J. Korean Math. Soc. 41 (2004), No. 4, pp. 681–715

FINITE ELEMENT APPROXIMATION AND COMPUTATIONS OF OPTIMAL DIRICHLET BOUNDARY CONTROL PROBLEMS FOR THE BOUSSINESQ EQUATIONS

HYUNG-CHUN LEE AND SOOHYUN KIM

ABSTRACT. Mathematical formulation and numerical solutions of an optimal Dirichlet boundary control problem for the Boussinesq equations are considered. The solution of the optimal control problem is obtained by adjusting of the temperature on the boundary. We analyze finite element approximations. A gradient method for the solution of the discrete optimal control problem is presented and analyzed. Finally, the results of some computational experiments are presented.

1. Introduction

Recently there has been substantial interest in control of fluid flows by virtue of its applications in flow separation, combustion, fluid structure interaction, design of novel submarine propulsion devices and modeling of nuclear reactors as industrial engineering progress. In consequence, many mathematicians and scientists have studied in mathematical analyses and computations of optimal control problems for fluid flows; see [2, 3, 5, 9, 11, 13, 14, 17, 25, 26, 29, 31, 32, 34, 35, 36] and references therein. In [31, 32], vorticity minimization problem was considered and optimality system was driven for the stationary Boussinesq equations with Neumann boundary control and in [29, 36] with Robin type boundary control. Velocity tracking problem was analyzed and given the numerical results in [23, 24, 25, 26, 27, 28] for time dependent Navier-Stokes equations and just presented the algorithm and computations in

Received March 31, 2003.
2000 Mathematics Subject Classification: Primary 35B40, 35B37, 35Q30, 65M60.
Key words and phrases: optimal control problem, Boussinesq equations.
This work was supported by KOSEF R01-2000-000-00008-0 and in part by the Ajou University under research facilities support program in the year 2003.

[36] for time dependent Boussinesq equations. In [34, 35], a tracking problems for the time-dependent 2D Boussinesq equations was studied with distributed controls. In [17, 22], controlling boundary temperature problems for stationary flow with Dirichlet and Neumann boundary temperature controls were considered in which a weakly temperature-fluid coupled system was considered. In [30], a Dirichlet boundary control problems for the Boussinesq equations was analyzed and a computational algorithm was presented.

In this article, we will address optimal control problems for steady incompressible flows whose motions are governed by the Boussinesq equations. We complete the analytical and numerical theory and get new computational results for a Dirichlet boundary optimal control problem of the Boussinesq equations with mixed boundary conditions.

We now write the 2-D non-dimensional stationary Boussinesq equations as follows:

$$(1.1) \qquad \begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + T\mathbf{g} + \mathbf{f} \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \\ -\kappa\Delta T + (\mathbf{u}\cdot\nabla)T = Q \quad \text{in } \Omega \end{cases}$$

with mixed boundary conditions

$$(1.2) \quad \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega, \quad T = h \text{ on } \Gamma_l, \quad T = g \text{ on } \Gamma_r, \quad \frac{\partial T}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma_N,$$

where Ω is the regular bounded open set in $\mathbb{R}^2$, with $\partial\Omega \in C^2$. In (1.2), $\Gamma_D = \partial\Omega \setminus \Gamma_N$, $\Gamma_l = \Gamma_D \setminus \Gamma_r$ and Γ_N is a regular nonempty open subset of $\partial\Omega$. In (1.1)-(1.2), $\mathbf{u}$, p and T denote the velocity, pressure and temperature fields, respectively, $\mathbf{f}$ a given body force, h a given function, Q a given heat source, and g a Dirichlet boundary control. The vector $\mathbf{g}$ is a unit vector in the direction of gravitational acceleration and $\kappa > 0$ the thermal conductivity parameter. In this paper we consider, for simplicity, the case of constant κ. The vector $\mathbf{n}$ denotes the outward unit normal to Ω and $\nu > 0$ denotes the kinematic viscosity.

The three functionals that we wish to minimize are given by

$$(1.3) \qquad \mathcal{J}_1(\mathbf{u}, T, p, g) = \frac{1}{2}\int_\Omega |\nabla\times\mathbf{u}|^2 \, d\mathbf{x} + \frac{\delta}{2}\| g \|_{H^1(\Gamma_r)},$$

$$(1.4) \qquad \mathcal{J}_2(\mathbf{u}, T, p, g) = \frac{1}{2}\int_{\Gamma_\sigma} |T - T_d|^2 \, d\mathbf{x} + \frac{\delta}{2}\| g \|_{H^1(\Gamma_r)},$$

and

$$\mathcal{J}_3(\mathbf{u}, T, p, g) = \frac{1}{2} \int_{\Omega} |\mathbf{u} - \mathbf{U}|^2 d\mathbf{x} + \frac{\delta}{2} \| g \|_{H^1(\Gamma_r)}. \tag{1.5}$$

Here Γ_σ is the portion of Ω where we want to match the temperature with T_d. The optimal control problems we consider are to seek state variables $(\mathbf{u}, p, T)$, and control g such that the functional (1.3), (1.4) or (1.5) is minimized subject to (1.1)-(1.2) where T_d and $\mathbf{U}$ are some desired temperature on the boundary Γ_σ and velocity distributions. The functional (1.3) measures the vorticity of the flow. The control of vorticity has significant applications in science and engineering such as control of turbulence and control of crystal growth process. The functional (1.4) effectively measures the difference between the temperature field T and a prescribed field T_d. By minimizing the functional (1.4) we can avoid any hot spots that may destroy the containers. This optimization problem may be applied to engine components, post-combustion chambers, and nuclear reactor piping. The first term in the functional (1.5) measures the L^2-distance between the candidate flow and the desired flow. Thus, the physical objective of this minimization problem is to match a desired flow field (in the L^2-sense) by adjusting the boundary temperature g. The real goal of these optimization problems is to minimize the first terms appearing in the definition of the functionals (1.3)-(1.5). The other terms in the cost functionals (1.3)-(1.5) are added to limit the cost of controls. The positive penalty parameter δ can be used to change the relative importance of the four terms appearing in the definitions of the functionals. The plan of the article is as follows. In Section 1.1 of this section we introduce the notations that will be used throughout the Section 2, 3, and 4. In §2, we analyze finite element approximations. In §3, a gradient method for the solution of the discrete optimal control problem is presented and analyzed. Finally in §4, the results of some computational experiments are presented.

1.1. Notations

We introduce some function spaces and their norms, along with some related notations used in subsequent sections; for details see [1]. Let Ω be a bounded domain of $\mathbb{R}^2$ with a C^2 boundary $\partial\Omega$. Let $L^2(\Omega)$ be the space of real-valued square integrable functions defined on Ω, and let $\| \cdot \|_{L^2(\Omega)}$ be the norm in this space. For any nonnegative integer m, we define the Sobolev space $H^m(\Omega)$ by

$$H^m(\Omega) = \left\{ u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \quad \text{for} \quad 0 \leq |\alpha| \leq m \right\},$$

where $D^\alpha u$ denotes the weak (or distributional) partial derivative and α is a multi-index, $|\alpha| = \sum_i \alpha_i$. Note that $H^0(\Omega) = L^2(\Omega)$. We equip $H^m(\Omega)$ with the norm

$$\|\mathbf{u}\|_m^2 = \sum_{|\alpha|\leq m} \|D^\alpha \mathbf{u}\|_{L^2(\Omega)}^2.$$

For vector valued functions, we define the Sobolev space $\mathbf{H}^m(\Omega)$ (in all cases, boldface indicates vector-valued) by

$$\mathbf{H}^m(\Omega) = \{\mathbf{u} = (u_1, u_2) | u_i \in H^m(\Omega), i = 1, 2\},$$

and its associated norm $\|\cdot\|_{\mathbf{H}^m(\Omega)}$ is given by

$$\|\mathbf{u}\|_{\mathbf{H}^m(\Omega)}^2 = \sum_{i=1}^{2} \|u_i\|_{H^m(\Omega)}^2.$$

We also define particular subspaces

$$L_0^2(\Omega) = \left\{ f \in L^2(\Omega) : \int_\Omega f \, d\mathbf{x} = 0 \right\},$$
$$\mathbf{H}_0^1(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathbf{u} = \mathbf{0} \text{ on } \Gamma\},$$

and

$$H_D^1 = \{S \in H^1(\Omega) : S = 0 \text{ on } \Gamma_D\}.$$

We make use of the well-known space $\mathbf{L}^4(\Omega)$ equipped with the norm $\|\cdot\|_{\mathbf{L}^4(\Omega)}$.

We also define the solenoidal spaces

$$\mathbf{V} = \{\mathbf{u} \in \mathbf{H}_0^1(\Omega) : \nabla \cdot \mathbf{u} = 0\}.$$

If Ω is bounded and has a C^2 boundary (these are the kinds of domains under consideration here), Sobolev's embedding theorem yields that $H^1(\Omega) \hookrightarrow\hookrightarrow L^4(\Omega)$, where $\hookrightarrow\hookrightarrow$ denotes compact embedding;i.e., a constant C exists such that

$$\|u\|_{L^4(\Omega)} \leq C\|u\|_{H^1(\Omega)}.$$

Obviously, a similar result holds for the spaces $\mathbf{H}^1(\Omega)$ and $\mathbf{L}^4(\Omega)$.

Finally, we restrict our computational domain Ω such that $\Gamma_l \cap \Gamma_r = \emptyset$ or $h = g$ at $\Gamma_l \cap \Gamma_r$ if $\Gamma_l \cap \Gamma_r \neq \emptyset$

1.2. Weak formulation and optimality system

We introduce the following bilinear and trilinear forms:

$$a_0(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, d\mathbf{x} \qquad \forall \mathbf{u},\mathbf{v} \in \mathbf{H}^1(\Omega),$$

$$a_1(T,S) = \int_\Omega \kappa\nabla T \cdot \nabla S d\mathbf{x} \qquad \forall T,S \in H^1(\Omega),$$

$$b_0(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\, d\mathbf{x} \qquad \forall \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{H}^1(\Omega),$$

$$b_1(\mathbf{u},T,S) = \int_\Omega (\mathbf{u}\cdot\nabla T) S\, d\mathbf{x} \qquad \forall \mathbf{u} \in \mathbf{H}^1(\Omega), \forall T,S \in H^1(\Omega),$$

$$c(\mathbf{u},q) = -\int_\Omega q\nabla\cdot\mathbf{u}\, d\mathbf{x} \qquad \forall \mathbf{u} \in \mathbf{H}^1(\Omega), \forall q \in L^2(\Omega).$$

We first note that the bilinear forms $a_0(\cdot,\cdot)$ and $a_1(\cdot,\cdot)$ are clearly continuous, i.e.,

$$|a_0(\mathbf{u},\mathbf{v})| \leq C\|\,\mathbf{u}\,\|_{\mathbf{H}^1(\Omega)}\|\,\mathbf{v}\,\|_{\mathbf{H}^1(\Omega)}, \tag{1.6}$$

and

$$|a_1(T,S)| \leq C\|\,T\,\|_{H^1(\Omega)}\|\,S\,\|_{H^1(\Omega)}, \tag{1.7}$$

$$|c(\mathbf{u},q)| \leq C\|\,\mathbf{u}\,\|_{\mathbf{H}^1(\Omega)}\|\,q\,\|_{L^2(\Omega)}. \tag{1.8}$$

We have the coercivity relations associated with $a_0(\cdot,\cdot)$ and $a_1(\cdot,\cdot)$

$$a_0(\mathbf{u},\mathbf{u}) = \|\,\mathbf{u}\,\|^2_{\mathbf{L}^2(\Omega)} \geq C_1\|\,\mathbf{u}\,\|^2_{\mathbf{H}^1(\Omega)} \qquad \forall \mathbf{u} \in \mathbf{H}^1_0(\Omega) \tag{1.9}$$

and

$$a_1(T,T) = \|\,T\,\|^2_{L^2(\Omega)} \geq C_2\|\,T\,\|^2_{H^1(\Omega)} \qquad \forall T \in H^1_D(\Omega) \tag{1.10}$$

which are direct consequences of Poincaré inequality.

LEMMA 1.1. *For every* $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{H}^1(\Omega)$ *and every* $T,S \in H^1(\Omega)$ *there are constants* C_1 *and* C_2 *such that*

$$|b_0(\mathbf{u},\mathbf{v},\mathbf{w})| \leq C_1\|\,\mathbf{u}\,\|_{\mathbf{H}^1(\Omega)}\|\,\mathbf{v}\,\|_{\mathbf{H}^1(\Omega)}\|\,\mathbf{w}\,\|_{\mathbf{H}^1(\Omega)}, \tag{1.11}$$

$$b_0(\mathbf{u},\mathbf{v},\mathbf{v}) = 0 \quad \textit{if } \mathbf{u} \in \mathbf{V}, \tag{1.12}$$

$$|b_1(\mathbf{u},T,S)| \leq C_2\|\,\mathbf{u}\,\|_{\mathbf{H}^1(\Omega)}\|\,T\,\|_{H^1(\Omega)}\|\,S\,\|_{H^1(\Omega)} \quad \forall \mathbf{u} \in \mathbf{V} \tag{1.13}$$

and

$$b_1(\mathbf{u},T,T) = 0 \quad \textit{if } \mathbf{u} \in \mathbf{V}. \tag{1.14}$$

Proof. These follow from the Cauachy-Schwarz inequality, Hölder's inequality, and various embedding results, in particular the continuous embeddings of $\mathbf{H}^1$ into $\mathbf{L}^4$ and $\mathbf{L}^2$ and H^1 into L^4 and L^2, respectively. □

The weak form of the constraint equations (1.1)-(1.2) is then given as follows: find $(\mathbf{u}, p, T, t) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H^1(\Omega) \times H^{-\frac{1}{2}}(\Gamma_r)$ such that

$$\nu a_0(\mathbf{u}, \mathbf{v}) + b_0(\mathbf{u}, \mathbf{u}, \mathbf{v}) + c(\mathbf{v}, p) = (T\mathbf{g}, \mathbf{v}) + \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{1.15}$$

$$c(\mathbf{u}, q) = 0 \quad \forall q \in L_0^2(\Omega), \tag{1.16}$$

$$a_1(T, S) + b_1(\mathbf{u}, T, S) - (t, S)_{\Gamma_r} = \langle Q, S \rangle \quad \forall S \in H_l^1(\Omega), \tag{1.17}$$

$$(T, R)_{\Gamma_r} = (g, R)_{\Gamma_r} \qquad \forall R \in H^{-\frac{1}{2}}(\Gamma_r), \tag{1.18}$$

and

$$T = h \qquad \text{on } \Gamma_l \tag{1.19}$$

where $t = \nabla T \cdot \mathbf{n}|_{\Gamma_r}$.

The analysis for Dirichlet boundary optimal control problems was studied in [30]. We describe the optimal control problem involving the functional (1.5) and state the optimality system.

We look for a $(\mathbf{u}, p, T, g) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H^1(\Omega) \times \mathcal{V}$ such that the cost functional

$$\mathcal{J}(\mathbf{u}, p, T, g) = \frac{1}{2} \int_\Omega |\mathbf{u} - \mathbf{U}|^2 d\mathbf{x} + \frac{\delta}{2} \| g \|_{H^1(\Gamma_r)} \tag{1.20}$$

is minimized subject to the constraints

$$\nu a_0(\mathbf{u}, \mathbf{v}) + b_0(\mathbf{u}, \mathbf{u}, \mathbf{v}) + c(\mathbf{v}, p) = (T\mathbf{g}, \mathbf{v}) + \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{1.21}$$

$$c(\mathbf{u}, q) = 0 \quad \forall q \in L_0^2(\Omega), \tag{1.22}$$

$$a_1(T, S) + b_1(\mathbf{u}, T, S) - (t, S)_{\Gamma_r} = \langle Q, S \rangle \quad \forall S \in H_l^1(\Omega), \tag{1.23}$$

$$T|_{\Gamma_l} = h, \quad (T, R)_{\Gamma_r} = (g, R)_{\Gamma_r} \quad \forall R \in H^{-\frac{1}{2}}(\Gamma_r) \tag{1.24}$$

where $t = \nabla T \cdot \mathbf{n}|_{\Gamma_r}$, and $\mathcal{V}$ is a nonempty, closed, and convex subset of $H^1(\Gamma_r)$.

The optimality system is as follows : find $(\mathbf{u}, p, T, \boldsymbol{\xi}, \psi, \theta) \in \mathbf{H}_0^1(\Omega) \times L_0^1(\Omega) \times H^1(\Omega) \times \mathbf{H}_0^1(\Omega) \times L_0^1(\Omega) \times H_D^1(\Omega)$ such that

$$(1.25)\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = T\mathbf{g} + \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega, \\ -\kappa\Delta T + (\mathbf{u}\cdot\nabla)T = Q & \text{in } \Omega, \\ T = h \text{ on } \Gamma_l, \quad T|_{\Gamma_r} = g, \quad \dfrac{\partial T}{\partial \mathbf{n}}\Big|_{\Gamma_N} = 0, & \end{cases}$$

$$(1.26)\begin{cases} -\nu\Delta\boldsymbol{\xi} - (\mathbf{u}\cdot\nabla)\boldsymbol{\xi} + \boldsymbol{\xi}\cdot(\nabla\mathbf{u})^T + \theta\nabla T + \nabla\phi = \mathbf{u} - \mathbf{U} & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{\xi} = 0 & \text{in } \Omega, \\ -\kappa\Delta\theta - (\mathbf{u}\cdot\nabla)\theta = \boldsymbol{\xi}\cdot\mathbf{g} & \text{in } \Omega, \\ \dfrac{\partial\theta}{\partial\mathbf{n}} = 0 & \text{on } \Gamma_N, \\ -\Delta_s g + g = -\dfrac{1}{\delta}\nabla\theta\cdot\mathbf{n}|_{\Gamma_r} & \text{on } \Gamma_r. \end{cases}$$

1.3. Quotation of some results concerning the approximation of a class of nonlinear problems

Here for the sake of completeness, we will state the relevant results specialized to our needs. The nonlinear problems considered in [7] and [15] are of the type

$$(1.27) \qquad F(\lambda, \psi) \equiv \psi + AG(\lambda, \psi) = 0$$

where $A \in \mathcal{L}(Y; X)$, G is a C^2 mapping from $\Lambda \times X$ into Y, where X and Y are Banach spaces and Λ is a compact interval of $\mathbb{R}$. We say that $\{(\lambda, \psi(\lambda)) : \lambda \in \Lambda\}$ is a branch of solutions of (1.27) if $\lambda \to \psi(\lambda)$ is a continuous function from Λ into X such that $F(\lambda, \psi(\lambda)) = 0$. The branch is called a *nonsingular branch* if we also have that $D_\psi F(\lambda, \psi(\lambda))$ is an isomorphism from X into X for all $\lambda \in \Lambda$. Here, D_ψ denotes the Fréchet derivative with respect to ψ. Approximations are defined by introducing a subspace $X^h \subset X$ and an approximating operator $A^h \in \mathcal{L}(Y; X^h)$. Then we seek $\psi^h \in X^h$ such that

$$(1.28) \qquad F^h(\lambda, \psi^h) \equiv \psi^h + A^h G(\lambda, \psi^h) = 0,$$

We will assume that there exists another Banach space Z, contained in Y, with continuous imbedding such that

$$(1.29) \qquad D_\psi G(\lambda, \psi) \in \mathcal{L}(X; Z), \qquad \forall\lambda \in \Lambda, \quad \forall\psi \in X.$$

Concerning the operator A^h, we assume the approximation properties

(1.30) $$\lim_{h\to 0} \| (A^h - A)y \|_X = 0 \qquad \forall y \in Y$$

and

(1.31) $$\lim_{h\to 0} \| A^h - A \|_{\mathcal{L}(Z;X)} = 0.$$

Note that (1.29) and (1.31) imply that the operator $D_\psi G(\lambda, \psi) \in \mathcal{L}(X, X)$ is compact. Moreover, (1.31) follows from (1.30) whenever the imbedding $Z \subset Y$ is compact.

Now we can state the first result of [7] and [15] that used in the sequel.

THEOREM 1.2. *Let X and Y be Banach spaces and Λ a compact subset of $\mathbb{R}$. Assume that G is a C^2 mapping from $\Lambda \times X$ into Y and that D^2G is bounded on all sets of $\Lambda \times X$. (D^2G represents second Fréchet derivative of G). Assume that (1.29)-(1.31) hold and $\{(\lambda, \psi(\lambda)) : \lambda \in \Lambda\}$ is a branch of nonsingular solutions of (1.27). Then, there exists a neighborhood $\mathcal{O}$ of the origin in X and for $h \le h_0$ small enough, a unique C^2 function $\lambda \in \Lambda \to \psi^h(\lambda) \in X^h$ such that $\{(\lambda, \psi^h(\lambda)) : \lambda \in \Lambda\}$ is a branch of nonsingular solutions of (1.28) and $\psi^h(\lambda) - \psi(\lambda) \in \mathcal{O}$ for all λ. Moreover, there exists a constant $C > 0$, independent of h and λ, such that*

(1.32) $$\| \psi^h(\lambda) - \psi(\lambda) \|_X \le C \| (A^h - A) G(\lambda, \psi(\lambda)) \|_X, \qquad \forall \lambda \in \Lambda.$$

For the second result, we have to introduce two other Banach spaces H and W, such that $W \subset X \subset H$, with continuous imbeddings and assume that

(1.33) $$\begin{gathered}\text{for all } w \subset W \text{ the operator } D_\psi G(\lambda, w) \text{ may be}\\ \text{extended as a linear operator of } \mathcal{L}(H;Y),\end{gathered}$$

and the mapping $w \to D_\psi G(\lambda, w)$ is continuous from W onto $\mathcal{L}(H;Y)$.

We also suppose that

(1.34) $$\lim_{h\to 0} \| A^h - A \|_{\mathcal{L}(Y;H)} = 0.$$

Then we may state the following additional result.

THEOREM 1.3. *Assume the hypotheses of Theorem 1.2.* and also *assume that* (1.33) *and* (1.34) *hold. Assume in addition that*

(1.35) $$\begin{gathered}\text{for each } \lambda \in \Lambda, \psi(\lambda) \in W \text{ and the function}\\ \lambda \to \psi(\lambda) \text{ is continuous from } \Lambda \text{ into } W\end{gathered}$$

and

(1.36) *for each $\lambda \in \Lambda$, $D_\psi F(\lambda, \psi(\lambda))$ is an isomorphism of H.*

Then, for $h \leq h_1$, sufficiently small, there exists a constant C, independent of h and λ, such that

$$(1.37)\quad \| \psi^h(\lambda) - \psi(\lambda) \|_H \leq C \| (A^h - A) G(\lambda, \psi(\lambda)) \|_H + \| \psi^h(\lambda) - \psi(\lambda) \|_X^2, \qquad \forall \lambda \in \Lambda.$$

2. Finite element approximation and error estimates

In this section we investigate a finite element discretization of the optimality system and an estimation of the approximation error. First we choose a family of the finite dimensional subspaces $\mathbf{V}^h \subset \mathbf{H}^1(\Omega), S^h \in L^2(\Omega)$. We let $\mathbf{V}_0^h = \mathbf{V}^h \cap \mathbf{H}_0^1(\Omega)$, $V_l^h = V^h \cap H_l^1(\Omega)$, and $S_0^h = S^h \cap L_0^2(\Omega)$. Let $O^h = V^h|_{\Gamma_r}$. For all choices of conforming finite element spaces, we then have and $O^h \in H^{-\frac{1}{2}}(\Gamma_r)$. Next, let $N^h = V^h|_{\Gamma_r}$. Again, for all choices of conforming finite element spaces V^h we have that $N^h \subset H^1(\Gamma_r)$. These families are parameterized by a parameter h that tends to zero; commonly, h is chosen to be some measure of the grid size. These finite-dimensional function spaces are defined on an approximate domain Ω_h. For simplicity we will state our results in this section by assuming $\Omega_h = \Omega$. We assume that these finite element spaces satisfy the following approximation properties: there exist an integer k and a constant C, independent of h,

$$(2.1)\quad \inf_{\mathbf{v}^h \in \mathbf{V}^h} \| \mathbf{v} - \mathbf{v}^h \|_{\mathbf{H}^1(\Omega)} \leq C h^m \| \mathbf{v} \|_{\mathbf{H}^{m+1}(\Omega)} \ \forall \mathbf{v} \in \mathbf{H}^{m+1}(\Omega),\ 1 \leq m \leq k,$$

$$(2.2)\quad \inf_{q^h \in S^h} \| q - q^h \|_{L^2(\Omega)} \leq C h^m \| q \|_{H^m(\Omega)} \ \forall q \in H^m(\Omega),\ 1 \leq m \leq k,$$

$$(2.3)\quad \inf_{T^h \in V^h} \| T - T^h \|_{H^1(\Omega)} \leq C h^m \| T \|_{H^{m+1}(\Omega)} \ \forall T \in H^{m+1}(\Omega),\ 1 \leq m \leq k,$$

$$(2.4)\quad \inf_{t^h \in O^h} \| t - t^h \|_{H^{-\frac{1}{2}}(\Gamma_r)} \leq C h^m \| t \|_{H^{m-\frac{1}{2}}(\Gamma_r)} \quad \forall t \in H^{m-\frac{1}{2}}(\Gamma_r),\ 1 \leq m \leq k,$$

$$(2.5)\quad \inf_{g^h \in N^h} \| g - g^h \|_{s,\Gamma_r} \leq C h^{m-s+\frac{1}{2}} \| g \|_{H^{m+\frac{1}{2}}(\Gamma_r)} \quad \forall g \in H^1(\Gamma_r), 1 \leq m \leq k,\ 0 \leq s \leq 1.$$

Here we may choose any pair of subspaces $\mathbf{V}^h$, V^h and S^h such that $\mathbf{V}_0^h$ V_l^h and S_0^h can be used for finding finite element approximations of solutions of Boussinesq equations. Thus, we make the following standard assumptions, which are exactly those employed in well-known finite element methods for the Navier-Stokes equations. Next, we assume the *inf-sup* condition: there exists a constant C, independent of h, such that

$$\inf_{0\neq q^h\in S_0^h}\sup_{0\neq \mathbf{v}^h\in \mathbf{V}_0^h}\frac{c(\mathbf{v}^h,q^h)}{\|\mathbf{v}^h\|_{\mathbf{H}^1(\Omega)}\|q^h\|_{L^2(\Omega)}}\geq C. \tag{2.6}$$

This condition assures the stability of finite element discretizations of the Navier-Stokes equations and also that of the optimality system (1.25)-(1.26). The reference [9], [15]-[25], [27], [28] may also be consulted for a catalogue of finite element subspaces that meet the requirements of the above approximation properties and the inf-sup condition. Once the approximating subspaces have been chosen, we seek $(\mathbf{u}^h, p^h, T^h, t^h, \boldsymbol{\xi}^h, \phi^h, \theta^h, \tau^h, g^h) \in \mathbf{V}_0^h\times S_0^h\times V^h\times O^h\times \mathbf{V}_0^h\times S_0^h\times V^h\times O^h\times N^h$ by solving the discrete optimality system of equations.

(2.7)

$$\begin{cases}\nu a_0(\mathbf{u}^h,\mathbf{v}^h)+b_0(\mathbf{u}^h,\mathbf{u}^h,\mathbf{v}^h)+c(\mathbf{v}^h,p)-(T^h\mathbf{g},\mathbf{v}^h)\\ \quad=\langle \mathbf{f},\mathbf{v}^h\rangle\ \forall \mathbf{v}^h\in\mathbf{V}_0^h,\\ c(\mathbf{u}^h,q^h)=0\quad \forall q^h\in S_0^h,\\ a_1(T^h,S^h)+b_1(\mathbf{u}^h,T^h,S^h)-(t^h,S^h)_{\Gamma_r}=\langle Q,S^h\rangle\quad \forall S^h\in V_l^h,\\ (T^h,\psi^h)_{\Gamma_D}-(g^h,\psi^h)_{\Gamma_r}=(h,\psi^h)_{\Gamma_l}\quad \forall\psi^h\in O^h,\end{cases}$$

(2.8)

$$\begin{cases}\nu a_0(\mathbf{w}^h,\boldsymbol{\xi}^h)+b_0(\mathbf{w}^h,\mathbf{u}^h,\boldsymbol{\xi}^h)+b_0(\mathbf{u}^h,\mathbf{w}^h,\boldsymbol{\xi}^h)+c(\mathbf{w}^h,\phi^h)\\ \quad=(\mathbf{u}^h-\mathbf{U},\mathbf{w}^h)-b_1(\mathbf{w}^h,T^h,\theta^h)\quad \forall\mathbf{w}^h\in\mathbf{V}^h,\\ c(\boldsymbol{\xi}^h,r^h)=0\quad \forall r^h\in S_0^h,\\ a_1(\theta^h,\varphi^h)+b_1(\mathbf{u}^h,\varphi^h,\theta^h)+(\tau^h,\varphi^h)_{\Gamma_r}=(\varphi^h\mathbf{g},\boldsymbol{\xi}^h)\quad\forall\varphi^h\in V_l^h,\\ (\theta^h,\chi^h)_{\Gamma_D}=0\quad\forall\chi^h\in O^h,\\ (\nabla_s g^h,\nabla_s z^h)_{\Gamma_r}+(g^h,z^h)_{\Gamma_r}=\frac{1}{\delta}(\tau^h,z^h)_{\Gamma_r}\quad\forall z^h\in N^h.\end{cases}$$

We concern ourselves with questions related to the accuracy of finite element approximations in this section. The error estimate makes use of the results of [7] and [15] concerning the approximation of a class of nonlinear problems.

We begin by recasting the optimality system (1.25)-(1.26) and its discretization (2.7)-(2.8) into a form that fits into the framework. Let $\lambda = \frac{1}{\nu}$; thus λ is the Reynolds number. Let

$$X = \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H^1(\Omega) \times H^{-\frac{1}{2}}(\Gamma_r) \times \mathbf{H}_0^1(\Omega)$$
$$\times L_0^2(\Omega) \times H^1(\Omega) \times H^{-\frac{1}{2}}(\Gamma_D) \times H^1(\Gamma_r),$$
$$Y = \mathbf{H}^{-1} \times H^{-1} \times \mathbf{H}^{-1} \times H^{-1} \times H^{-\frac{1}{2}}(\Gamma_r),$$
$$Z = \mathbf{L}^{\frac{3}{2}}(\Omega) \times L^2(\Omega) \times \mathbf{L}^{\frac{3}{2}}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_r),$$
$$X^h = \mathbf{V}_0^h \times S_0^h \times V^h \times O^h \times \mathbf{V}_0^h \times S_0^h \times V^h \times O^h \times N^h.$$

Note that $Z \subset Y$ with a compact imbedding. Let the operator $A \in \mathcal{L}(Y : X)$ be defined as the following:

$$A(\Xi, Q, \eta, P, \Theta) = (\mathbf{u}, p, T, t, \boldsymbol{\xi}, \phi, \theta, \tau, g) \in X \quad \text{if and only if}$$

$$\text{(2.9)} \qquad \begin{cases} \nu a_0(\mathbf{u}, \mathbf{v}) + c(\mathbf{v}, p) = \langle \Xi, \mathbf{v} \rangle & \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ c(\mathbf{u}, q) = 0 & \forall q \in L_0^2(\Omega), \end{cases}$$

$$\text{(2.10)} \qquad \begin{cases} a_1(T, S) - (t, S)_{\Gamma_r} = \langle Q, S \rangle & \forall S \in H^1(\Omega), \\ (T, \psi)_{\Gamma_D} = (\Theta, \psi)_{\Gamma_D} & \forall \psi \in H^{-\frac{1}{2}}(\Gamma_D), \end{cases}$$

$$\text{(2.11)} \qquad \begin{cases} \nu a_0(\mathbf{w}, \boldsymbol{\xi}) + c(\mathbf{w}, \phi) = \langle \eta, \mathbf{w} \rangle & \forall \mathbf{w} \in \mathbf{H}_0^1(\Omega), \\ c(\boldsymbol{\xi}, r) = 0 & \forall r \in L_0^2(\Omega), \end{cases}$$

$$\text{(2.12)} \qquad \begin{cases} a_1(\theta, \varphi) + (\tau, \varphi)_{\Gamma_r} = \langle P, \varphi \rangle & \forall \varphi \in H^1(\Omega), \\ (\theta, \chi)_{\Gamma_r} = 0 & \forall \chi \in L_0^2(\Gamma_r), \end{cases}$$

$$\text{(2.13)} \qquad (\nabla_s g, \nabla_s z)_{\Gamma_r} + (g, z)_{\Gamma_r} = \frac{1}{\delta}(\tau, z)_{\Gamma_r} \qquad \forall z \in H^1(\Gamma_r).$$

Note that the system is weakly coupled. Analogously, the operator $A^h \in \mathcal{L}(Y : X^h)$ be defined as the following:
$A^h(\Xi, Q, \eta, P, \Theta) = (\mathbf{u}^h, p^h, T^h, t^h, \boldsymbol{\xi}^h, \phi^h, \theta^h, \tau^h, g^h) \in X^h$ if and only if

$$\text{(2.14)} \qquad \begin{cases} \nu a_0(\mathbf{u}^h, \mathbf{v}^h) + c(\mathbf{v}^h, p^h) = \langle \Xi, \mathbf{v}^h \rangle & \forall \mathbf{v}^h \in \mathbf{V}_0^h \\ c(\mathbf{u}^h, q^h) = 0 & \forall q^h \in S_0^h \end{cases}$$

$$\text{(2.15)} \qquad \begin{cases} a_1(T^h, S^h) - (t^h, S^h)_{\Gamma_r} = \langle Q, S^h \rangle & \forall S^h \in V^h \\ (T^h, \psi^h)_{\Gamma_D} = (\Theta, \psi^h)_{\Gamma_D} & \forall \psi^h \in O^h \end{cases}$$

$$(2.16)\qquad \begin{cases} \nu a_0(\mathbf{w}^h, \boldsymbol{\xi}^h) + c(\mathbf{w}^h, \phi^h) = \langle \eta, \mathbf{w}^h \rangle & \forall \mathbf{v}^h \in \mathbf{V}_0^h \\ c(\boldsymbol{\xi}^h, r^h) = 0 \quad \forall q^h \in S_0^h \end{cases}$$

$$(2.17)\qquad \begin{cases} a_1(\theta^h, \varphi^h) + (\tau^h, \varphi^h)_{\Gamma_r} = \langle P, \varphi^h \rangle & \forall \varphi^h \in V^h \\ (\theta^h, \chi^h)_{\Gamma_D} = 0 \quad \forall \chi^h \in O^h \end{cases}$$

$$(2.18)\qquad (\nabla_s g^h, \nabla_s z^h)_{\Gamma_r} + (g^h, z^h)_{\Gamma_r} = \frac{1}{\delta}(\tau^h, z^h)_{\Gamma_r} \qquad \forall z^h \in N^h.$$

This system is weakly coupled in the same sense as the system (2.9)-(2.13).

Let Λ denote a compact subset of $\mathbb{R}$. Next we define the *nonlinear* mapping $G : \Lambda \times X \to Y$ as follows: $G(\lambda, (\mathbf{u}, p, T, t, \boldsymbol{\xi}, \phi, \theta, \tau, g)) = (\Xi, Q, \eta, P, \Theta)$ for $\lambda \in \Lambda$, $(\mathbf{u}, p, T, t, \boldsymbol{\xi}, \phi, \theta, \tau, g) \in X$ and $(\Xi, Q, \eta, P, \Theta) \in Y$ if and only if

(2.19)

$$\begin{cases} \langle \Xi, \mathbf{v} \rangle = \lambda b_0(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \lambda(T\mathbf{g}, \mathbf{v}) - \lambda(\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega) \\ \langle Q, S \rangle = \lambda b_1(\mathbf{u}, T, S) \quad \forall S \in H^1(\Omega) \\ \langle \Theta, \psi \rangle_{\Gamma_D} = -\lambda(h, \psi)_{\Gamma_l} - \lambda(g, \psi)_{\Gamma_r} \quad \forall \psi \in \mathbf{H}^{\frac{1}{2}}(\Gamma_D) \\ \langle \eta, \mathbf{w} \rangle = \lambda b_0(\mathbf{w}, \mathbf{u}, \boldsymbol{\xi}) + \lambda b_0(\mathbf{u}, \mathbf{w}, \boldsymbol{\xi}) - \lambda b_1(\mathbf{w}, T, \theta) - (\mathbf{u} - \mathbf{U}, \mathbf{w}) \\ \qquad \forall \mathbf{w} \in \mathbf{H}^1(\Omega) \\ \langle P, \varphi \rangle = \lambda b_1(\mathbf{u}, \varphi, \theta) - \lambda(\varphi, \mathbf{g} \cdot \boldsymbol{\xi}) \qquad \forall \varphi \in H^1(\Omega) \end{cases}$$

It is easily seen that the optimality system (1.25)-(1.26) is equivalent to

$$(2.20)\qquad (\mathbf{u}, p, T, t, \boldsymbol{\xi}, \phi, \theta, \tau, g) + AG(\lambda, (\mathbf{u}, p, T, t, \boldsymbol{\xi}, \phi, \theta, \tau, g)) = 0$$

and that the discrete optimality system (2.7)-(2.8) is equivalent to

$$(2.21)\qquad \begin{aligned} &(\mathbf{u}^h, p^h, T^h, t^h, \boldsymbol{\xi}^h, \phi^h, \theta^h, \tau^h, g^h) \\ &+ AG(\lambda, (\mathbf{u}^h, p^h, T^h, t^h, \boldsymbol{\xi}^h, \phi^h, \theta^h, \tau^h, g^h)) = 0. \end{aligned}$$

Thus we have recast our continuous and discrete optimality problems into a form that enables us to apply Theorem 1.2 and 1.3.

REMARK 1. It can be shown that for almost all values of Reynolds number, i.e, for almost all data and values of the viscosity ν, the optimality system (1.25)-(1.26), or equation of (2.20), is nonsingular, i.e., is locally unique. Thus, it is reasonable to assume that the optimality system has branches of nonsingular solutions. In order to apply the previous theorems, we need to estimate the approximation properties of operator A^h.

PROPOSITION 2.1. *The problem* (2.9)-(2.13) *has a unique solution belonging to* X. *Assume that the finite element spaces* $\mathbf{V}_0^h$ *and* S_0^h *satisfy* (2.1)-(2.2) *and* (2.6) *and the finite element spaces* V^h, O^h, N^h *satisfy* (2.3)-(2.5). *Then, the problem* (2.14)-(2.18) *has a unique solution belonging to* X^h. *Let* $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{T}, \tilde{t}, \tilde{\boldsymbol{\xi}}, \tilde{\phi}, \tilde{\theta}, \tilde{\tau}, \tilde{g})$ *and* $(\tilde{\mathbf{u}}^h, \tilde{p}^h, \tilde{T}^h, \tilde{t}^h, \tilde{\boldsymbol{\xi}}^h, \tilde{\phi}^h, \tilde{\theta}^h, \tilde{\tau}^h, \tilde{g}^h)$ *denote the solution of* (2.9)-(2.13) *and* (2.14)-(2.18), *respectively. Then we also have that*

$$(2.22)\quad \begin{aligned} &\| \tilde{\mathbf{u}} - \tilde{\mathbf{u}}^h \|_{\mathbf{H}^1(\Omega)} + \| \tilde{p} - \tilde{p}^h \|_{L^2(\Omega)} + \| \tilde{T} - \tilde{T}^h \|_{H^1(\Omega)} \\ &+ \| \tilde{t} - \tilde{t}^h \|_{H^{-\frac{1}{2}}(\Gamma_r)} + \| \tilde{\boldsymbol{\xi}} - \tilde{\boldsymbol{\xi}}^h \|_{\mathbf{H}^1(\Omega)} + \| \tilde{\phi} - \tilde{\phi}^h \|_{L^2(\Omega)} \\ &+ \| \tilde{\theta} - \tilde{\theta}^h \|_{H^1(\Omega)} + \| \tilde{\tau} - \tilde{\tau}^h \|_{H^{-\frac{1}{2}}(\Gamma_r)} + \| \tilde{g} - \tilde{g}^h \|_{H^1(\Gamma_r)} \\ &\longrightarrow 0 \quad \text{as} \quad h \to 0. \end{aligned}$$

If, in addition,

$$\begin{aligned} &(\tilde{\mathbf{u}}, \tilde{p}, \tilde{T}, \tilde{t}, \tilde{\boldsymbol{\xi}}, \tilde{\phi}, \tilde{\theta}, \tilde{\tau}, \tilde{g}) \\ &\in \mathbf{H}_0^{m+1}(\Omega) \times H^m \cap L_0^2(\Omega) \times H^{m+1}(\Omega) \times H^m(\Omega) \cap L_0^2(\Omega) \times \mathbf{H}_0^{m+1}(\Omega) \\ &\quad \times H^m \cap L_0^2(\Omega) \times H^{m+1}(\Omega) \times H^m(\Omega) \cap L_0^2(\Omega) \times H^1(\Gamma_c), \end{aligned}$$

then there exists a constant C, *independent of* h, *such that*

$$(2.23)\quad \begin{aligned} &\| \tilde{\mathbf{u}} - \tilde{\mathbf{u}}^h \|_{\mathbf{H}^1(\Omega)} + \| \tilde{p} - \tilde{p}^h \|_{L^2(\Omega)} + \| \tilde{T} - \tilde{T}^h \|_{H^1(\Omega)} \\ &+ \| \tilde{t} - \tilde{t}^h \|_{H^{-\frac{1}{2}}(\Gamma_r)} + \| \tilde{\boldsymbol{\xi}} - \tilde{\boldsymbol{\xi}}^h \|_{\mathbf{H}^1(\Omega)} + \| \tilde{\phi} - \tilde{\phi}^h \|_{L^2(\Omega)} \\ &+ \| \tilde{\theta} - \tilde{\theta}^h \|_{H^1(\Omega)} + \| \tilde{\tau} - \tilde{\tau}^h \|_{H^{-\frac{1}{2}}(\Gamma_r)} + \| \tilde{g} - \tilde{g}^h \|_{H^1(\Gamma_r)} \\ &\leq Ch^m \Big(\| \tilde{\mathbf{u}} \|_{\mathbf{H}^{m+1}(\Omega)} + \| \tilde{p} \|_{H^m(\Omega)} + \| \tilde{T} \|_{H^{m+1}(\Omega)} \\ &+ \| \tilde{\boldsymbol{\xi}} \|_{\mathbf{H}^{m+1}(\Omega)} + \| \tilde{\phi} \|_{H^m(\Omega)} + \| \tilde{\theta} \|_{H^{m+1}(\Omega)} \Big) . \end{aligned}$$

Proof. First, it is well known [7] that the two Stokes problems (2.9) and (2.11) have a unique solution $(\tilde{\mathbf{u}}, \tilde{p})$ and $(\tilde{\boldsymbol{\xi}}, \tilde{\phi})$ belonging to $\mathbf{H}_0^1(\Omega) \times \mathbf{L}_0^2(\Omega)$, respectively. Also, the discrete Stokes problems (2.14) and (2.16) have a unique solution $(\tilde{\mathbf{u}}^h, \tilde{p}^h)$ and $(\tilde{\boldsymbol{\xi}}^h, \tilde{\phi}^h)$ belonging to $\mathbf{V}_0^h \times S_0^h$, respectively. Moreover, we have that

$$\| \tilde{\mathbf{u}} - \tilde{\mathbf{u}}^h \|_{\mathbf{H}^1(\Omega)} + \| \tilde{p} - \tilde{p}^h \|_{L^2(\Omega)} \longrightarrow 0$$

and

$$\| \tilde{\boldsymbol{\xi}} - \tilde{\boldsymbol{\xi}}^h \|_{\mathbf{H}^1(\Omega)} + \| \tilde{\phi} - \tilde{\phi}^h \|_{L^2(\Omega)} \longrightarrow 0$$

as $h \to 0$, and if in addition, $(\tilde{\mathbf{u}}, \tilde{p}) \in \mathbf{H}_0^{m+1}(\Omega) \times \mathbf{H}^m \cap L_0^2(\Omega)$ and $(\tilde{\boldsymbol{\xi}}, \tilde{\phi}) \in \mathbf{H}_0^{m+1}(\Omega) \times \mathbf{H}^m \cap L_0^2(\Omega)$, we have that

$$\| \tilde{\mathbf{u}} - \tilde{\mathbf{u}}^h \|_{\mathbf{H}^1(\Omega)} + \| \tilde{p} - \tilde{p}^h \|_{L^2(\Omega)} \leq Ch^m \left(\| \tilde{\mathbf{u}} \|_{\mathbf{H}^{m+1}(\Omega)} + \| \tilde{p} \|_{H^m(\Omega)} \right)$$

and

$$\| \tilde{\boldsymbol{\xi}} - \tilde{\boldsymbol{\xi}}^h \|_{\mathbf{H}^1(\Omega)} + \| \tilde{\phi} - \tilde{\phi}^h \|_{L^2(\Omega)} \leq Ch^m \left(\| \tilde{\boldsymbol{\xi}} \|_{\mathbf{H}^{m+1}(\Omega)} + \| \tilde{\phi} \|_{H^m(\Omega)} \right).$$

Next, it is also well known that the two second order elliptic problems (2.10) and (2.12) have a unique solution $(\tilde{T}, \tilde{t})$ and $(\tilde{\theta}, \tilde{\tau})$ belonging to $H^1(\Omega) \times H^{-\frac{1}{2}}(\Gamma_D)$, respectively. From the Babuska's theory, the discrete second order elliptic problems (2.15) and (2.17) have a unique solution $(\tilde{T}^h, \tilde{t}^h)$ and $(\tilde{\theta}^h, \tilde{\tau}^h)$ belonging to $V^h \times O^h$, respectively. Moreover, we have that

$$\| \tilde{T} - \tilde{T}^h \|_{H^1(\Omega)} + \| \tilde{t} - \tilde{t}^h \|_{H^{-\frac{1}{2}}(\Gamma_r)} \to 0 \tag{2.24}$$

and

$$\| \tilde{\theta} - \tilde{\theta}^h \|_{H^1(\Omega)} + \| \tilde{\tau} - \tilde{\tau}^h \|_{H^{-\frac{1}{2}}(\Gamma_r)} \to 0 \quad \text{as} \quad h \to 0, \tag{2.25}$$

and if in addition $(\tilde{T}, \tilde{t}) \in H^{m+1}(\Omega) \times H^{m-\frac{1}{2}}(\Gamma_r)$ and $(\tilde{\theta}, \tilde{\tau}) \in H^{m+1}(\Omega) \times H^{m-\frac{1}{2}}(\Gamma_r)$, we have that

$$\| \tilde{T} - \tilde{T}^h \|_{H^1(\Omega)} + \| \tilde{t} - \tilde{t}^h \|_{H^{-\frac{1}{2}}(\Gamma_r)} \leq Ch^m \| \tilde{T} \|_{H^{m+1}(\Omega)}, \tag{2.26}$$

$$\| \tilde{\theta} - \tilde{\theta}^h \|_{H^1(\Omega)} + \| \tilde{\tau} - \tilde{\tau}^h \|_{H^{-\frac{1}{2}}(\Gamma_r)} \leq Ch^m \| \tilde{\theta} \|_{H^{m+1}(\Omega)}. \tag{2.27}$$

Note that the problem (2.13) is a well-known second-order elliptic problem. Thus, we have that the problems (2.13) and (2.18) both have unique solutions and that

$$\| \tilde{g} \|_{H^{m+1}(\Gamma_r)} \leq C \| \tilde{\tau} \|_{H^{m-\frac{1}{2}}(\Gamma_r)} \leq C \| \tilde{\theta} \|_{H^{m+1}(\Omega)}, \tag{2.28}$$

(2.29)

$$\| \tilde{g} - \tilde{g}^h \|_{H^1(\Gamma_r)} \leq C \left(\| \tilde{g} - \hat{g}^h \|_{H^1(\Gamma_r)} + \| \tilde{\tau} - \tilde{\tau}^h \|_{H^{-\frac{1}{2}}(\Gamma_r)} \right) \quad \forall \hat{g}^h \in N^h.$$

Using (2.5), (2.24) and (2.25), we have that

$$\| \tilde{g} - \tilde{g}^h \|_{H^1(\Gamma_r)} \to 0 \quad \text{as} \quad h \to 0, \tag{2.30}$$

and using (2.5), (2.26) and (2.27), we conclude that

$$\| \tilde{g} - \tilde{g}^h \|_{H^1(\Gamma_r)} \leq Ch^m \| \tilde{\theta} \|_{H^{m+1}(\Omega)}. \tag{2.31}$$

□

THEOREM 2.2. *Assume that Λ is a compact interval of $\mathbb{R}_+$ and that there exists a branch $\{(\lambda, \psi(\lambda) := (\mathbf{u}, p, T, t, \boldsymbol{\xi}, \phi, \theta, \tau, g)) \in \Lambda \times X\}$ of nonsingular solutions of the optimality system (1.25)-(1.26). Assume that the finite elements spaces $\mathbf{V}^h$, S_0^h, O^h, V^h, N^h satisfy the conditions (2.1)-(2.5) and the finite elements spaces $\mathbf{V}^h$, S_0^h satisfy the inf-sup condition (2.6). Then, there exists a neighborhood $\mathcal{O}$ of the origin in X and, for $h \leq h_0$, small enough, a unique branch $\{(\lambda, \psi^h(\lambda) := (\mathbf{u}^h, p^h, T^h, t^h, \boldsymbol{\xi}^h, \phi^h, \theta^h, \tau^h, g^h)) \in \Lambda \times X^h\}$ of solutions of the discrete optimality system (2.7)-(2.8) such that $\psi^h(\lambda) - \psi(\lambda) \in \mathcal{O}$ for all $\lambda \in \Lambda$. Moreover,*

(2.32)

$$\begin{aligned}
&\| \psi^h(\lambda) - \psi(\lambda) \|_X \\
&= \| \tilde{\mathbf{u}} - \tilde{\mathbf{u}}^h \|_{\mathbf{H}^1(\Omega)} + \| \tilde{p} - \tilde{p}^h \|_{L^2(\Omega)} + \| \tilde{T} - \tilde{T}^h \|_{H^1(\Omega)} \\
&\quad + \| \tilde{t} - \tilde{t}^h \|_{H^{-\frac{1}{2}}(\Gamma_r)} + \| \tilde{\boldsymbol{\xi}} - \tilde{\boldsymbol{\xi}}^h \|_{\mathbf{H}^1(\Omega)} + \| \tilde{\phi} - \tilde{\phi}^h \|_{L^2(\Omega)} \\
&\quad + \| \tilde{\theta} - \tilde{\theta}^h \|_{H^1(\Omega)} + \| \tilde{\tau} - \tilde{\tau}^h \|_{H^{-\frac{1}{2}}(\Gamma_r)} + \| \tilde{g} - \tilde{g}^h \|_{H^1(\Gamma_r)} \longrightarrow 0
\end{aligned}$$

as $h \to 0$, uniformly in $\lambda \in \Lambda$. If, in addition, $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{T}, \tilde{t}, \tilde{\boldsymbol{\xi}}, \tilde{\phi}, \tilde{\theta}, \tilde{\tau}, \tilde{g}) \in \mathbf{H}^{m+1}(\Omega) \times H^m \cap L_0^2(\Omega) \times H^{m+1}(\Omega) \times H^{m-\frac{1}{2}}(\Gamma_D) \times \mathbf{H}^{m+1}(\Omega) \times H^m \cap L_0^2(\Omega) \times H^{m+1}(\Omega) \times H^{m-\frac{1}{2}}(\Gamma_D) \times H^1(\Gamma_c)$ for $\lambda \in \Lambda$, then there exists a constant C, independent of h, such that

$$\begin{aligned}
&\| \tilde{\mathbf{u}} - \tilde{\mathbf{u}}^h \|_{\mathbf{H}^1(\Omega)} + \| \tilde{p} - \tilde{p}^h \|_{L^2(\Omega)} + \| \tilde{T} - \tilde{T}^h \|_{H^1(\Omega)} \\
&\quad + \| \tilde{t} - \tilde{t}^h \|_{H^{-\frac{1}{2}}(\Gamma_r)} + \| \tilde{\boldsymbol{\xi}} - \tilde{\boldsymbol{\xi}}^h \|_{\mathbf{H}^1(\Omega)} + \| \tilde{\phi} - \tilde{\phi}^h \|_{L^2(\Omega)} \\
&\quad + \| \tilde{\theta} - \tilde{\theta}^h \|_{H^1(\Omega)} + \| \tilde{\tau} - \tilde{\tau}^h \|_{H^{-\frac{1}{2}}(\Gamma_r)} + \| \tilde{g} - \tilde{g}^h \|_{H^1(\Gamma_r)} \\
&\leq Ch^m \Big(\| \tilde{\mathbf{u}} \|_{\mathbf{H}^{m+1}(\Omega)} + \| \tilde{p} \|_{H^m(\Omega)} + \| \tilde{T} \|_{H^{m+1}(\Omega)} \\
&\quad + \| \tilde{\boldsymbol{\xi}} \|_{\mathbf{H}^{m+1}(\Omega)} + \| \tilde{\phi} \|_{H^m(\Omega)} + \| \tilde{\theta} \|_{H^{m+1}(\Omega)} \Big)
\end{aligned} \tag{2.33}$$

uniformly in $\lambda \in \Lambda$.

Proof. Clearly, G is a C^∞ polynomial map from $\Lambda \times X$ into Y. Therefore, using (1.6)-(1.8), (1.11), and (1.13), it is easily shown that

$D^2G(\lambda, \cdot)$ is bounded on all bounded sets of X. Now, given $(\mathbf{u}, p, T, t, \boldsymbol{\xi}, \phi, \theta, \tau, g) \in X$, a direct computation yields that $(\tilde{\Xi}, \tilde{Q}, \tilde{\eta}, \tilde{P}, \tilde{\Theta}) \in Y$ satisfies

$$(\tilde{\Xi}, \tilde{Q}, \tilde{\eta}, \tilde{P}, \tilde{\Theta}) = D_\psi G(\lambda, (\mathbf{u}, p, T, t, \boldsymbol{\xi}, \phi, \theta, \tau, g))(\bar{\mathbf{u}}, \bar{p}, \bar{T}, \bar{t}, \bar{\boldsymbol{\xi}}, \bar{\phi}, \bar{\theta}, \bar{\tau}, \bar{g})$$

for $(\bar{\mathbf{u}}, \bar{p}, \bar{T}, \bar{t}, \bar{\boldsymbol{\xi}}, \bar{\phi}, \bar{\theta}, \bar{\tau}, \bar{g}) \in X$ if and only if

$$\begin{aligned}
\langle \tilde{\Xi}, \mathbf{v} \rangle &= b_0(\mathbf{u}, \bar{\mathbf{u}}, \mathbf{v}) + b_0(\bar{\mathbf{u}}, \mathbf{u}, \mathbf{v}) - (\bar{T}\mathbf{g}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega) \\
\langle \tilde{Q}, S \rangle &= b_1(\bar{\mathbf{u}}, T, S) + b_1(\mathbf{u}, \bar{T}, S) \quad \forall S \in H^1(\Omega) \\
\langle \tilde{\eta}, \mathbf{w} \rangle &= b_0(\mathbf{w}, \bar{\mathbf{u}}, \boldsymbol{\xi}) + b_0(\mathbf{w}, \mathbf{u}, \bar{\boldsymbol{\xi}}) + b_0(\bar{\mathbf{u}}, \mathbf{w}, \boldsymbol{\xi}) + b_0(\mathbf{u}, \mathbf{w}, \bar{\boldsymbol{\xi}}) \\
&\quad - b_1(\mathbf{w}, \bar{T}, \theta) - b_1(\mathbf{w}, T, \bar{\theta}) \quad \forall \mathbf{w} \in \mathbf{H}_0^1(\Omega) \\
\langle \tilde{P}, \varphi \rangle &= b_1(\bar{\mathbf{u}}, \phi, \varphi) + b_1(\mathbf{u}, \bar{\phi}, \varphi) - (\bar{\phi}, \varphi)_{\Gamma_\sigma} - (\varphi \mathbf{g}, \bar{\boldsymbol{\xi}}) \quad \forall \varphi \in H^1(\Omega) \\
\langle \tilde{\theta}, \psi \rangle &= -(\bar{g}, \psi)_{\Gamma_r} \quad \forall \psi \in H^{\frac{1}{2}}(\Gamma_D)
\end{aligned}$$

Thus, it follows from (1.6)-(1.8), (1.11), and (1.13) that $D_\psi G(\lambda, (\mathbf{u}, p, T, t, \boldsymbol{\xi}, \phi, \theta, \tau, g)) \in \mathcal{L}(X, Y)$. On the other hand, since $(\mathbf{u}, p, T, t, \boldsymbol{\xi}, \phi, \theta, \tau, g) \in X$ and $(\bar{\mathbf{u}}, \bar{p}, \bar{T}, \bar{t}, \bar{\boldsymbol{\xi}}, \bar{\phi}, \bar{\theta}, \bar{\tau}, \bar{g}) \in X$, by the Sobolev imbedding theorem, $T, \theta, \bar{T}, \bar{\theta} \in L^2(\Omega)$, $\mathbf{u}, \boldsymbol{\xi}, \bar{\mathbf{u}}, \bar{\boldsymbol{\xi}} \in \boldsymbol{L}^6(\Omega)$ and $\frac{\partial \mathbf{u}}{\partial x_j}, \frac{\partial \mathbf{v}}{\partial x_j}, \frac{\partial \bar{\mathbf{u}}}{\partial x_j}, \frac{\partial \bar{\mathbf{v}}}{\partial x_j} \in \mathbf{L}^2(\Omega)$ for $j = 1, 2$. Then it follows that $(\tilde{\Xi}, \tilde{Q}, \tilde{\eta}, \tilde{P}, \tilde{\Theta}) \in Z$ and that for $(\mathbf{u}, p, T, t, \boldsymbol{\xi}, \phi, \theta, \tau, g) \in X$,

$$D_\psi G(\lambda, (\mathbf{u}, p, T, t, \boldsymbol{\xi}, \phi, \theta, \tau, g)) \in \mathcal{L}(X, Z)$$

Next, we turn to the approximation properties of the operator T. From Proposition 2.1, we have that (1.30) holds. Since the imbedding of Z into Y is compact, (1.31) follows from (1.30), and the (2.32) follows from Theorem 1.2. Also from Proposition 2.1, we may conclude that there exists a constant C, independent of h, such that

$$\begin{aligned}
&\| (T - T^h) G(\lambda, \psi(\lambda)) \|_X \\
&\le Ch^m \Big(\| \mathbf{u} \|_{\mathbf{H}^{m+1}(\Omega)} + \| p \|_{H^m(\Omega)} \\
&\quad + \| T \|_{H^{m+1}(\Omega)} + \| \boldsymbol{\xi} \|_{\mathbf{H}^{m+1}(\Omega)} + \| \phi \|_{H^m(\Omega)} + \| \theta \|_{H^{m+1}(\Omega)} \Big).
\end{aligned}$$

Then (2.33) follows from Theorem 1.2. □

Now, using Theorem 1.3. we derive an estimate for the error of $\mathbf{u}^h$ and $\boldsymbol{\xi}^h, T^h, \theta^h, g^h$ in the L^2-norm. Since $G(\lambda, \psi(\lambda))$ does not depend on p, t, τ, or ϕ, we redefine $X = \mathbf{H}^1(\Omega) \times H^1(\Omega) \times \mathbf{H}^1(\Omega) \times H^1(\Omega) \times H^1(\Gamma_c)$ and $X^h = \mathbf{V}^h \times V^h \times \mathbf{V}^h \times V^h \times N^h$, Y and Z remain as before.

THEOREM 2.3. *Assume the hypotheses of Theorem 2.2. Then exists a constant C, independent of h such that*

$$\|\mathbf{u}^h - \mathbf{U}\|_{L^2(\Omega)} + \|T^h - T\|_{L^2(\Omega)} + \|\theta^h - \theta\|_{L^2(\Omega)} + \|g^h - g\|$$
$$\leq Ch^{m+\frac{1}{2}} \Big(\|\mathbf{u}(\lambda)\|_{\mathbf{H}^{m+1}(\Omega)} + \|p(\lambda)\|_{H^m(\Omega)} + \|T(\lambda)\|_{H^{m+1}(\Omega)}$$
$$+\|\boldsymbol{\xi}(\lambda)\|_{\mathbf{H}^{m+1}(\Omega)} + \|\phi(\lambda)\|_{H^m(\Omega)} + \|\theta(\lambda)\|_{H^{m+1}(\Omega)}\Big).$$

Proof. We must verify that (1.33)-(1.36) hold in our setting; the approximation properties (2.1) and the results of Theorem 1.ξ Theorem 2.2 easily lead to the conclusion. In similar methods with we can verify (1.33)-(1.36).

REMARK 2. By other means, it can be shown [29] that actuall

$$\|\mathbf{u}^h - \mathbf{U}\|_{\mathbf{L}^2(\Omega)} + \|T^h - T\|_{L^2(\Omega)} + \|\boldsymbol{\xi}^h - \boldsymbol{\xi}\|_{\mathbf{L}^2(\Omega)} + \|\theta^h - \theta\|$$
$$\leq Ch^{m+1} \Big(\|\mathbf{u}(\lambda)\|_{\mathbf{H}^{m+1}(\Omega)} + \|p(\lambda)\|_{H^m(\Omega)} + \|T(\lambda)\|_{H^{m+1}(\Omega)}$$
$$+\|\boldsymbol{\xi}(\lambda)\|_{\mathbf{H}^{m+1}(\Omega)} + \|\phi(\lambda)\|_{H^m(\Omega)} + \|\theta(\lambda)\|_{H^{m+1}(\Omega)}\Big).$$

3. Numerical algorithm

In this section we present a computational algorithm using a si gradient method. The optimal control problem (1.20)-(1.24) is eq lent to the following minimization problem: Find $g \in H^1(\Gamma_r)$ such $\mathcal{K}(g) := \mathcal{J}(\mathbf{u}(g), p(g), T(g), g)$ is minimized where $(\mathbf{u}(g), p(g), T($ is defined as the solution of (1.15)-(1.19).
The classical simple gradient algorithm proceeds as follow:

$$\text{Given} \quad g^{(0)};$$
$$\text{define} \quad g^{(n+1)} = g^{(n)} - \frac{\rho}{\delta}\frac{d\mathcal{K}(g^{(n)})}{dg^{(n)}} \quad \text{recursively,} \tag{3.1}$$

where $\frac{\rho}{\delta}$ is a step size.

Let $\hat{g}$ be a solution of the minimization problem $\min_g \mathcal{K}(g)$. Th

$$\frac{d\mathcal{K}(\hat{g})}{d\hat{g}} = \frac{d\mathcal{J}(\mathbf{u}(\hat{g}), p(\hat{g}), T(\hat{g}), \hat{g})}{d\hat{g}} = 0 \tag{3.2}$$

For each fixed g, the derivative $\frac{d\mathcal{K}(g)}{dg} \cdot z$ for every direction $z \in H^1(\Gamma_r)$ may be easily computed

$$(3.3) \quad \frac{d\mathcal{K}(g)}{dg} \cdot z = \delta(\nabla g, \nabla z)_{\Gamma_r} + \delta(g, z) + (\mathbf{u} - \mathbf{U}, \bar{\mathbf{u}}) \quad \forall z \in H^1(\Gamma_r),$$

where for each $z \in H^1(\Gamma_r)$, $\bar{\mathbf{u}} \in \mathbf{H}^1(\Omega)$ is the solution of

$$(3.4)$$
$$\nu a_0(\bar{\mathbf{u}}, \mathbf{v}) + b_0(\bar{\mathbf{u}}, \mathbf{u}, \mathbf{v}) + b_0(\bar{\mathbf{u}}, \bar{\mathbf{u}}, \mathbf{v}) + c(\mathbf{v}, \bar{p}) = (\bar{T}\mathbf{g}, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)$$
$$(3.5) \qquad c(\bar{\mathbf{u}}, q) = 0 \qquad \forall q \in L_0^2(\Omega)$$
$$(3.6) \qquad a_1(\bar{T}, S) + b_1(\bar{\mathbf{u}}, T, S) + b_1(\mathbf{u}, \bar{T}, S) = 0 \qquad \forall S \in H_D^1(\Omega)$$
$$(3.7) \qquad \bar{T} = z \quad \text{on } \Gamma_r \qquad \bar{T} = h \quad \text{on } \Gamma_l,$$

Let $(\mathbf{u}, p, T, t) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H^1(\Omega) \times H^{-\frac{1}{2}}(\Gamma_r)$ be the solution of (1.15)-(1.19) and let $(\boldsymbol{\xi}, \phi, \theta, \tau) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H^1(\Omega) \times H^{-\frac{1}{2}}(\Gamma_r)$ be defined as the solution of the adjoint problem

$$\nu a_0(\boldsymbol{\xi}, \mathbf{w}) + b_0(\mathbf{u}, \mathbf{w}, \boldsymbol{\xi}) + c(\mathbf{w}, \phi)$$
$$(3.8) \qquad = (\mathbf{u} - \mathbf{U}, \mathbf{w}) - b_1(\mathbf{w}, T, \theta) \quad \forall \mathbf{w} \in \mathbf{H}_0^1(\Omega)$$
$$(3.9) \qquad c(\boldsymbol{\xi}, r) = 0 \quad \forall r \in L_0^2(\Omega)$$
$$(3.10) \qquad a_1(\theta, \varphi) + b_1(\mathbf{u}, \varphi, \theta) + (\tau, \varphi)_{\Gamma_D} = (\boldsymbol{\xi}, \mathbf{g}\varphi) \quad \forall \varphi \in H_D^1(\Omega)$$
$$(3.11) \qquad (\theta, \chi)_{\Gamma_D} = 0 \quad \forall \chi \in H^{-\frac{1}{2}}(\Gamma_D)$$

Setting $(\mathbf{v}, q, S) = (\boldsymbol{\xi}, \phi, \theta)$ in (3.4)-(3.7) and $(\mathbf{w}, r, \varphi) = (\bar{\mathbf{u}}, \bar{p}, \bar{T})$ in (3.8)-(3.11), we have that

$$(3.12) \qquad (\tau, z)_{\Gamma_r} = (\mathbf{u} - \mathbf{U}, \bar{u}).$$

Thus, from the necessary condition (3.2), we see that optimal value of the control g satisfies

$$(\nabla g, \nabla z)_{\Gamma_r} + (g, z)_{\Gamma_r} = \frac{1}{\delta}(\tau, z)_{\Gamma_r} \qquad \forall z \in H^1(\Gamma_r)$$

Collecting the above results, we may also obtain the optimality system

(3.13)
$$\begin{cases} \nu a_0(\mathbf{u},\mathbf{v}) + b_0(\mathbf{u},\mathbf{u},\mathbf{v}) + c(\mathbf{v},p) = (T\mathbf{g},\mathbf{v}) + \langle \mathbf{f},\mathbf{v}\rangle & \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega) \\ c(\mathbf{u},q) = 0 & \forall q \in L_0^2(\Omega) \\ a_1(T,S) + b_1(\mathbf{u},T,S) - (t,S)_{\Gamma_r} = \langle Q,S\rangle & \forall S \in H_l^1(\Omega) \\ T = h \quad \text{on } \Gamma_l \quad (T,R)_{\Gamma_r} = (g,R)_{\Gamma_r} & \forall R \in H^{-\frac{1}{2}}(\Gamma_r) \\ \nu a_0(\boldsymbol{\xi},\mathbf{w}) + b_0(\mathbf{u},\mathbf{w},\boldsymbol{\xi}) + c(\mathbf{w},\phi) & \\ \quad = (\mathbf{u}-\mathbf{U},\mathbf{w}) - b_1(\mathbf{w},T,\theta) & \forall \mathbf{w} \in \mathbf{H}_0^1(\Omega) \\ c(\boldsymbol{\xi},r) = 0 & \forall r \in L_0^2(\Omega) \\ a_1(\theta,\varphi) + b_1(\mathbf{u},\varphi,\theta) + (\tau,\varphi)_{\Gamma_D} = (\boldsymbol{\xi}\cdot\mathbf{g},\varphi) & \forall \varphi \in H_l^1(\Omega) \\ (\theta,\chi)_{\Gamma_D} = 0 & \forall \chi \in H^{-\frac{1}{2}}(\Gamma_D) \end{cases}$$

From (3.12) and the definition of τ, for fixed g, the derivative $\frac{d\mathcal{K}(g)}{dg}$ may be computed

$$\frac{d\mathcal{K}(g)}{dg} = -\delta_1\Delta_s g + \delta_2 g + \nabla\theta^{(n)}\cdot\mathbf{n}|_{\Gamma_r}.$$

Here, δ_1 and δ_2 are used to change the relative importance of the two terms appearing in the definition $\frac{d\mathcal{K}(g)}{dg}$ and δ will be the maximum of δ_1 and δ_2.

Thus, (3.1) may be replaced by

for $n = 0,1,2,\ldots,$

set
$$\begin{aligned} g^{(n+1)} &= g^{(n)} - \frac{\rho}{\delta}(-\delta_1\Delta_s g^{(n)} + \delta_2 g^{(n)} + \nabla\theta^{(n)}\cdot\mathbf{n}|_{\Gamma_r}) \\ &= \Big(1-\rho\frac{\delta_2}{\delta}\Big)g^{(n)} + \rho\frac{\delta_1}{\delta}\Delta_s g^{(n)} + \frac{\rho}{\delta}\nabla\theta^{(n)}\cdot\mathbf{n}|_{\Gamma_r}, \end{aligned}$$

where $\theta^{(n)}$ is determined from $g^{(n)}$ through the relations

$$\text{(3.14)} \qquad \begin{cases} a_1(T^{(n)},S) + b_1(\mathbf{u}^{(n-1)},T^{(n)},S) = \langle Q,S\rangle \quad \forall S \in H^1(\Omega) \\ T^{(n)} = g^{(n-1)} \text{ on } \Gamma_r, \quad T^{(n)} = h \text{ on } \Gamma_l, \end{cases}$$

$$(3.15)\quad \begin{cases} \nu a_0(\mathbf{u}^{(n)},\mathbf{v}) + b_0(\mathbf{u}^{(n)},\mathbf{u}^{(n)},\mathbf{v}) + c(\mathbf{v},p^{(n)}) - (T^{(n)}\mathbf{g},\mathbf{v}) \\ \qquad = \langle \mathbf{f},\mathbf{v}\rangle \quad \forall \mathbf{v}\in \mathbf{H}_0^1(\Omega) \\ c(\mathbf{u}^{(n)},q) = 0 \quad \forall q \in L_0^2(\Omega) \end{cases}$$

$$(3.16)\quad \begin{cases} a_1(\theta^{(n)},\varphi) + b_1(\mathbf{u}^{(n)},\varphi,\theta^{(n)}) = (\varphi\mathbf{g},\boldsymbol{\xi}^{(n-1)}) \quad \forall \varphi \in H^1(\Omega) \\ \theta^{(n)} = 0 \quad \text{on } \Gamma_D \end{cases}$$

$$(3.17)\quad \begin{cases} a_0(\mathbf{w},\boldsymbol{\xi}^{(n)}) + b_0(\mathbf{w},\mathbf{u}^{(n)},\boldsymbol{\xi}^{(n)}) + b_0(\mathbf{u}^{(n)},\mathbf{w},\boldsymbol{\xi}^{(n)}) + c(\mathbf{w},\phi^{(n)}) \\ \qquad = (\mathbf{u}^{(n)} - \mathbf{U},\mathbf{w}) - b_1(\mathbf{w},T^{(n)},\theta^{(n)}) \quad \forall \mathbf{w}\in \mathbf{H}_0^1(\Omega) \\ c(\boldsymbol{\xi}^{(n)},r) = 0 \quad \forall r \in L_0^2(\Omega). \end{cases}$$

The optimality system of equations (2.7)-(2.8) consists of three groups of equations: the state equations for $(\mathbf{u},p,T)$, the adjoint state equations for $(\boldsymbol{\xi},\phi,\theta)$, and the optimality condition for g. We may construct an iterative method, i.e., to iterate among the three groups of equations so that at each iteration we are dealing with a smaller size system of equations, besides, $(\mathbf{u}^{(n)},p^{(n)})$ and $(\boldsymbol{\xi}^{(n)},\phi^{(n)})$ are solved with $T^{(n)}$ and $\theta^{(n)}$ computed from the heat equations with $\mathbf{u}^{(n-1)}$ and $\boldsymbol{\xi}^{(n-1)}$ at each state equations, respectively.

A simple gradient method is given by follows

1. choose an initial guess $g^{(0)}$;
2. for each $n \geq 1$,
   (a) solve for $(\mathbf{u}^{(n)},p^{(n)},T^{(n)})$ from the state equation with $g^{(n-1)}$

$$(3.18)\quad \begin{cases} a_1(T^{(n)},S) + b_1(\mathbf{u}^{(n-1)},T^{(n)},S) = \langle Q,S\rangle \qquad \forall S \in H^1(\Omega) \\ T^{(n)} = g^{(n-1)} \text{ on } \Gamma_r, \qquad T^{(n)} = h \text{ on } \Gamma_l, \end{cases}$$

$$(3.19)\quad \begin{cases} \nu a_0(\mathbf{u}^{(n)},\mathbf{v}) + b_0(\mathbf{u}^{(n)},\mathbf{u}^{(n)},\mathbf{v}) + c(\mathbf{v},p^{(n)}) - (T^{(n)}\mathbf{g},\mathbf{v}) \\ \qquad = \langle \mathbf{f},\mathbf{v}\rangle \quad \forall \mathbf{v}\in \mathbf{H}_0^1(\Omega) \\ c(\mathbf{u}^{(n)},q) = 0 \quad \forall q \in L_0^2(\Omega), \end{cases}$$

   (b) solve for $(\boldsymbol{\xi}^{(n)},\phi^{(n)},\theta^{(n)})$ from the adjoint state equation with $(\mathbf{u}^{(n)},\, p^{(n)},T^{(n)})$

$$(3.20)\quad \begin{cases} a_1(\theta^{(n)},\varphi) + b_1(\mathbf{u}^{(n)},\varphi,\theta^{(n)}) = (\varphi\mathbf{g},\boldsymbol{\xi}^{(n-1)}) \qquad \forall \varphi \in H^1(\Omega) \\ \theta^{(n)} = 0 \qquad \text{on } \Gamma_D \end{cases}$$

$$
(3.21)\quad \begin{cases} a_0(\mathbf{w},\boldsymbol{\xi}^{(n)}) + b_0(\mathbf{w},\mathbf{u}^{(n)},\boldsymbol{\xi}^{(n)}) + b_0(\mathbf{u}^{(n)},\mathbf{w},\boldsymbol{\xi}^{(n)}) + c(\mathbf{w},\phi^{(n)}) \\ \qquad = (\mathbf{u}^{(n)} - \mathbf{U},\mathbf{w}) - b_1(\mathbf{w},T^{(n)},\theta^{(n)}) \quad \forall \mathbf{w} \in \mathbf{H}_0^1(\Omega) \\ \qquad c(\boldsymbol{\xi}^{(n)},r) = 0 \quad \forall r \in L_0^2(\Omega), \end{cases}
$$

(c) solve for $g^{(n)}$ from the optimality condition

$$
(3.22)\qquad g^{(n)} = \Big(1 - \rho\frac{\delta_2}{\delta}\Big)g^{(n-1)} + \rho\frac{\delta_1}{\delta}\Delta_s g^{(n-1)} + \frac{\rho}{\delta}\nabla\theta^{(n)}\cdot\mathbf{n}|_{\Gamma_r}.
$$

The convergence of the algorithm (3.18)-(3.22) is a direct consequence of the following lemma.

LEMMA 3.1. *Let $\mathcal{K}$ be a real-valued functional on a Hilbert space X with norm $\|\cdot\|_X$ and scalar product $(\cdot,\cdot)_X$. Suppose that there exist two constants m and M such that*

(i) $\mathcal{K}$ has a local minimum at a point $\bar{x}$ is of class C^2 in an open ball B centered at $\bar{x}$.

(ii) $\forall u \in B, \quad \forall (x,y) \in X \times X, \mathcal{K}''(u)\cdot(x,y) \le M\| x \|_X \| y \|_Y,$

(iii) $\forall u \in B, \quad \forall x \in X, \mathcal{K}''(u)\cdot(x,x) \ge m\| x \|_X^2.$

Let $\boldsymbol{R}$ denotes the Riesz map, i.e. $\langle f,x\rangle = (\boldsymbol{R}f,x)_X$ for all $x \in X$ and all $f \in X^*$. Choose $x^{(0)} \in B$ and choose a sequence $\{\rho_n\}$ such that $0 < \rho_* \le \rho_n \le \rho^* < 2m/M^2$. Then, the sequence $\{x^{(n)}\}$ defined by

$$
x^{(n)} = x^{(n-1)} - \rho_n R\mathcal{K}'(x^{(n-1)}) \qquad \textit{for} \qquad n = 1,2,\ldots,
$$

converges to $\bar{x}$. Furthermore, if $B = X$ and $\bar{x}$ is a global minimum, then the gradient algorithm converges to $\bar{x}$ for any initial value $x^{(0)}$.

Proof. See. e.g.,[8]. □

THEOREM 3.2. *Let $(\mathbf{u}^{(n)},p^{(n)},T^{(n)},\boldsymbol{\xi}^{(n)},\phi^{(n)},\theta^{(n)},g^{(n)})$ be the solution of (3.14)-(3.17) and $(\mathbf{u},p,T,\boldsymbol{\xi},\phi,\theta,g)$ the solution of (3.13). Then, if δ is sufficiently large, $g^{(n)} \to g$ and thus, $(\mathbf{u}^{(n)},p^{(n)},T^{(n)},\boldsymbol{\xi}^{(n)},\phi^{(n)},\theta^{(n)})$ $\to (\mathbf{u},p,T,\boldsymbol{\xi},\phi,\theta)$ in $\mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)\times H^1(\Omega)\times \mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)\times H_D^1(\Omega)$ as $n \to \infty$.*

Proof. In (3.14)-(3.17), we have the fixed parameter $\rho = \frac{1}{\kappa\delta}$. For each $g \in H^1(\Gamma_r)$, the second Fŕechet-derivative $\mathcal{K}''(g)\cdot(z,w)$ may be computed by $\mathcal{K}''(g)\cdot(z,w) = \delta(\nabla w,\nabla z)_{\Gamma_r} + \delta(w,z) + (\bar{\bar{\mathbf{u}}},\bar{\mathbf{u}})$, where

$\bar{\bar{\mathbf{u}}} \in \mathbf{H}^1(\Omega)$, and $\bar{\mathbf{u}} \in \mathbf{H}^1(\Omega)$ are the solutions of

(3.23)

$$\begin{cases} \nu a_0(\bar{\bar{\mathbf{u}}}, \mathbf{v}) + b_0(\bar{\bar{\mathbf{u}}}, \mathbf{u}, \mathbf{v}) + b_0(\bar{\mathbf{u}}, \bar{\mathbf{u}}, \mathbf{v}) + b_0(\bar{\mathbf{u}}, \bar{\mathbf{u}}, \mathbf{v}) + b_0(\mathbf{u}, \bar{\bar{\mathbf{u}}}, \mathbf{v}) \\ \qquad + c(\mathbf{v}, \bar{\bar{p}}) = (\bar{\bar{T}}\mathbf{g}, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega) \\ \qquad c(\bar{\bar{\mathbf{u}}}, q) = 0 \qquad \forall q \in L_0^2(\Omega) \\ a_1(\bar{\bar{T}}, S) + b_1(\bar{\bar{\mathbf{u}}}, T, S) + b_1(\bar{\mathbf{u}}, \bar{T}, S) + b_1(\bar{\mathbf{u}}, \bar{T}, S) \\ \qquad + b_1(\mathbf{u}, \bar{\bar{T}}, S) = 0 \qquad \forall S \in H_D^1(\Omega) \\ \bar{\bar{T}} = w \quad \text{on } \Gamma_r \qquad \bar{\bar{T}} = h \quad \text{on } \Gamma_l, \end{cases}$$

and of

$$(3.24) \quad \begin{cases} \nu a_0(\bar{\mathbf{u}}, \mathbf{v}) + b_0(\bar{\mathbf{u}}, \mathbf{u}, \mathbf{v}) + b_0(\bar{\mathbf{u}}, \bar{\mathbf{u}}, \mathbf{v}) + c(\mathbf{v}, \bar{p}) \\ \qquad = (\bar{T}\mathbf{g}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega) \\ \qquad c(\bar{\mathbf{u}}, q) = 0 \quad \forall q \in L_0^2(\Omega) \\ a_1(\bar{T}, S) + b_1(\bar{\mathbf{u}}, T, S) + b_1(\mathbf{u}, \bar{T}, S) = 0 \quad \forall S \in H_D^1(\Omega) \\ \bar{T} = z \quad \text{on } \Gamma_r \qquad \bar{T} = h \quad \text{on } \Gamma_l. \end{cases}$$

One can easily have that $\| \bar{\bar{\mathbf{u}}} \| \le C \| w \|_{1,\Gamma_r}$ and $\| \bar{\mathbf{u}} \| \le C \| z \|_{1,\Gamma_r}$, where the value of the constant C depends only on Ω. Then,

$$\begin{aligned} \mathcal{K}''(g) \cdot (z, w) &\le \delta \| w \|_{1,\Gamma_r} \| z \|_{1,\Gamma_r} + C \| \bar{\bar{\mathbf{u}}} \|_1 \| \bar{\mathbf{u}} \|_1 \\ &\le \delta \| w \|_{1,\Gamma_r} \| z \|_{1,\Gamma_r} + C \| w \|_{1,\Gamma_r} \| z \|_{1,\Gamma_r} \\ &= (\delta + C) \| w \|_{1,\Gamma_r} \| z \|_{1,\Gamma_r} \end{aligned}$$

and

$$\mathcal{K}''(g) \cdot (z, w) = \delta \| z \|_{1,\Gamma_r}^2 + \int |\bar{\mathbf{u}}|^2 d\Omega \ge \delta \| z \|_{1,\Gamma_r}^2.$$

Setting $M = \kappa\delta + C$ and $m = \delta$, we have, if $\delta > \frac{C}{(\sqrt{2}-1)}$, $\frac{2m}{M^2} > \rho = \frac{1}{\delta}$. The other hypotheses of Lemma 3.1 are easily shown to be valid. Hence, from that lemma we obtain that $g^{(n)} \to g$ in $H^1(\Gamma_r)$ as $n \to \infty$. The desired convergence results follow from the a priori estimate in ([30], Proposition 2.3) □

Now we discuss the numerical solution of the optimal control problem. To carry out the computation we discretized the problem using the finite element method. We use the Taylor-Hood finite element, that is, the piecewise quadratic element for the velocity and the temperature and bilinear element for the pressure defined on a triangle mesh. We use the mesh size $h = 1/16$ for all computations.

Since the equations (3.15) are nonlinear, we use the Newton's method based on exact Jacobian. Let us denote the finite element spaces by $\mathbf{V}^h \subset \mathbf{H}_0^1(\Omega)$, $V^h \subset H^1(\Omega)$, and $W^h \subset L^2(\Omega)$ for velocity, temperature, and pressure, respectively. The approximate problem for (3.14)-(3.17) is given as follows:

1. initialize $(\mathbf{u}^0, p^0, T^0, \boldsymbol{\xi}^0, \phi^0, \theta^0, g^0)$;

2. for each $n = 1, 2, \ldots$, solve the state equations using previous solution $(\mathbf{u}^{(n-1)}, g^{(n-1)})$;

   (a) find $T^{(n)} \in V^h$ such that

$$\begin{cases} a_1(T^{(n)}, S^h) + b_1(\mathbf{u}^{(n-1)}, T^{(n)}, S^h) = \langle Q, S^h \rangle \quad \forall S^h \in V^h \\ \qquad T^{(n)} = g^{(n-1)} \text{ on } \Gamma_r, \quad T^{(n)} = h \text{ on } \Gamma_l, \end{cases}$$

   (b) find $(\mathbf{u}^{(n)}, p^{(n)}) \in \mathbf{V}^h \times W^h$ such that

$$\begin{cases} a_0(\mathbf{u}^{(n)}, \mathbf{v}^h) + b_0(\mathbf{u}^{(n-1)}, \mathbf{u}^{(n)}, \mathbf{v}^h) + b_0(\mathbf{u}^{(n)} - \mathbf{u}^{(n-1)}, \mathbf{u}^{(n-1)}, \mathbf{v}^h) \\ \qquad + c(\mathbf{v}^h, p^{(n)}) - (T^{(n)}\mathbf{g}, \mathbf{v}^h) = \langle \mathbf{f}, \mathbf{v}^h \rangle \quad \forall \mathbf{v}^h \in \mathbf{V}^h \\ \qquad c(\mathbf{u}^{(n)}, q^h) = 0 \quad \forall q^h \in W^h, \end{cases}$$

3. solve adjoint equations using previous solution $(\mathbf{u}^{(n)}, T^{(n)}, \boldsymbol{\xi}^{(n-1)})$;

   (a) find $\theta^{(n)} \in V^h$ such that

$$\begin{cases} a_1(\theta^{(n)}, \varphi^h) + b_1(\mathbf{u}^{(n)}, \varphi^h, \theta^{(n)}) = (\varphi^h \mathbf{g}, \boldsymbol{\xi}^{(n-1)}) \qquad \forall \varphi^h \in V^h \\ \qquad \theta^{(n)} = 0 \quad \text{on } \Gamma_D, \end{cases}$$

   (b) find $(\boldsymbol{\xi}^{(n)}, \phi^{(n)}) \in \mathbf{V}^h \times W^h$ such that

$$\begin{cases} a_0(\mathbf{w}^h, \boldsymbol{\xi}^{(n)}) + b_0(\mathbf{w}^h, \mathbf{u}^{(n)}, \boldsymbol{\xi}^{(n)}) + b_0(\mathbf{u}^{(n)}, \mathbf{w}^h, \boldsymbol{\xi}^{(n)}) + c(\mathbf{w}^h, \phi^{(n)}) \\ \qquad = (\mathbf{u}^{(n)} - \mathbf{U}, \mathbf{w}^h) - b_1(\mathbf{w}^h, T^{(n)}, \theta^{(n)}) \quad \forall \mathbf{w}^h \in \mathbf{V}^h \\ \qquad c(\boldsymbol{\xi}^{(n)}, r^h) = 0 \quad \forall r^h \in W^h, \end{cases}$$

4. find $g^{(n)}$ such that

$$g^{(n)} = \Big(1 - \rho\frac{\delta_2}{\delta}\Big) g^{(n-1)} + \rho\frac{\delta_1}{\delta}\Delta_s g^{(n-1)} + \frac{\rho}{\delta}\nabla\theta^{(n)} \cdot \mathbf{n}|_{\Gamma_r}.$$

At each Newton's iteration, we solve the linear system of equations by Gaussian eliminations for banded matrices. Since quadratic convergence of Newton's method is valid only within a contraction ball, we normally first perform a few(usually 3 or 4 times) simple successive iterations and then switch to the Newton's method. The simple successive iterations are defined by

$$\begin{cases} a_0(\mathbf{u}^{(n)}, \mathbf{v}^h) + b_0(\mathbf{u}^{(n-1)}, \mathbf{u}^{(n)}, \mathbf{v}^h) + c(\mathbf{v}^h, p) - (T^{(n)}\mathbf{g}, \mathbf{v}^h) \\ \qquad\qquad = \langle \mathbf{f}, \mathbf{v}^h \rangle \quad \forall \mathbf{v}^h \in \mathbf{V}^h, \\ c(\mathbf{u}^{(n)}, q^h) = 0 \quad \forall q^h \in W^h. \end{cases}$$

In the case of the uncontrolled Navier-Stokes equations, the solution is unique for a small Reynold number and the simple successive approximations converges globally and linearly. (See [16])

In the same way, one can study the optimal control problems for the functionals (1.3) and (1.4) with the same Dirichlet boundary temperature control.

To decide ρ, we let $\rho = 1$ and choose the prescribed tolerance τ and perform the following steps in each iteration $k = 1, \ldots,$

1. if $\mathcal{K}(k) \geq \mathcal{K}(k-1)$, set $\rho = .5\rho$ and go to the beginning of the iterations; otherwise, go to next step;
2. if $|\mathcal{K}(k) - \mathcal{K}(k-1)|/|\mathcal{K}(k)| > \tau$, set $\rho = 1.5\rho$ and go to beginning of the iterations; otherwise, stop.

4. Computational results

In this section we test three examples involving the functionals (1.3)-(1.5) with Dirichlet boundary temperature controls using the simple gradient algorithm studied in section 3. Let us consider that the domain Ω is the unit square $(0,1)\times(0,1) \in \mathbb{R}^2$. Let $\Gamma_l = 0\times(0,1)$, $\Gamma_r = 1\times(0,1)$, $\Gamma_b = (0,1) \times 0$, and $\Gamma_t = (0,1) \times 1$. The state variables satisfies the following Boussinesque equations:

$$-\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = T\mathbf{g} \qquad \text{in } \Omega \tag{4.1}$$

$$\nabla\cdot\mathbf{u} = 0 \qquad \text{in } \Omega \tag{4.2}$$

$$-\Delta T + (\mathbf{u}\cdot\nabla)T = 0 \qquad \text{in } \Omega \tag{4.3}$$

In our computation, we take the Reynolds number to be $1(\nu = 1)$, the thermal conductivity $\kappa = 1$, body force $\mathbf{f} = (0,0)^T$.

4.1. The velocity matching problem

Let us consider the following problem: minimize

$$\mathcal{J}(\mathbf{u}, p, T, g) = \frac{1}{2}\int_\Omega |\mathbf{u} - \mathbf{U}|^2\, d\mathbf{x} + \frac{\delta_1}{2}\int_{\Gamma_r} |\nabla_s g|^2 ds + \frac{\delta_2}{2}\int_{\Gamma_r} |g|^2 ds. \tag{4.4}$$

TABLE 1. $\|\mathbf{u} - \mathbf{U}\|$ with Dirichlet boundary control g ($\|\mathbf{u} - \mathbf{U}\| = 1.3106\text{E-}03$ without control)

δ	10^{-5}	10^{-6}	10^{-7}	10^{-8}
$\delta_2 = 0,\ \delta_1 = \delta$	1.2239E-03	4.5733E-04	2.1447E-04	1.9509E-05
$\delta_1 = \delta_2 = \delta$	1.0179E-03	1.0405E-03	3.6571E-04	1.8564E-05
$\delta_1 = 0,\ \delta_2 = \delta$	7.6038E-04	2.0532E-04	4.4535E-05	8.2875E-06
$\delta_1 = \delta_2 \times 10^{-1},\ \delta_2 = \delta$	8.2249E-04	3.7175E-04	3.6571E-04	4.1400E-06
$\delta_1 = \delta_2 \times 10^{-2},\ \delta_2 = \delta$	7.4068E-04	2.0324E-04	4.0100E-05	6.6839E-06

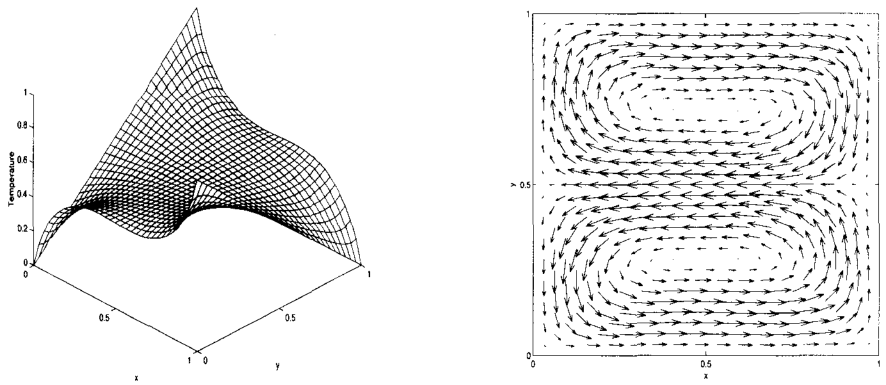

FIGURE 1. Desired temperature field(left) and velocity field(right)

TABLE 2. $\|\nabla \times \mathbf{u}\|$ with Dirichlet boundary control g ($\|\nabla \times \mathbf{u}\| = 1.0111E - 02$ without control)

δ	10^{-5}	10^{-6}	10^{-7}	10^{-8}
$\delta_2 = 0,\ \delta_1 = \delta$	2.2333E-03	1.9230E-03	1.5705E-03	1.4628E-03
$\delta_1 = \delta_2 = \delta$	2.1968E-03	1.9270E-03	1.5709E-03	1.4631E-03
$\delta_1 = 0,\ \delta_2 = \delta$	1.5964E-03	1.4028E-03	1.3568E-03	1.3516E-03
$\delta_1 = \delta_2 \times 10^{-1},\ \delta_2 = \delta$	1.6797E-03	1.5748E-03	1.4641E-03	1.3841E-03
$\delta_1 = \delta_2 \times 10^{-2},\ \delta_2 = \delta$	1.6400E-03	1.4764E-03	1.3810E-03	1.3514E-03

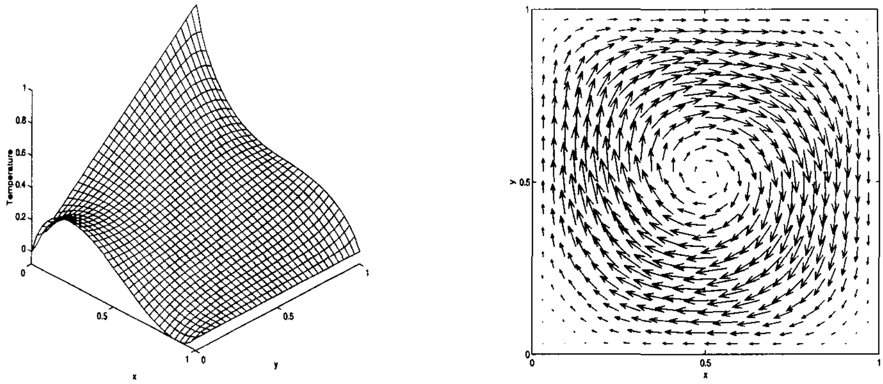

FIGURE 2. Uncontrolled temperature field(left) and velocity field(right)

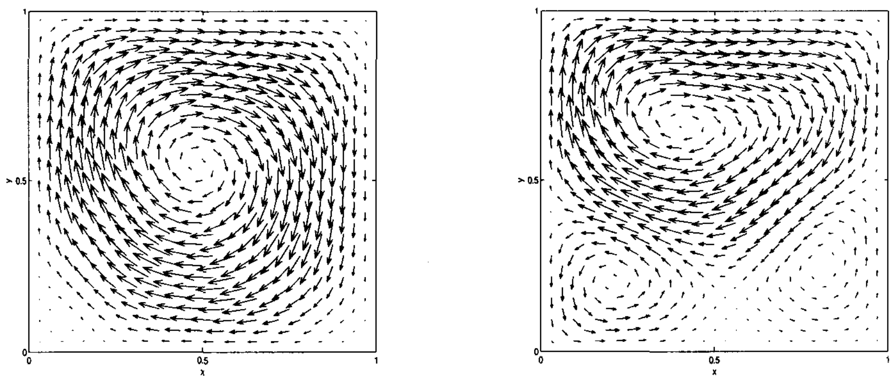

FIGURE 3. Velocity fields with $\delta = \delta_2 = 10^{-5}, \delta_1 = \delta_2 \times 10^{-2}$(left) and $\delta = \delta_2 = 10^{-6}$, $\delta_1 = \delta_2 \times 10^{-2}$(right)

subject to the equations (4.1)-(4.3) with boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega, \qquad \frac{\partial T}{\partial \mathbf{n}} = \cos(\pi x)\cos(\pi y) \quad \text{on } \Gamma_b \cup \Gamma_t,$$

and

$$T = y \quad \text{on } \Gamma_l, \qquad T = g \quad \text{on } \Gamma_r.$$

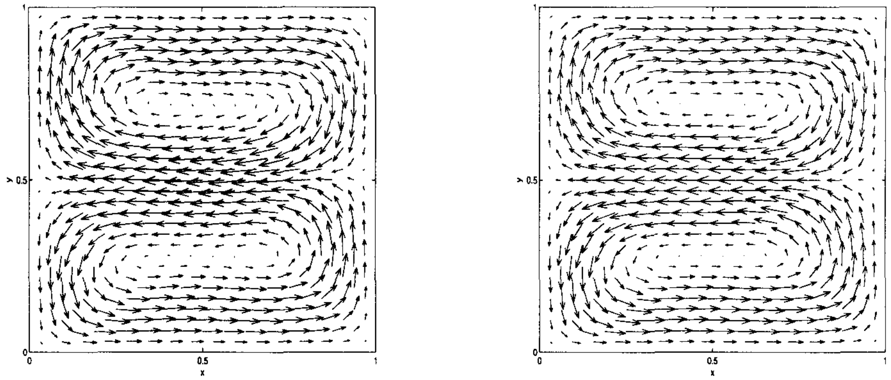

FIGURE 4. Velocity fields with $\delta = \delta_2 = 10^{-7}, \delta_1 = \delta_2 \times 10^{-2}$(left) and $\delta = \delta_2 = 10^{-8}$, $\delta_1 = \delta_2 \times 10^{-2}$(right)

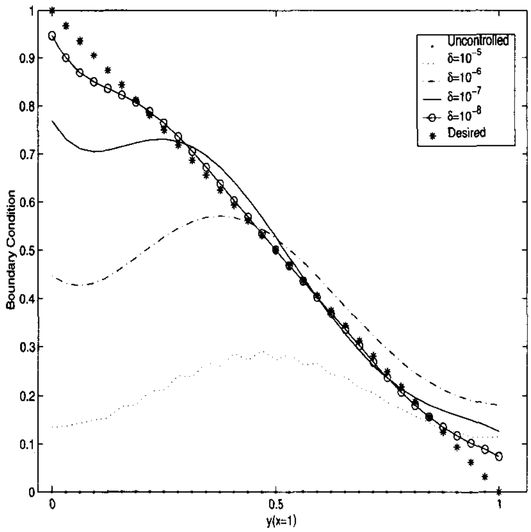

FIGURE 5. Boundary controls on Γ_r, $\delta_2 = \delta$, $\delta_1 = \delta_2 \times 10^{-2}$

Here the desired velocity $\mathbf{U}$ satisfies the equations (4.1)-(4.3) with the same data as chosen previously and with the additional boundary condition $T = 1-y$ on Γ_r. We plot the desired velocity field in Figure 1(right) and corresponding temperature field(left). We want to match the velocity field $\mathbf{u}$ with this $\mathbf{U}$ by adjusting the temperature on the boundary Γ_r. When $g = 0$ we say that this problem is an uncontrolled problem. The numerical solution of the uncontrolled problem is shown in Figure 2.

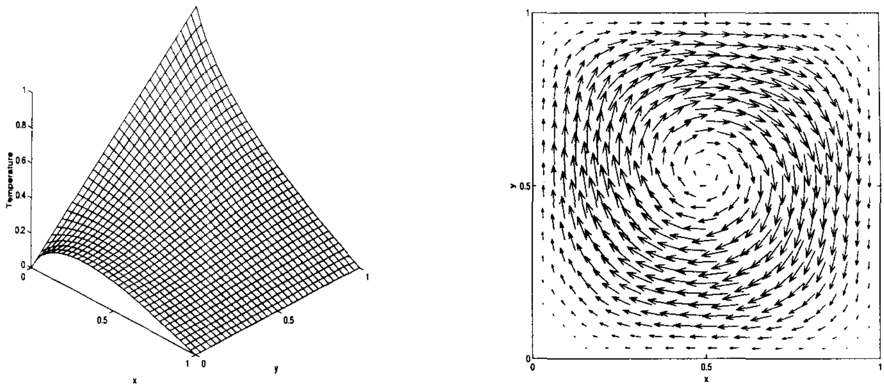

FIGURE 6. Uncontrolled temperature field(left) and velocity field(right)

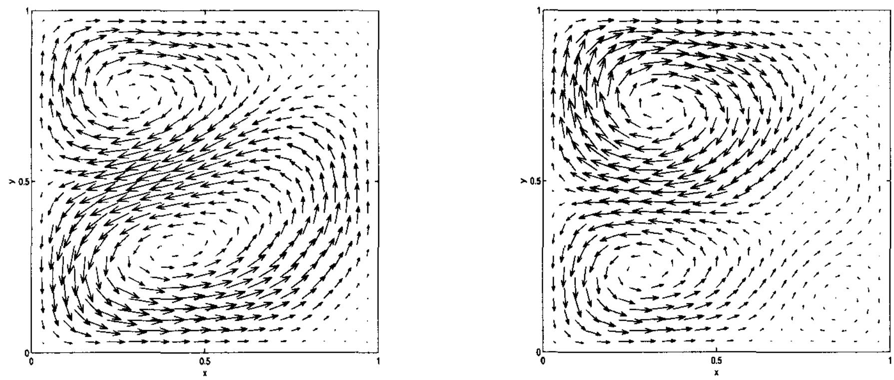

FIGURE 7. Velocity fields with $\delta = \delta_1 = \delta_2 = 10^{-5}$(left), 10^{-6}(right)

As shown in Table 1, the smaller δ is, the smaller the L^2 distance of $\mathbf{u}$ and $\mathbf{U}$ is. The optimal velocity fields $\mathbf{u}$ are given in Figure 3 and Figure 4 with various values of $\delta, \delta_1, \delta_2$. From these figures we can easily check the problem is well controlled by using the method described before with the chosen data. In Figure 5, we plot the approximate optimal controls g^h on the boundary Γ_r. It is more effective including the surface gradient of boundary control g in the cost functional, especially in this case, with $\delta_2 = \delta$, $\delta_1 = \delta_2 \times 10^{-2}$.

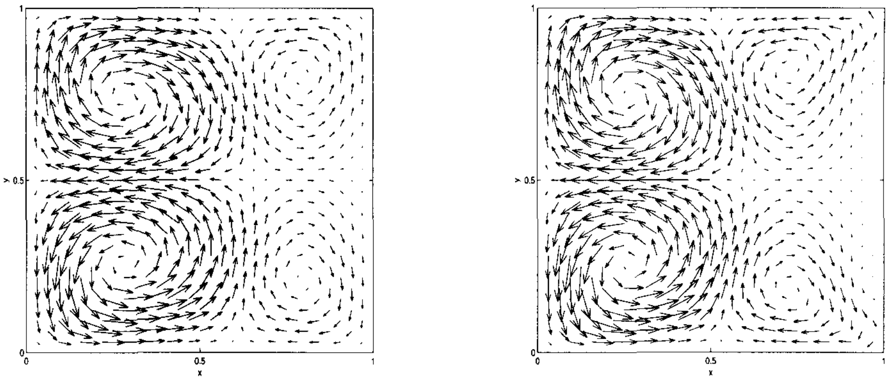

FIGURE 8. Velocity fields with $\delta = \delta_1 = \delta_2 = 10^{-7}$(left), 10^{-8}(right)

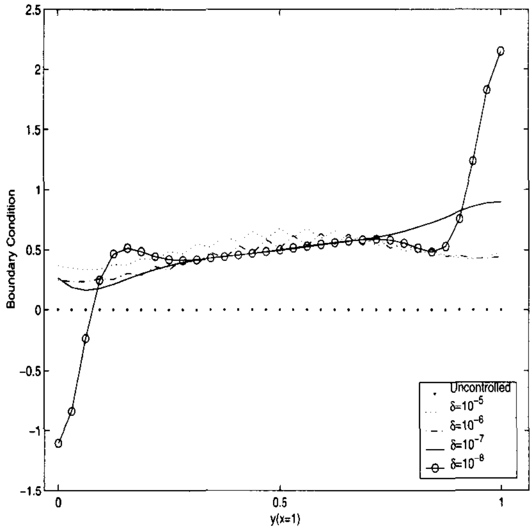

FIGURE 9. Boundary controls on Γ_r, $\delta = \delta_1 = \delta_2$(right)

4.2. The vorticity minimization problem

Let us consider the vorticity minimization problem. The cost functional is the following

$$(4.5)\quad \mathcal{J}(\mathbf{u},p,T,g) = \frac{1}{2}\int_\Omega |\nabla \times \mathbf{u}|^2\, d\mathbf{x} + \frac{\delta_1}{2}\int_{\Gamma_r} |\nabla_s g|^2 ds + \frac{\delta_2}{2}\int_{\Gamma_r} |g|^2 ds.$$

TABLE 3. $\| T - T_d \|$ with Dirichlet boundary control g ($\| T - T_d \| = 1.0810E - 01$ without control)

δ	10^{-3}	10^{-4}	10^{-5}	10^{-6}
$\delta_2 = 0,\ \delta_1 = \delta$	6.1527E-02	3.6424E-02	1.9880E-02	1.2322E-02
$\delta_1 = \delta_2 = \delta$	6.2822E-02	3.6723E-02	2.1981E-02	1.2764E-02
$\delta_1 = 0,\ \delta_2 = \delta$	3.4562E-02	3.4250E-02	1.4495E-02	1.4472E-02
$\delta_1 = \delta_2 \times 10^{-1},\ \delta_2 = \delta$	3.9235E-02	3.6723E-02	1.2956E-02	1.4175E-02
$\delta_1 = \delta_2 \times 10^{-2},\ \delta_2 = \delta$	2.8527E-02	3.6723E-02	1.4173E-02	1.4448E-02

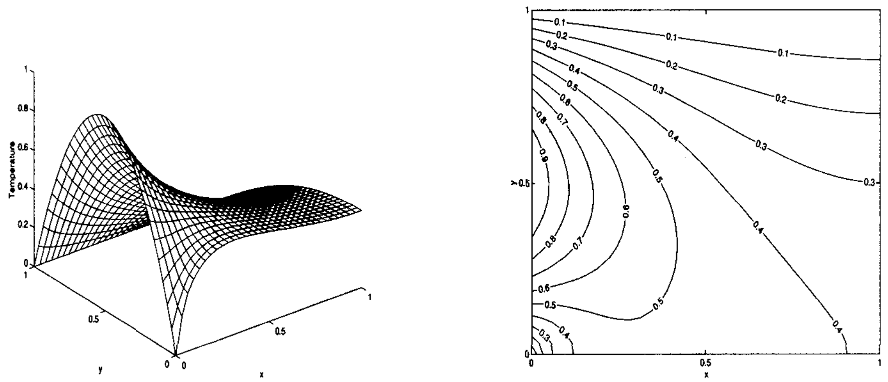

FIGURE 10. Temperature surface plot(left) and contour plot(right) for the uncontrolled prblem

and the state variables satisfy the equations (4.1)-(4.3) with the boundary conditions $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$, $\frac{\partial T}{\partial \mathbf{n}} = 0$ on $\Gamma_b \cup \Gamma_t$, $T = y$ on Γ_l, $T = g_D$ on Γ_r. The uncontrolled solution of this problem is plotted in Figure 6. Table 2 shows that the vorticity of $\mathbf{u}$ decreases as δ goes to 0 and δ_1 is much smaller than δ_2 when $\delta_2 = \delta$ is fixed. We achieved a reduction of 86.6% in the L^2 norm of the velocity when $\delta_1 = 0,\ \delta_2 = \delta = 10^{-8}$. Figure 7 and 8 give the controlled velocity fields $\mathbf{u}^h$ with various values of δ. We plot the optimal boundary controls in Figure 9(right). As expected, we can see the vorticity reduces when the temperature field is flat in Figure 9(left).

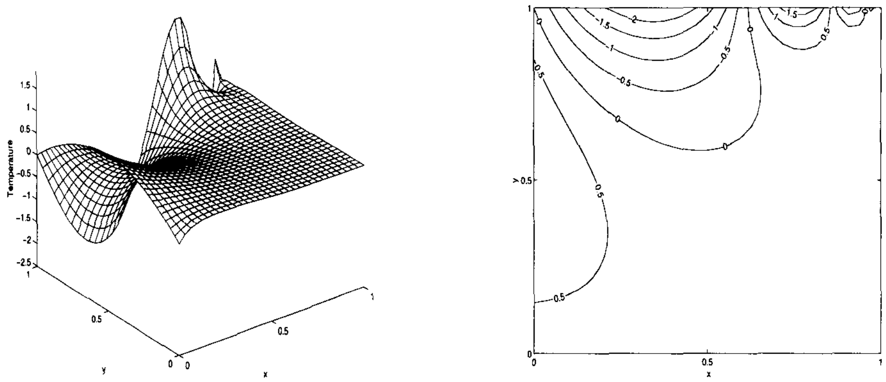

FIGURE 11. Temperature surface plot(left) and contour plot(right) $\delta = \delta_2 = 10^{-6}$ and $\delta_1 = \delta_2 \times 10^{-1}$

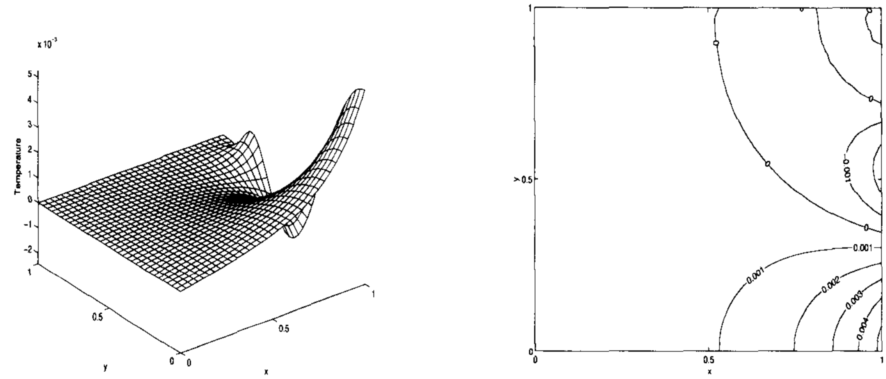

FIGURE 12. Adjoint temperature surface plot and contour plot with $\delta = \delta_2 = 10^{-6}$ and $\delta_1 = \delta_2 \times 10^{-1}$

4.3. The temperature matching problem

Let us consider the minimization problem involving the cost functional (1.5). The cost functional is the following

$$(4.6)\quad \mathcal{J}(\mathbf{u},p,T,g) = \frac{1}{2}\int_{\Gamma_r} |T - T_d|^2 d\mathbf{x} + \frac{\delta_1}{2}\int_{\Gamma_r} |\nabla_s g|^2 ds + \frac{\delta_2}{2}\int_{\Gamma_r} |g|^2 ds.$$

and the boundary conditions are $\mathbf{u} = (4y(1-y), 0)$ on Γ_l, $\mathbf{u} = \mathbf{0}$ on $\Gamma_b \cup \Gamma_t \cup \Gamma_r$, $\frac{\partial T}{\partial \mathbf{n}} = 0$ on $\Gamma_b \cup \Gamma_r$, $T = 4y(1-y)$ on Γ_l, $T = g_D$ on Γ_t. The numerical solution of the uncontrolled problem is shown in Figure 10,

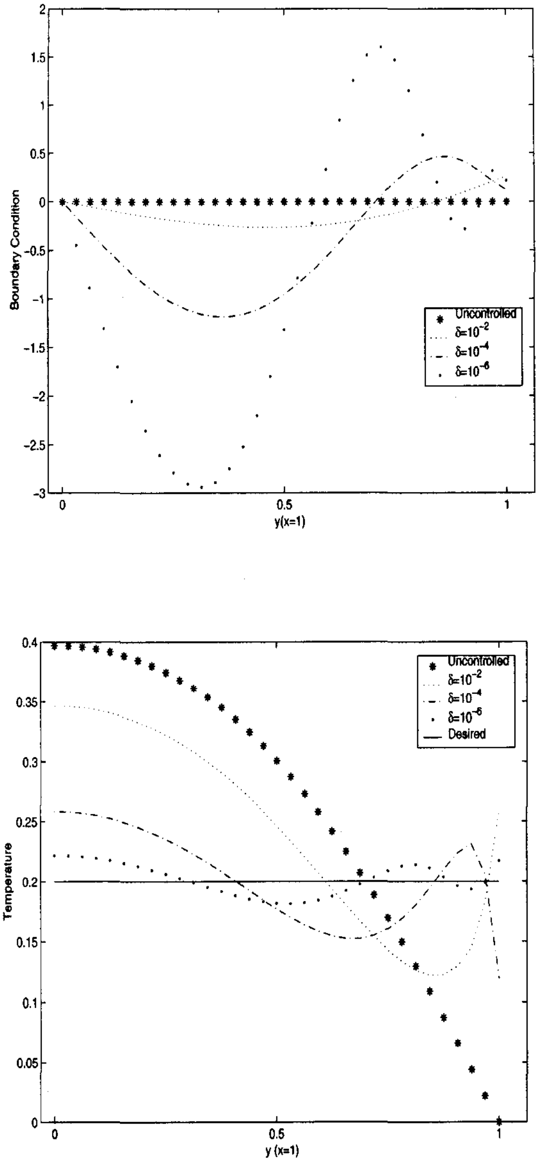

FIGURE 13. Optimal boundary controls on Γ_r(left) and temperature distributions on Γ_r

in which one can see that the temperature is distributed from 0 to 0.4. One can choose any reasonable desired temperature T_d, but we choose the parameter T_d=0.2 which is the average temperature on Γ_r. For the various choices of the parameter δ_1 and δ_2 appearing in the functional (4.6), the computations were performed. We report some numerical results in Table 3. In Figure 11 and 12, we plot the controlled surface and contour plots of the temperature T and adjoint state θ for each case. In Figure 13, we plot the approximate optimal control g^h on the boundary Γ_t and the temperature distribution on Γ_r with $\delta = \delta_1 = \delta_2$.

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Department of Mathematics
Ajou University
Suwon 442-749, Korea
E-mail: hclee@ajou.ac.kr