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NONLINEAR VARIATIONAL EVOLUTION INEQUALITIES WITH NONLOCAL CONDITIONS

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ABSTRACT. The existence of solutions for the nonlinear functional differential equation with nonlocal conditions governed by the variational inequality is studied. The regularity and a variation of solutions of the equation are also given.

1. Introduction

Let H and V be two complex Hilbert spaces. Assume that V is dense subspace in H and the injection of V into H is continuous.

In this paper, we deal with the existence and regularity for solutions of the following nonlinear functional differential equation with the nonlocal initial condition governed by the variational inequality in Hilbert spaces:

$$\text{(VIP)} \quad \begin{cases} (\dfrac{dx(t)}{dt} + Ax(t), x(t) - z) + \phi(x(t)) - \phi(z) \\ \leq (f(t, x(t)) + k(t), x(t) - z), \text{ a.e., } 0 < t \leq T, \quad z \in V \\ x(0) = x_0 - g(t_1, \dots t_p, x(\cdot)). \end{cases}$$

The norms on V, H and the duality pairing between V^* and V will be denoted by $||\cdot||$, $|\cdot|$ and $(\cdot,\cdot)$, respectively. Let $\phi : V \to (-\infty, +\infty]$ be a lower semicontinuous, proper convex function and $g : L^2(0,T;V) \to H$ be assumed to be uniformly Lipschitz continuous. Let A be the operator associated with a sesquilinear form $a(\cdot,\cdot)$ defined on $V \times V$ satisfying Gårding's inequality:

$$(Au, v) = a(u, v), \quad u, \ v \in V$$

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which is assumed to satisfy

$$(Au, u) \geq \omega_1 ||u||^2 - \omega_2 |u|^2$$

where $\omega_1 > 0$ and ω_2 is a real number. Then A generates an analytic semigroup in both H and V^* (see [9; Theorem 3.6.1]) and so the equation (VIP) may be considered as an equation in H as well as in V^*. The nonlinear operator f from $[0, T] \times V$ to H is assumed to be uniformly Lipschitz continuous with respect to the second variable. Noting the definition of the subdifferential operator $\partial\phi$, the problem (VIP) is represented by the following nonlinear functional differential problem on H

$$\text{(NDE)} \quad \begin{cases} \dfrac{dx(t)}{dt} + Ax(t) + \partial\phi(x(t)) \ni f(t, x(t)) + k(t), & 0 < t \leq T, \\ x(0) = x_0 - g(x). \end{cases}$$

The existence and regularity for solutions of the parabolic variational inequality in the linear case($f \equiv 0$) without nonlocal initial conditions was investigated by Brézis [5] (also see section 4.3.2 in Barbu [4]). Jackson [8] showed the existence and uniqueness of solutions to semilinear nonlocal parabolic equations and Byszewski and Akca [7] studied the existence of mild and classical solutions of nonlocal Cauchy problem for a semilinear functional differential evolution equation. Aizicovici, Gao, Mckibben [1, 2] have studied for differential equations governed by m-accretive operators under various compactness assumptions. For some other results on nonlocal problems see the bibliographies of [1, 2, 3, 6, 7].

In this note, with some general condition of the Lipschitz continuity of nonlinear operator f from $[0, T] \times V$ to H, we established the problem for the wellposedness and regularity of solutions of nonlinear variational evolution inequalities with nonlocal conditions by converting the problem into the contraction principle and the norm estimate of a solution of the above nonlinear equation on $L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \cap C([0, T]; H)$. Consequently, an example illustrated the applicability of our work is given in the last section.

2. Preliminaries

Let V and H be complex Hilbert spaces forming Gelfand triple $V \subset H \subset V^*$ with pivot space H. For the sake of simplicity, we may consider

$$(2.1) \qquad ||u||_* \leq |u| \leq ||u||, \quad u \in V$$

where $||\cdot||_*$ is the norm of the element of V^*. Let $a(\cdot,\cdot)$ be a bounded sesquilinear form defined in $V\times V$ and satisfying Gårding's inequality

$$\text{Re } a(u,u)\geq \omega_1||u||^2-\omega_2|u|^2, \tag{2.2}$$

where $\omega_1>0$ and ω_2 is a real number.

Let A be the operator associated with the sesquilinear form $a(\cdot,\cdot)$:

$$(Au,v)=a(u,v),\quad u,\ v\in V.$$

Then A is a bounded linear operator from V to V^* and $-A$ generates an analytic semigroup in both of H and V^* as is seen in [10; Theorem 3.6.1]. The realization for the operator A in H which is the restriction of A to

$$D(A)=\{u\in V; Au\in H\}$$

is also denoted by A.

The following L^2-regularity for the abstract linear parabolic equation

$$\begin{cases}\dfrac{dx(t)}{dt}+Ax(t)=k(t), & 0<t\leq T,\\ x(0)=x_0\end{cases} \tag{LE}$$

has a unique solution x in $[0,T]$ for each $T>0$ if $x_0\in(D(A),H)_{1/2,2}$ and $k\in L^2(0,T;H)$ where $(D(A),H)_{1/2,2}$ is the real interpolation space between $D(A)$ and H. Moreover, we have

$$\begin{aligned}||x||_{L^2(0,T;D(A))\cap W^{1,2}(0,T,H)}\leq C_2(&||x_0||_{(D(A),H)_{1/2,2}}\\ &+||k||_{L^2(0,T;H)})\end{aligned} \tag{2.3}$$

where C_2 depends on T and M(see [9]).

If an operator A is bounded linear from V to V^* associated with the sesquilinear form $a(\cdot,\cdot)$ then it is easily seen that

$$H=\{x\in V^*:\int_0^T||Ae^{tA}x||_*^2dt<\infty\},$$

for the time $T>0$. Therefore, in terms of the intermediate theory we can see that

$$(V,V^*)_{1/2,2}=H$$

and obtain the following results.

PROPOSITION 2.1. *Let $x_0 \in H$ and $k \in L^2(0,T;V^*)$, $T > 0$. Then there exists a unique solution x of* (LE) *belonging to*

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

and satisfying

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \leq C_2(|x_0| + ||k||_{L^2(0,T;V^*)}), \tag{2.4}$$

where C_2 is a constant depending on T.

Let $\phi : V \to (-\infty, +\infty]$ be a lower semicontinuous, proper convex function. Then the subdifferential operator $\partial\phi$ of ϕ is defined by

$$\partial\phi(x) = \{x^* \in V^*; \phi(x) \leq \phi(y) + (x^*, x - y), \quad y \in V\}.$$

First, let us concern with the following perturbation of subdifferential operator:

$$\text{(VE)} \qquad \begin{cases} \dfrac{dx(t)}{dt} + Ax(t) + \partial\phi(x(t)) \ni k(t), & 0 < t \leq T, \\ x(0) = x_0. \end{cases}$$

Using the regularity for the variational inequality of parabolic type as seen in [4; section 4.3] we have the following results on the equation (VE). We denote the closure in H of the set $D(\phi) = \{u \in V : \phi(u) < \infty\}$ by $\overline{D(\phi)}$ and the minimal segment of $\partial\phi$ by $(\partial\phi)^0$.

PROPOSITION 2.2. 1) *Let $k \in L^2(0,T;V^*)$ and $x_0 \in \overline{D(\phi)}$. Then the equation* (VE) *has a unique solution*

$$x \in L^2(0,T;V) \cap C([0,T];H),$$

which satisfies

$$x'(t) = (k(t) - Ax(t) - \partial\phi(x(t)))^0$$

and

$$||x||_{L^2\cap C} \leq C_3(1 + |x_0| + ||k||_{L^2(0,T;V^*)}) \tag{2.5}$$

where C_3 is a positive constant and $L^2 \cap C = L^2(0,T;V) \cap C([0,T];H)$.

2) *Let A be symmetric and let us assume that there exist $h \in H$ such that for every $\epsilon > 0$ and any $y \in D(\phi)$*

$$J_\epsilon(y + \epsilon h) \in D(\phi) \text{ and } \phi(J_\epsilon(y + \epsilon h)) \leq \phi(y)$$

where $J_\epsilon = (I + \epsilon A)^{-1}$. Then for $k \in L^2(0, T; H)$ and $x_0 \in \overline{D(\phi)} \cap V$ the equation (VE) *has a unique solution*

$$x \in L^2(0, T; D(A)) \cap W^{1,2}(0, T; H) \cap C([0, T]; H),$$

which satisfies

$$||x||_{L^2 \cap W^{1,2} \cap C} \leq C_3(1 + ||x_0|| + ||k||_{L^2(0,T;H)}). \tag{2.6}$$

Here, we remark that if $D(A)$ endowed with the graph norm of A to H is compactly embedded in V and $x \in L^2(0, T; D(A))$(or the semigroup operator $S(t)$ generated by A is compact), the following embedding

$$L^2(0, T; D(A)) \cap W^{1,2}(0, T; H) \subset L^2(0, T; V)$$

is compact in view of the interpolation theory. Hence, the mapping $k \mapsto x$ is compact from $L^2(0, T; H)$ to $L^2(0, T; V)$, which is also applicable to optimal control problem.

3. Existence of solutions

In this section, we consider the existence and regularity for solutions of (NDE) under the following basic assumptions:

(H1) Let f be a nonlinear valued mapping from $[0, T] \times V$ into H. We assume that

$$|f(t, z_1) - f(t, z_2)| \leq L||z_1 - z_2||, \quad z_1, \ z_2 \in V.$$

(H2) For $0 < T' \leq T$, $g : L^2(0, T'; V) \to H$ is assumed to be uniformly Lipschitz continuous, namely that there is a constant $M > 0$ such that

$$|g(x_1) - g(x_2)| \leq M||x_1 - x_2||_{L^2(0,T';V)}, \quad x_1, \ x_2 \in L^2(0, T'; V)$$

and

$$g(x) \in \overline{D(\phi)}, \quad x \in L^2(0, T'; V).$$

We assume the following inequality condition.

(H3) $$2\omega_1 - M^2 e^{2\omega_2 T} > 0.$$

The condition (H3) shows some information for the relation among the parameters, ω_1, ω_2, M and T. We shall prove that the existence of solutions of (NDE) is guaranteed if ω_2, M and T is sufficiently small. It may be possible to prove the existence of solutions at a large time T in case where the system is assumed to satisfy certain inequality, for instance, $\omega_2 < 0$. If the nonlinear term f is a reaction rate function, then the value of f means the velocity of generating a chemical material and Lipschitz constant is the accelerative rate of its reaction. We note that the Lipschitz constant M of the nonlinear term f has no effect on the limits of the existence of solutions of (NDE).

The following Lemma is from Brézis [5; Lemma A.5].

LEMMA 3.1. *Let $m \in L^1(0,T;\mathcal{R})$ satisfying $m(t) \geq 0$ for all $t \in (0,T)$ and $a \geq 0$ be a constant. Let b be a continuous function on $[0,T] \subset \mathcal{R}$ satisfying the following inequality:*

$$\frac{1}{2}b^2(t) \leq \frac{1}{2}a^2 + \int_0^t m(s)b(s)ds, \quad t \in [0,T].$$

Then,

$$|b(t)| \leq a + \int_0^t m(s)ds, \quad t \in [0,T].$$

We establish the following results on the solvability of (NDE).

THEOREM 3.1. *Let the assumptions* (H1), (H2) *and* (H3) *be satisfied. Assume that $k \in L^2(0,T;H)$ and $x_0 \in \overline{D(\phi)}$. Then, the equation* (NDE) *has a unique solution*

(3.1) $$x \in L^2(0,T;V) \cap C([0,T];H) \cap W^{1,2}(0,T;H)$$

and there exists a constant C_4 depending on T such that

(3.2) $$||x||_{L^2 \cap C \cap W^{1,2}} \leq C_4(1 + |x_0| + ||k||_{L^2(0,T;V^*)}).$$

Proof. Let us fix $0 < T_0 \leq T$ so that

$$\frac{L^2 + 2LM}{4\omega_2}(e^{2\omega_2 T_0} - 1) < \omega_1 - \frac{1}{2}M^2 e^{2\omega_2 T}. \tag{3.3}$$

where $\omega_i (i = 1, 2)$, L and M are constants in (2.2), (H1) and (H2), respectively.

Noting that $x_0 + g(x) \in \overline{D(\phi)}$ for any $x \in L^2(0, T_0; V)$ by (H2), the following equation

$$\begin{cases} \dfrac{dy(t)}{dt} + Ay(t) + \partial\phi(y(t)) \ni f(t, x(t)) + k(t), \quad 0 < t \leq T_0, \\ y(0) = x_0 + g(x) \end{cases}$$

has a unique solution $y \in L^2(0, T_0; V) \cap C([0, T_0]; H)$ in virtue of Proposition 2.2.

For $i = 1,\ 2$, we consider the following equation.

$$\begin{cases} \dfrac{dy_i(t)}{dt} + Ay_i(t) + \partial\phi(y_i(t)) \ni f(t, x_i(t)) + k(t), \quad 0 < t \leq T_0, \\ y_i(0) = x_0 + g(x_i). \end{cases} \tag{3.4}$$

We are going to show that a well known extension of the contraction principle gives the existence of a unique solution of (NDE) if the condition (3.3) is satisfied. Let $y_1,\ y_2$ be the solutions of (3.4) with x, replaced by $x_1,\ x_2 \in L^2(0, T_0; V)$, respectively. From (3.4) it follows that

$$\begin{aligned} &\frac{d}{dt}(y_1(t) - y_2(t)) + A(y_1(t) - y_2(t)) + \partial\phi(y_1(t)) - \partial\phi(y_2(t)) \\ &\ni f(t, x_1(t)) - f(t, x_2(t)), \quad t > 0. \end{aligned}$$

Acting on both sides by $y_1(t) - y_2(t)$ and using the monotonicity of $\partial\phi$, we have

(3.5)

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|y_1(t) - y_2(t)|^2 + \omega_1 ||y_1(t) - y_2(t)||^2 \\ &\leq \omega_2 |y_1(t) - y_2(t)|^2 + (f(t, x_1(t)) - f(t, x_2(t)), y_1(t) - y_2(t)) \\ &\leq \omega_2 |y_1(t) - y_2(t)|^2 + L||x_1(t) - x_2(t)||\, |y_1(t) - y_2(t)|. \end{aligned}$$

Putting

$$G(t) = L||x_1(t) - x_2(t)||\, |y_1(t) - y_2(t)|,$$

and integrating (3.5) over $(0,t)$, this yields that

(3.6)
$$\frac{1}{2}|y_1(t)-y_2(t)|^2+\omega_1\int_0^t ||y_1(s)-y_2(s)||^2 ds$$
$$\leq \frac{1}{2}|g(x_1)-g(x_2)|^2+\omega_2\int_0^t |y_1(s)-y_2(s)|^2 ds+\int_0^t G(s)ds.$$

From (3.6) it follows that

$$(3.7)\qquad \frac{d}{dt}\{e^{-2\omega_2 t}\int_0^t |y_1(s)-y_2(s)|^2 ds\}$$
$$=2e^{-2\omega_2 t}\{\frac{1}{2}|y_1(t)-y_2(t)|^2-\omega_2\int_0^t |y_1(s)-y_2(s)|^2 ds\}$$
$$\leq 2e^{-2\omega_2 t}(\frac{1}{2}|g(x_1)-g(x_2)|^2+\int_0^t G(s)ds).$$

Thus, integrating (3.7) over $(0,t)$ we have

$$e^{-2\omega_2 t}\int_0^t |y_1(s)-y_2(s)|^2 ds$$
$$\leq \frac{1-e^{-2\omega_2 t}}{2\omega_2}|g(x_1)-g(x_2)|^2+2\int_0^t e^{-2\omega_2\tau}\int_0^\tau G(s)dsd\tau$$
$$=\frac{1-e^{-2\omega_2 t}}{2\omega_2}|g(x_1)-g(x_2)|^2+2\int_0^t\int_s^t e^{-2\omega_2\tau}d\tau G(s)ds$$
$$=\frac{1-e^{-2\omega_2 t}}{2\omega_2}|g(x_1)-g(x_2)|^2+\frac{1}{\omega_2}\int_0^t (e^{-2\omega_2 s}-e^{-2\omega_2 t})G(s)ds,$$

hence, we get

(3.8)
$$\omega_2\int_0^t |y_1(s)-y_2(s)|^2 ds\leq \frac{e^{2\omega_2 t}-1}{2}|g(x_1)-g(x_2)|^2$$
$$+\int_0^t (e^{2\omega_2(t-s)}-1)G(s)ds.$$

Combining (3.6) with (3.8) it holds that

$$
\begin{aligned}
(3.9) \qquad & \frac{1}{2}|y_1(t)-y_2(t)|^2+\omega_1\int_0^t ||y_1(s)-y_2(s)||^2 ds \\
&\leq \frac{e^{2\omega_2 t}}{2}|g(x_1)-g(x_2)|^2+\int_0^t e^{2\omega_2(t-s)}G(s)ds \\
&= \frac{e^{2\omega_2 t}}{2}|g(x_1)-g(x_2)|^2 \\
&+\int_0^t e^{2\omega_2(t-s)}L||x_1(s)-x_2(s)||\,|y_1(s)-y_2(s)|ds,
\end{aligned}
$$

which implies

$$
\begin{aligned}
& \frac{1}{2}(e^{-2\omega_2 t}|y_1(t)-y_2(t)|)^2+\omega_1 e^{-2\omega_2 t}\int_0^t ||y_1(s)-y_2(s)||^2 ds \\
&\leq \frac{1}{2}|g(x_1)-g(x_2)|^2 \\
&+L\int_0^t e^{-\omega_2 s}||x_1(s)-x_2(s)||e^{-\omega_2 s}|y_1(s)-y_2(s)|ds.
\end{aligned}
$$

Here, by using Lemma 3.1, we obtain that

$$
\begin{aligned}
(3.10) \qquad e^{-\omega_2 t}|y_1(t)-y_2(t)| &\leq |g(x_1)-g(x_2)| \\
&+\int_0^t Le^{-\omega_2 s}||x_1(s)-x_2(s)||ds.
\end{aligned}
$$

From (3.9) and (3.10) it follows that

$$
\begin{aligned}
& \frac{1}{2}|y_1(t)-y_2(t)|^2+\omega_1\int_0^t ||y_1(s)-y_2(s)||^2 ds \\
&\leq \frac{M^2 e^{2\omega_2 t}}{2}||x_1-x_2||^2_{L^2(0,T_0;V)} \\
&+L\int_0^t e^{2\omega_2(t-s)}||x_1(s)-x_2(s)||(\int_0^s Le^{\omega_2(s-\tau)}||x_1(\tau)-x_2(\tau)||d\tau \\
&+e^{\omega_2 s}|g(x_1)-g(x_2)|)ds \\
&= \frac{M^2 e^{2\omega_2 t}}{2}||x_1-x_2||^2_{L^2(0,T_0;V)}
\end{aligned}
$$

$$+ L^2 e^{2\omega_2 t} \int_0^t e^{-\omega_2 s} ||x_1(s) - x_2(s)|| \int_0^s e^{-\omega_2 \tau} ||x_1(\tau) - x_2(\tau)|| d\tau ds$$
$$+ LM e^{2\omega_2 t} |g(x_1) - g(x_2)| \int_0^t e^{-\omega_2 s} ||x_1(s) - x_2(s)|| ds$$
$$= \frac{M^2 e^{2\omega_2 t}}{2} ||x_1 - x_2||^2_{L^2(0,T_0;V)} + I + II,$$

$$\begin{aligned} I &= L^2 e^{2\omega_2 t} \int_0^t \frac{1}{2} \frac{d}{ds} \{ \int_0^s e^{-\omega_2 \tau} ||x_1(\tau) - x_2(\tau)|| d\tau \}^2 ds \\ &= \frac{1}{2} L^2 e^{2\omega_2 t} \{ \int_0^t e^{-\omega_2 \tau} ||x_1(\tau) - x_2(\tau)|| d\tau \}^2 \\ &\le \frac{1}{2} L^2 e^{2\omega_2 t} \int_0^t e^{-2\omega_2 \tau} d\tau \int_0^t ||x_1(\tau) - x_2(\tau)||^2 d\tau \\ &\le \frac{L^2}{4\omega_2} (e^{2\omega_2 t} - 1) \int_0^t ||x_1(s) - x_2(s)||^2 ds, \\ II &\le \frac{LM(e^{2\omega_2 t} - 1)}{2\omega_2} ||x_1 - x_2||^2_{L^2(0,T_0;V)}. \end{aligned}$$

Hence, we conclude that

$$\begin{aligned} &\frac{1}{2} |y_1(t) - y_2(t)|^2 + \omega_1 \int_0^t ||y_1(s) - y_2(s)||^2 ds \\ &\le \frac{M^2 e^{2\omega_2 t}}{2} ||x_1 - x_2||^2_{L^2(0,T_0;V)} \\ &+ \frac{L^2 + 2LM}{4\omega_2} (e^{2\omega_2 T_0} - 1) ||x_1 - x_2||^2_{L^2(0,T_0;V)}. \end{aligned} \tag{3.11}$$

Starting from initial value $x_0(t) = x_0 + g(x_0)$, consider a sequence $\{x_n(\cdot)\}$ satisfying

$$\begin{cases} \frac{d}{dt} x_{n+1}(t) + A x_{n+1}(t) + \partial\phi(x_{n+1}(t)) \ni f(t, x_n(t)) + k(t), \ 0 < t \le T_0, \\ x_{n+1}(0) = x_0 + g(x_n). \end{cases}$$

Then the inequality (3.11) implies that for $0 \le t \le T_0$,

$$\begin{aligned} &\frac{1}{2} |x_{n+1}(t) - x_n(t)|^2 + \omega_1 \int_0^t ||x_{n+1}(s) - x_n(s)||^2 ds \\ &\le \frac{M^2 e^{2\omega_2 t}}{2} ||x_n - x_{n-1}||^2_{L^2(0,T_0;V)} \end{aligned} \tag{3.12}$$

$$+\frac{L^2+2LM}{4\omega_2}(e^{2\omega_2 T_0}-1)||x_n-x_{n-1}||^2_{L^2(0,T_0;V)}.$$

So by virtue of the condition (3.3), we have that

$$\sum_{n=1}^{\infty}||x_{n+1}-x_n||_{L^2(0,T_0;V)}<\infty,$$

it gives that there exists $x(\cdot)\in L^2(0,T;V)$ such that

$$x_n(\cdot)\to x(\cdot)\quad \text{in}\quad L^2(0,T_0;V),$$

and hence, from (3.12) there exists $x(\cdot)\in C([0,T];H)$ such that

$$x_n(\cdot)\to x(\cdot)\quad \text{in}\quad C([0,T_0];H).$$

Next, we establish a variation of constant formula (3.2) of a solution of (NDE). Let y be the solution of

$$\begin{cases}\dfrac{dy(t)}{dt}+Ay(t)+\partial\phi(y(t))\ni k(t), & 0<t\le T_0,\\ y(0)=x_0.\end{cases}$$

Then, since

$$\frac{d}{dt}(x(t)-y(t))+A(x(t)-y(t))+\partial\phi(x(t))-\partial\phi(y(t))\ni f(t,x(t)),$$

by multiplying by $x(t)-y(t)$ and using the monotonicity of $\partial\phi$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|x(t)-y(t)|^2+\omega_1||x(t)-y(t)||^2\\ \le \omega_2|x(t)-y(t)|^2+L||x(t)||\,|x(t)-y(t)|.\end{aligned}\tag{3.13}$$

By integrating on (3.13) over $(0,t)$ we have

(3.14)

$$\begin{aligned}&\frac{1}{2}|x(t)-y(t)|^2+\omega_1\int_0^t||x(s)-y(s)||^2ds\\ &\le \frac{1}{2}|g(x)|^2+\omega_2\int_0^t|x(s)-y(s)|^2ds+L\int_0^t||x(s)||\,|x(s)-y(s)|ds.\end{aligned}$$

By the procedure similar to (3.11) the inequality

$$\frac{1}{2}|x(t)-y(t)|+\omega_1\int_0^t ||x(s)-y(s)||^2 ds$$
$$\leq \frac{M^2e^{2\omega_2 t}}{2}||x||^2_{L^2(0,T_0;V)}$$
$$+L\int_0^t e^{2\omega_2(t-s)}||x(s)||(\int_0^s Le^{\omega_2(s-\tau)}||x(\tau)||d\tau + e^{\omega_2 s}|g(x)|)ds$$

implies

$$\frac{1}{2}|x(t)-y(t)|^2+\omega_1\int_0^t ||x(s)-y(s)||^2 ds$$
$$\leq \frac{M^2e^{2\omega_2 t}}{2}||x||^2_{L^2(0,T_0;V)}+\frac{L^2+2LM}{4\omega_2}(e^{2\omega_2 T_0}-1)||x||^2_{L^2(0,T_0;V)}.$$

Put

$$N := (\omega_1-\frac{1}{2}M^2e^{2\omega_2 T})-\frac{L^2+2LM}{4\omega_2}(e^{2\omega_2 T_0}-1).$$

Then it holds

$$||x-y||_{L^2(0,T_0;V)}\leq N^{1/2}||x||_{L^2(0,T_0;V)}$$

and hence, from (2.5) in Proposition 2.2, we have that

$$(3.15)\qquad \begin{aligned} &||x||_{L^2(0,T_0;V)}\\ &\leq \frac{1}{1-N^{1/2}}||y||_{L^2(0,T_0;V)}\\ &\leq C_4(1+||x_0||+||k||_{L^2(0,T_0;V^*)}) \end{aligned}$$

for some positive constant C_4. Furthermore, acting on both side of (NDE) by $x'(t)$ and by using

$$\frac{d}{dt}\phi(x(t))=(g(t),\frac{d}{dt}x(t)),\quad \text{a.e.}\quad 0<t,$$

for all $g(t)\in\partial\phi(x(t))$, it holds

(3.16)

$$\int_0^t |x'(t)|^2+\frac{1}{2}(Ax(t),x(t))+\phi(x(t))$$
$$\leq \frac{1}{2}(Ax_0,x_0)+\phi(x_0)+\int_0^t |f(s,x(s)+k(s)|\,|x'(s)|ds,$$

thus, noting that $x \in L^2(0, T_0; V)$, we obtain the norm estimate of x in $W^{1,2}(0, T_0; H)$ satisfying (3.3).

Since the condition (3.3) is independent of initial values, from the fact that a solution x of (NDE) belongs to

$$L^2(0, T_0; V) \cap C([0, T_0]; H) \cap W^{1,2}(0, T_0, H),$$

and $\phi(x(nT_0)) < \infty$, we can solve the equation in $[T_0, 2T_0]$ with the initial value $x(T_0)$ and obtain an analogous estimate to (3.11). By proceeding this process, the solution of (NDE) can be extended the internal $[0, T]$, i.e., for the initial $x(nT_0)$ in the interval $[nT_0, (n+1)T_0]$ where n is a natural number, as analogous estimate (3.11) holds for the solution to the time T in $[0, (n+1)T_0]$. Furthermore, the estimate (3.2) is easily obtained from (3.11) and (3.16). □

Finally, we also obtain the asymptotic property of solutions of (NDE) as follows.

THEOREM 3.2. *Let the assumptions* (H1), (H2) *and* (H3) *be satisfied and* $(x_0, k) \in H \times L^2(0, T; H)$. *Then the solution* x *of the equation* (NDE) *belongs to* $x \in L^2(0, T; V) \cap C([0, T]; H)$ *and the mapping*

$$H \times L^2(0, T; H) \ni (x_0, k) \mapsto x \in L^2(0, T; V) \cap C([0, T]; H)$$

is continuous.

Proof. If $(x_0, k) \in H \times L^2(0, T; H)$ then x belongs to $L^2(0, T; V) \cap C([0, T]; H)$ form Theorem 3.1. Let $(x_{0i}, k_i) \in H \times L^2(0, T; H)$ and x_i be the solution of (NDE) with (x_{0i}, k_i) in place of (x_0, k) for $i = 1, 2$. Multiplying on (NDE) by $x_1(t) - x_2(t)$, we have

(3.17)
$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|x_1(t) - x_2(t)|^2 + \omega_1||x_1(t) - x_2(t)||^2 \\ &\leq \omega_2|x_1(t) - x_2(t)|^2 + |f(t, x_1(t)) - f(t, x_2(t))|\,|x_1(t) - x_2(t)| \\ &\quad + |k_1(t) - k_2(t)|\,|x_1(t) - x_2(t)|. \end{aligned}$$

Put

$$H(t) = (L||x_1(t) - x_2(t)|| + |k_1(t) - k_2(t)|)|x_1(t) - x_2(t)|.$$

Then, by the similar way to (3.9) we have

$$\omega_2 \int_0^t |x_1(s) - x_2(s)|^2 ds \le \frac{1}{2}(e^{2\omega_2 t} - 1)(|x_{01} - x_{02}|^2 + |g(x_1) - g(x_2)|^2) + \int_0^t (e^{2\omega_2(t-s)} - 1)H(s)ds.$$

Combining this and (3.17) it holds that

(3.18)
$$\frac{1}{2}|x_1(t) - x_2(t)|^2 + \omega_1 \int_0^t ||x_1(s) - x_2(s)||^2 ds \le \frac{1}{2}e^{2\omega_2 t}(|x_{01} - x_{02}|^2 + |g(x_1) - g(x_2)|^2) + \int_0^t e^{2\omega_2(t-s)}H(s)ds.$$

By Lemma 3.1, we obtain

(3.19)
$$e^{-\omega_2 t}|x_1(t) - x_2(t)| \le |x_{01} - x_{02}| + |g(x_1) - g(x_2)| + \int_0^t e^{-\omega_2 s}(L||x_1(s) - x_2(s)|| + |k_1(s) - k_2(s)|)ds.$$

Thus,

$$H(t) \le (L||x_1(t) - x_2(t)|| + |k_1(t) - k_2(t)|)e^{\omega_2 t}(|x_{01} - x_{02}| + |g(x_1) - g(x_2)|) + \int_0^t e^{\omega_2(t-s)}(L||x_1(s) - x_2(s)|| + |k_1(s) - k_2(s)|)ds.$$

Let $T_1 < T$ be such that

$$\omega_1 - \frac{1}{2}M^2 e^{2\omega_2 T_1} - \frac{L^2}{2\omega_2}(e^{2\omega_2 T_1} - 1) - LMe^{\omega_2 T_1}(\frac{e^{2\omega_2 T_1} - 1}{2\omega_2})^{1/2} > 0.$$

From (3.18) and (3.19) it follows that

(3.20)

$$\begin{aligned}
&\frac{1}{2}|x_1(t)-x_2(t)|^2+\omega_1\int_0^t ||x_1(s)-x_2(s)||^2 ds \\
&\leq \frac{1}{2}e^{2\omega_2 t}|x_{01}-x_{02}|^2+\frac{1}{2}M^2e^{2\omega_2 t}||x_1-x_2||^2_{L^2(0,T_1;V)} \\
&\quad+\int_0^t e^{2\omega_2(t-s)}(L||x_1(s)-x_2(s)||+|k_1(s)-k_2(s)|) \\
&\qquad\cdot e^{\omega_2 s}(|x_{01}-x_{02}|+|g(x_1)-g(x_2)|)ds \\
&\quad+\int_0^t e^{2\omega_2(t-s)}(L||x_1(s)-x_2(s)||+|k_1(s)-k_2(s)|) \\
&\qquad\cdot\int_0^s e^{\omega_2(s-\tau)}(L||x_1(\tau)-x_2(\tau)||+|k_1(\tau)-k_2(\tau)|)d\tau ds.
\end{aligned}$$

The third term of the right of (3.20) is estimated as

$$\frac{(e^{2\omega_2 t}-1)}{4\omega_2}\int_0^t 2(L^2||x_1(s)-x_2(s)||^2+|k_1(s)-k_2(s)|^2)ds. \tag{3.21}$$

We can choose a constant $c>0$ such that

$$\begin{aligned}
&\omega_1-\frac{1}{2}M^2e^{2\omega_2T_1}-\frac{L^2}{2\omega_2}(e^{2\omega_2T_1}-1) \\
&-LMe^{\omega_2T_1}\left(\frac{e^{2\omega_2T_1}-1}{2\omega_2}\right)^{1/2}-cLe^{2\omega_2T_1}>0
\end{aligned}$$

and

$$|x_{01}-x_{02}|\,||x_1(s)-x_2(s)||\leq\frac{1}{2c}|x_{01}-x_{02}|^2+\frac{c}{2}||x_1(s)-x_2(s)||^2.$$

Thus, the second term of the right of (3.20) is estimated as

(3.22)

$$\begin{aligned}
&\frac{e^{2\omega_2T_1}-1}{4c\omega_2}|x_{01}-x_{02}|^2 \\
&+e^{2\omega_2T_1}\frac{c}{2}\int_0^{T_1}(L||x_1(s)-x_2(s)||^2+|k_1(s)-k_2(s)|^2)ds
\end{aligned}$$

$$+ LMe^{\omega_2 T_1}(\frac{e^{2\omega_2 T_1}-1}{2\omega_2})^{1/2}\int_0^{T_1}||x_1(s)-x_2(s)||^2 ds$$
$$+ Le^{2\omega_2 T_1}\frac{c}{2}\int_0^{T_1}||x_1(s)-x_2(s)||^2 ds$$
$$+ \int_0^{T_1}\frac{1}{L}e^{2\omega_2(t-s)}|k_1(s)-k_2(s)|^2 ds.$$

Hence, from (3.20) to (3.22) it follows that there exists a constant $C > 0$ such that

$$|x_1(T_1)-x_2(T_1)|^2 + \int_0^{T_1}||x_1(s)-x_2(s)||^2 ds \le C(|x_{01}-x_{02}|^2 + \int_0^{T_1}|k_1(s)-k_2(s)|^2 ds). \tag{3.23}$$

Suppose $(x_{0n}, k_n) \to (x_0, k)$ in $H \times L^2(0, T_1; V^*)$, and let x_n and x be the solutions (NDE) with (x_{0n}, k_n) and (x_0, k), respectively. Then, by virtue of (3.23), we see that $x_n \to x$ in $L^2(0, T_1, V) \cap C([0, T_1]; H)$. This implies that $x_n(T_1) \to x(T_1)$ in H. Therefore the same argument shows that $x_n \to x$ in

$$L^2(T_1, \min\{2T_1, T\}; V) \cap C([T_1, \min\{2T_1, T\}]; H).$$

Repeating this process, we conclude that $x_n \to x$ in $L^2(0, T; V) \cap C([0, T]; H)$. □

4. Example

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We take $V = W_0^{1,2}(\Omega)$, $H = L^2(\Omega)$. Let a_{ij} be a real valued function for each $i, j = 1, \ldots, n$. Assume that $a_{ij} = a_{ji}$ are continuous and bounded on $\overline{\Omega}$ and $\{a_{ij}(x)\}$ is positive definite uniformly in Ω, i.e., there exists a positive number δ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge \delta|\xi|^2 \tag{4.1}$$

for all $x \in \overline{\Omega}$ and all real vectors ξ. Let $b_i \in L^\infty(\Omega)$ and $c \in L^\infty(\Omega)$. Put $\beta_i = \sum_{j=1}^{n}\partial a_{ij}/\partial x_j + b_i$, then $\beta \in L^\infty(\Omega)$. For each $u, v \in V$, we put

$$(4.2) \qquad a(u,v)=\int_\Omega\{\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_i}\overline{\frac{\partial v}{\partial x_j}}+\sum_{i=1}^n\beta_i\frac{\partial u}{\partial x_i}\overline{v}+cu\overline{v}\}dx.$$

Since a_{ij} is real symmetric, by (4.1) the inequality

$$(4.3) \qquad \sum_{i,j=1}^n a_{ij}(x)\zeta_i\overline{\zeta_j}\ge\delta|\zeta|^2$$

holds for all complex vectors $\zeta=(\zeta_1,\dots,\zeta_n)$. On the other hand, by this hypothesis, there exists a certain number K such that $|\beta_i(x)|\le K$ and $|c(x)|\le K$ hold almost everywhere. Hence,

$$\begin{aligned}\operatorname{Re}a(u,v)&\ge\int_\Omega\delta\sum_{i=1}^n|\frac{\partial u}{\partial x_i}|^2dx-K\int_\Omega\sum_{i=1}^n|\frac{\partial u}{\partial x_i}||u|dx-K\int_\Omega|u|^2dx\\&\ge\delta\int_\Omega\sum_{i=1}^n|\frac{\partial u}{\partial x_i}|^2dx-K\int_\Omega\sum_{i=1}^n(\frac{\epsilon}{2}|\frac{\partial u}{\partial x_i}|^2+\frac{1}{2\epsilon}|u|^2)dx\\&\quad-K\int_\Omega|u|^2dx\\&=(\delta-\frac{\epsilon}{2}K)\sum_{i=1}^n\int_\Omega|\frac{\partial u}{\partial x_i}|^2dx-(\frac{nK}{2\epsilon}+K)\int_\Omega|u|^2dx.\end{aligned}$$

By choosing $\epsilon=\delta K^{-1}$, we obtain

$$\begin{aligned}\operatorname{Re}a(u,v)&\ge\frac{\delta}{2}\sum_{i=1}^n\int_\Omega|\frac{\partial u}{\partial x_i}|^2dx-(\frac{nK^2}{2\delta}+K)\int_\Omega|u|^2dx\\&=\frac{\delta}{2}||u||_V^2-(\frac{nK^2}{2\delta}+K+\frac{\delta}{2})||u||_H^2.\end{aligned}$$

Let $x\in L^2(0,T';V)$ for $T'<T$ and $\chi(s)$ be defining function of $[0,T']$. Theorem 3.1 can be applied for $g:[0,T]^p\times L^2(0,T';V)\to H$ defined by the formula

$$g(t_1,\dots,t_p,x)=\sum_{i=1}^p\frac{1}{l_i}\int_{t_i-l_i}^{t_i}h_i(t_i,s)\chi(s)\frac{(x(s)-x_0)}{1+|x(s)-x_0|}ds,$$

where $l_i > 0 (i = 1, \dots, p)$ are constants such that $0 < t_1 - l_1 < t_1 < t_2 < \cdots < t_p < T$ and $t_{i-1} \le t_i - l_i (i = 2, 3, \dots, p)$ and $h_i(t,s)$ is measurable function from $\mathcal{R} \times \mathcal{R}$ into $\mathcal{R}$ such that

$$\text{ess sup}\{\int_0^t |h_i(t,s)|^2 ds : t > 0\}^{\frac{1}{2}} = C < \infty, \quad i = 1, \dots, p.$$

Then by using the Hölder inequality it holds that

$$\begin{aligned} &|g(x) - g(y)| \\ &\le \sum_{i=1}^p \frac{1}{l_i} \int_{t_i - l_i}^{t_i} \frac{h_i(t_i, s)\chi(s)(1 + 2|y(s) - x_0|)|x(s) - y(s)|}{(1 + |x(s) - x_0|)(1 + |y(s) - x_0|)} ds \\ &\le \sum_{i=1}^p \frac{1}{l_i} \int_{t_i - l_i}^{t_i} 2h_i(t_i, s)\chi(s)|x(s) - y(s)| ds \\ &\le \sum_{i=1}^p \frac{2C\sqrt{l_i}}{l_i} ||x - y||_{L^2(0,T';V)}, \end{aligned}$$

and hence, the assumption (H2) holds. Thus, if

$$\delta - \{\sum_{i=1}^p \frac{2C\sqrt{l_i}}{l_i} exp((\frac{nK^2}{2\delta} + K + \frac{\delta}{2})T)\}^2 > 0,$$

the equation (NDE) has a unique solution in

$$L^2(0,T; W_0^{1,2}(\Omega)) \cap C([0,T]; L^2(\Omega)) \cap W^{1,2}(0,T; L^2(\Omega)).$$

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