

## 4. Covering spaces inducing a $(m,s)$-diagram

In this section we realize a manifold $M$ which admits a $(2,1)$-splitting induced by a $(m,s)$-diagram as a covering space of $\mathbb{S}^3$ by the polyhedron description.

THEOREM 4.1. *For integers $m, s$ not both even, there is a 3-dimensional manifold which is an $n$-fold covering over the $(2,1)$-splitting of $(\mathbb{S}^3, K)$ induced by a $(m,s)$-diagram.*

PROOF. We consider a tessellation on the boundary of 3-ball, which can be regarded as a polyhedron $\mathcal{P}(m,s,n)$, consisting of $n$ $m$-gons $F_i$ in the northern hemisphere, $n$ $s$-gons $K_i$ in the southern hemisphere, $n$ $m$-gons $\overline{F}_i$ and $n$ $s$-gons $\overline{K}_i$ in the equatorial zone. Then $\mathcal{P}(m,s,n)$ has $4n$ faces, $n(m+s)+2n$ edges, $n(m+s)-2n+2$ vertices(see Figure 8).

We now consider two cases, $m = 2k+1, s = 2\ell+1$ and $m = 2k+1, s = 2\ell$ separately.

For $m = 2k+1, s = 2\ell$, the oriented edges can be labelled as shown in Figure 10. Then the oriented edges fall into $2n$ classes: $x_i, y_i$, $i = 1, \ldots, n$. Moreover the boundary cycle of the $m$-gons $F_i$ and $\overline{F}_{i+n-2}$ is $y_i^{-1} x_{i+2k}^{-1} x_{i+2k-1} \cdots x_{i+3}^{-1} x_{i+2} x_{i+1}^{-1} x_i$ and the boundary cycle of the $s$-gons $K_i$ and $\overline{K}_i$ is $x_{i+2k+2}(y_i y_{i+1}^{-1})^{\ell-1} y_i$ where the indices are taken mod $n$.

Note that the set of all the faces splits into pairs of faces with the same sequences of oriented boundary edges. We define the face identification such that the corresponding oriented edges on polygons carrying the

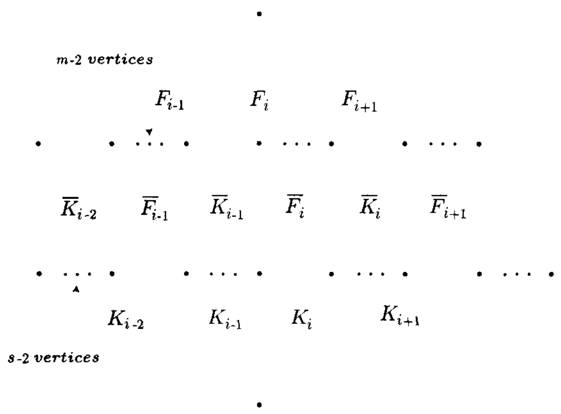



FIGURE 8. $\mathcal{P}(m, s, n)$

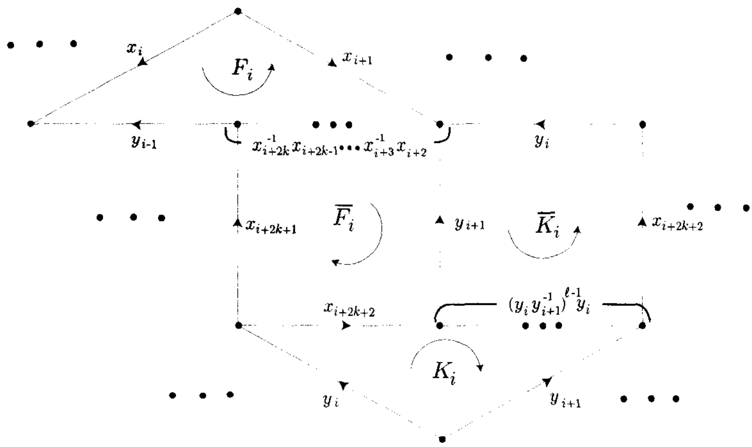



FIGURE 9

same label are identified as follows: for each $i = 1, \ldots, n$,

$$\begin{cases} F_i \Longleftrightarrow \overline{F}_{i+n-2} \\ K_i \Longleftrightarrow \overline{K}_i \end{cases}.$$

The resulting complex $\mathcal{M}(m, s, n)$ has one vertex, $2n$ 1-cells, $2n$ 2-cells and one 3-cell. Then we have a closed connected orientable 3-manifold $\mathcal{M}(m, s, n)$ since its Euler characteristic equals zero.

To complete the proof, we simply note that $\mathcal{M}(m, s, n)$ is an $n$-fold cyclic covering over the orbifold $\mathcal{O}(m, s)$ whose Heegaard diagram is a $(m, s)$-diagram shown in Figure 2. In fact, $\mathcal{O}(m, s)$ is the quotient space of $\mathcal{M}(m,s,n)$ by the action of the cyclic group of rotations induced by the cylindrical symmetry of the polyhedron $\mathcal{P}(m,s,n)$. The singular set $(m-1)+\frac{1}{s-1}$ is the image in the quotient of the axis of symmetry of the rotation.

For the case $m=2k+1, s=2\ell+1$, we can apply the similar argument. The only difference are the edge orientation and labelling, and face identification as follow; The boundary cycle of the $m$-gons $F_i$ and $\overline{F}_i$ is $y_i(x_{i+1}^{-1}x_i)^k$ and the boundary cycle of the $s$-gons $K_i$ and $\overline{K}_i$ is $x_i(y_{i+1}^{-1}y_i)^\ell$ with the indices taken mod $n$(see Figure 9).

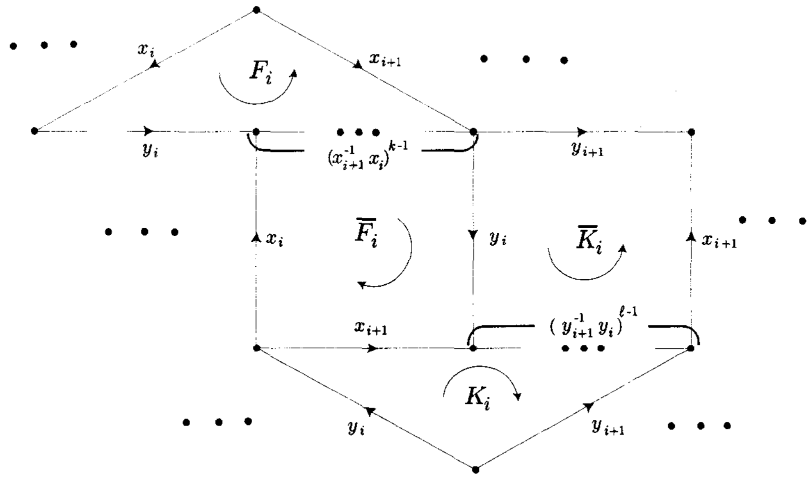



FIGURE 10

For the face identification, we define as follows:

$$\begin{cases} F_i \Longleftrightarrow \overline{F}_i \\ K_i \Longleftrightarrow \overline{K}_i \end{cases}.$$

□

We note that $\mathcal{P}(m,s,n)$ has a unique vertex and so the relators of the fundamental group of $\mathcal{M}(m,s,n)$ can be obtained by running around the boundaries of the $2n$ 2-cells of $\mathcal{P}(m,s,n)$. Hence the fundamental group of $\mathcal{M}(2k+1,2\ell+1,n)$ is

$$\langle x_1,\ldots,x_n,y_1,\ldots,y_n \mid y_i(x_{i+1}^{-1}x_i)^k=1,\ x_{i+1}(y_{i+1}^{-1}y_i)^\ell=1,\ i \bmod n\rangle.$$

Similarly, for $\mathcal{M}(2k+1,2\ell,n)$, we have

$$\langle x_1,\ldots,x_n,y_1,\ldots,y_n \mid y_i^{-1}x_{i+2k}^{-1}x_{i+2k-1}\cdots x_{i+3}^{-1}x_{i+2}x_{i+1}^{-1}x_i,$$
$$x_{i+2\ell+2}(y_iy_{i+1}^{-1})^{\ell-1}y_i=1,\ i \bmod n\rangle.$$

THEOREM 4.2. *The fundamental group of $\mathcal{M}(2k+1, 2\ell+1, n)$ has the following cyclic presentation with $n$ generators:*

$$\begin{aligned}&\pi_1(\mathcal{M}(2k+1,2\ell+1,n))\\ =\ &\langle y_1,\dots,y_n | y_i((y_{i+1}^{-1}y_i)^\ell(y_{i-1}^{-1}y_i)^\ell)^k = 1, i=1,\dots,n\rangle.\end{aligned}$$

PROOF. Note that the fundamental group of $\pi_1(\mathcal{M}(2k+1, 2\ell+1, n))$ has the following presentation:

$$\begin{aligned}&\pi_1(\mathcal{M}(2k+1,2\ell+1,n))\\ =\ &\langle x_1,\dots,x_n,y_1,\dots,y_n \mid R_{2i+1}=1,\ R_{2i}=1,\ i \bmod n\rangle.\end{aligned}$$

where $R_{2i+1} = y_i(x_{i+1}^{-1}x_i)^k$ and $R_{2i} = x_{i+1}(y_{i+1}^{-1}y_i)^\ell$.

We simply isolate $R_{2i} = x_{i+1}(y_{i+1}^{-1}y_i)^\ell$ from the presentation with $x_{i+1} = (y_i^{-1}y_{i+1})^\ell$ for $i = 1,\dots,n$. Then we have the following cyclic presentation:

$$\begin{aligned}&\pi_1(\mathcal{M}(2k+1,2\ell+1,n))\\ =\ &\langle y_1,\dots,y_n | y_i((y_{i+1}^{-1}y_i)^\ell(y_{i-1}^{-1}y_i)^\ell)^k, i=1,\dots,n\rangle.\end{aligned}$$

We can get a similar presentation for $\mathcal{M}(2k+1, 2\ell, n)$. □

For odd integers $m = 2k+1, s = 2\ell+1$, we have the following presentation for the homology groups;

COROLLARY 4.3.

$$\begin{aligned}&H_1(\mathcal{M}(2k+1,2\ell+1,n)) \cong G(2k+1,2\ell+1,n)^{ab}\\ \cong\ &\begin{cases}\mathbb{Z}_N\times\mathbb{Z}_{(4\ell^2+k^2)N}, & N=\frac{1}{k}f(n')g(n') \quad \text{for } n=2n'\\ \mathbb{Z}_N\times\mathbb{Z}_N, & N=g(n) \quad \text{for } n=2n'+1.\end{cases}\end{aligned}$$

*Here $f(n), g(n)$ are the generalized Fibonacci-Lucas numbers defined by the equation*

$$h(n+2) = \ell^2 h(n) + kh(n+1),$$

*and the initial values*

$$f(0)=0, f(1)=k, \text{ and } g(0)=2, g(1)=k$$

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