J. Korean Math. Soc. **41** (2004), No. 4, pp. 747–754

PSEUDO ALMOST PERIODIC SOLUTIONS FOR DIFFERENTIAL EQUATIONS INVOLVING REFLECTION OF THE ARGUMENT*

Daxiong Piao

Abstract. In this paper we investigate the existence and uniqueness of almost periodic and pseudo almost periodic solution for nonlinear differential equation with reflection of argument. For the case of almost periodic forced term , we consider the frequency modules of the solutions.

1. Introduction

The differential equations involving reflection of argument have applications in the study of stability of differential-difference equations, see Sarkovskii [1] , and such equations show very interesting properties by themselves, so many authors worked on them. Wiener and Afidabizadeh [2] initiated to study boundary value problems involving reflection of the argument. Cupta [3, 4] investigated two point boundary value problems for this kind of equations. Afidabizadeh, Huang, and Wiener [5] studied the existence of unique bounded solution of

$$\dot{x}(t) = f(t, x(t), x(-t)). \tag{1.1}$$

They proved that $x(t)$ is almost periodic by assuming the existence of bounded solution $x(t)$ of (1.1). Our present paper is mainly motivated by above reference [5] and Zhang [6], and devoted to investigate the existence and uniqueness of almost periodic and pseudo almost periodic

Received April 30, 2003.
2000 Mathematics Subject Classification: 34C27, 34K13, 34K14.
Key words and phrases: almost periodic solution, pseudo almost periodic solutions differential equations, reflection of the argument, fixed point method.
*Supported by Natural Science Foundation of Liaoning Province (Grant No. 001084), Science Foundation of OUC and National Science Foundation of China (Grant No. 10371010).

solution of the equation

$$\dot{x}(t) + ax(t) + bx(-t) = f(t, x(t), x(-t)), b \neq 0, t \in R, \tag{1.2}$$

where $f(t, x, y)$ is almost periodic or pseudo almost periodic on t uniformly with respect to x and y in any compact subset of R^2. Pseudo almost periodic function (the definition will be given later)is a new generalization of almost periodic function. It was introduced by Zhang [7]. In [7], Zhang also discussed it's applications to some differential equations. After that some literatures discussed pseudo almost periodic solutions for various differential equations, for example [6]-[11] .

To do our main business, we need to consider linear differential equation first

$$\dot{x}(t) + ax(t) + bx(-t) = g(t), b \neq 0, t \in R, \tag{1.3}$$

where $g(t)$ is continuous on R. Let $y(t) = x(-t)$, then (1.3) is changed into system

$$\dot{x} = -ax - by + g(t), \qquad \dot{y} = bx + ay - g(-t), \tag{1.4}$$

which is in a Hamilton system in form with Hamiltonian function

$$H(x, y, t) = \frac{1}{2}bx^2 + \frac{1}{2}by^2 + axy - g(-t)x - g(t)y.$$

So we may say that some first order scalar differential equations can also generate Hamilton systems.

To this end, we give some definitions for our business.

Definition 1.1. [12]-[14] A fuction $f : R \to R$ is almost periodic , if the ϵ–translation of f

$$\mathbf{T}(f, \epsilon) = \{\tau \in R : |f(t + \tau) - f| < \epsilon, \forall t \in R\}$$

is relatively dense in R . We denote the set all such functions by $AP(R)$.

Definition 1.2. [12]-[14] A function $F : R \times R^2 \to R$ is almost periodic for t uniformly on R^2 , if for any compact $W \subset R^2$, the ϵ–translation of F

$$\mathbf{T}(F, \epsilon, W) = \{\tau \in R : |F(t+\tau, x, y) - F(t, x, y)| < \epsilon, \forall (t, x, y) \in R \times W\}$$

is relatively dense in R . We denote the set of all such functions by $AP(R \times R^2)$.

We denote by $PAP_0(R)$ the set

$$\left\{\varphi \in C(R) : \lim_{t \to} \frac{1}{2t} \int_{-t}^{t} |\varphi(s)| ds = 0\right\},$$

and by $PAP_0(R \times R^2)$ the set

$$\left\{\varphi \in C(R \times R^2) : \lim_{t\to} \frac{1}{2t} \int_{-t}^{t} |\varphi(s,x,y)| ds = 0, \forall (x,y) \in \Omega, \ \forall \Omega \subset R^2 \right\}.$$

where Ω is compact in R^2.

DEFINITION 1.3. [6, 7] A function $F : R \times R^2 \to R$ is called pseudo almost periodic for t uniformly on R^2 if

$$F = G + \Phi$$

where $G \in AP(R \times R^2), \Phi \in PAP_0(R \times R^2)$. Denote by $PAP(R \times R^2)$ the set of all such functions .

2. Main results

Our main results can be stated as follows.

THEOREM 2.1. *For any $g(t) \in PAP(R), \lambda^2 = a^2 - b^2, \lambda > 0$, Eq. (2.1) has a unique pseudo almost periodic solution $x(t)$. Furthermore if $g(t) \in AP(R)$, then Eq. (1.3) has a unique almost periodic solution $x(t)$ and* $\mod(x) = \mod(g)$.

THEOREM 2.2. *Suppose $f(t,x,y) \in PAP(R \times R^2)$ and satisfies Lipschitz condition*

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \le L(|x_1 - x_2| + |y_1 - y_2|)$$

for any $(x_1,y_1),(x_2,y_2) \in R^2$, where $L < \frac{2\lambda^2}{|\lambda-a|+|\lambda+a|+2|b|}, \lambda^2 = a^2 - b^2, \lambda > 0$. Then Eq. (1.1) has a unique pseudo almost periodic solution $x(t)$.In addition , if $f(t,x,y) \in AP(R\times R^2)$, then Eq. (1.1) has a unique almost periodic solution $x(t)$,and $\mod(x) = \mod(f)$.

Before proving above two theorems, we state a useful lemma which can be easily proven.

LEMMA 2.3. *If $g(t) \in AP(R)$, then $g(-t) \in AP(R)$. If $g(t) \in PAP(R)$, then $g(-t) \in PAP(R)$. Furthermore if $g(t) \in AP(R)$ and τ is an ϵ-translation of $g(t)$, then τ is also an ϵ-translation of $g(-t)$, and* $\mod(g(t)) = \mod(g(-t))$.

We refer the readers to good books [12]-[14] for the basic results on the almost periodic functions.

Proof of theorem 2.1. Existence. From Lemma 2 and Lemma 3 of [5], we can derive

$$\begin{aligned} x(t) &= -\frac{1}{2\lambda}\left[e^{\lambda t}\int_t^{\infty} e^{-\lambda s}((\lambda-a)g(s)+bg(-s))ds\right] \\ &\quad +\frac{1}{2\lambda}\left[e^{-\lambda t}\int_{-\infty}^{t} e^{\lambda s}((\lambda+a)g(s)-bg(-s))ds\right] \end{aligned}$$

is a particular solution of Eq.(1.3) for any $g(t) \in PAP(R)$. Now we show $x(t) \in PAP(R)$. Suppose $g(t) = h(t) + \varphi(t), h(t) \in AP(R), g(t) \in PAP_0(R)$. Let

$$\begin{aligned} H(t) = &-\frac{1}{2\lambda}\left[e^{\lambda t}\int_t^{\infty} e^{-\lambda s}((\lambda-a)h(s)+bh(-s))ds\right] \\ &+\frac{1}{2\lambda}\left[e^{-\lambda t}\int_{-\infty}^{t} e^{\lambda s}((\lambda+a)h(s)-bh(-s))ds\right] \end{aligned}$$

and

$$\begin{aligned} \Phi(t) = &-\frac{1}{2\lambda}\left[e^{\lambda t}\int_t^{\infty} e^{-\lambda s}((\lambda-a)\varphi(s)+b\varphi(-s))ds\right] \\ &+\frac{1}{2\lambda}\left[e^{-\lambda t}\int_{-\infty}^{t} e^{\lambda s}((\lambda+a)\varphi(s)-b\varphi(-s))ds\right] \end{aligned}$$

then $x(t) = H(t) + \Phi(t)$. For $\tau \in T(h(t), \epsilon)$, we have

$$\begin{aligned} &|H(t+\tau)-H(t)| \\ = &\left|-\frac{1}{2\lambda}\left[e^{\lambda(t+\tau)}\int_{t+\tau}^{\infty} e^{-\lambda s}((\lambda-a)h(s)+bh(-s))ds\right]\right. \\ &+\frac{1}{2\lambda}\left[e^{-\lambda(t+\tau)}\int_{-\infty}^{t+\tau} e^{\lambda s}((\lambda+a)h(s)-bh(-s))ds\right] \\ &+\frac{1}{2\lambda}\left[e^{\lambda t}\int_t^{\infty} e^{-\lambda s}((\lambda-a)h(s)+bh(-s))ds\right] \\ &\left.-\frac{1}{2\lambda}\left[e^{-\lambda t}\int_{-\infty}^{t} e^{\lambda s}((\lambda+a)h(s)-bh(-s))ds\right]\right| \end{aligned}$$

$$
\begin{aligned}
= & \left|-\frac{1}{2\lambda}\left[\int_t^{\infty} e^{\lambda(t-s)}((\lambda-a)h(s+\tau)+bh(-(s+\tau))ds\right]\right. \\
& +\frac{1}{2\lambda}\left[\int_t^{\infty} e^{\lambda(t-s)}((\lambda-a)h(s)+bh(-s))ds\right] \\
& +\frac{1}{2\lambda}\left[\int_{\infty}^{t} e^{\lambda(s-t)}((\lambda+a)h(s+\tau)-bh(-(s+\tau)))ds\right] \\
& \left.-\frac{1}{2\lambda}\left[\int_{-\infty}^{t} e^{\lambda(s-t)}((\lambda+a)h(s)-bh(-s))ds\right]\right| \\
= & \left|-\frac{1}{2\lambda}\int_t^{\infty} e^{\lambda(t-s)}\left[((\lambda-a)(h(s+\tau)-h(s))\right.\right. \\
& \left.\left.+b(h(-(s+\tau)-h(-s))\right]ds\right| \\
& +\left|-\frac{1}{2\lambda}\int_{-\infty}^{t} e^{\lambda(t-s)}\left[((\lambda+a)(h(s+\tau)-h(s))\right.\right. \\
& \left.\left.+b(h(-(s+\tau)-h(-s))\right]ds\right| \\
\leq & \frac{1}{2\lambda}\int_t^{\infty} e^{\lambda(t-s)}(|\lambda-a|+|b|)\epsilon ds+\frac{1}{2\lambda}\int_{-\infty}^{t} e^{\lambda(s-t)}(|\lambda+a|+|b|)\epsilon ds \\
\leq & \frac{|\lambda-a|+|\lambda+a|+2|b|}{2\lambda^2}\epsilon.
\end{aligned}
$$

So $H(t)\in AP(R)$, and $\operatorname{mod}(H)\subset \operatorname{mod}(h)$.

On the other hand

$$
\begin{aligned}
& \frac{1}{2T}\int_{-T}^{T}|\Phi(t)|dt \\
\leq & \frac{1}{4|\lambda|T}\int_{-T}^{T}dt\int_t^{\infty} e^{\lambda(t-s)}(|\lambda-a||\varphi(s)|+|b||\varphi(-s)|)ds \\
& +\frac{1}{4|\lambda|T}\int_{-T}^{T}dt\int_{-\infty}^{t} e^{\lambda(s-t)}(|\lambda+a||\varphi(s)|+|b||\varphi(-s)|)ds \\
= & \frac{1}{4|\lambda|T}\int_{-T}^{T}ds\int_{-T}^{s} e^{\lambda(t-s)}(|\lambda-a||\varphi(s)|+|b||\varphi(-s)|)dt \\
& +\frac{1}{4|\lambda|T}\int_{T}^{\infty}ds\int_{-T}^{T} e^{\lambda(t-s)}(|\lambda-a||\varphi(s)|+|b||\varphi(-s)|)dt
\end{aligned}
$$

$$+\frac{1}{4|\lambda|T}\int_{-T}^{T}ds\int_{s}^{T}e^{\lambda(s-t)}(|\lambda+a||\varphi(s)|+|b||\varphi(-s)|)dt$$
$$+\frac{1}{4|\lambda|T}\int_{-\infty}^{-T}ds\int_{-T}^{T}e^{\lambda(s-t)}(|\lambda+a||\varphi(s)|+|b||\varphi(-s)|)dt.$$

For convenience, we denote the last four terms by I_1, I_2, I_3 and I_4 respectively. Since $\frac{1}{2T}\int_{-T}^{T}|\varphi(t)|dt = 0$, and $\varphi(t)$ is continuous and bounded on R,

$$\begin{aligned} I_1 &= \frac{1}{4|\lambda|T}\int_{-T}^{T}e^{-\lambda s}(|\lambda-a||\varphi(s)|+|b||\varphi(-s)|)ds\int_{-T}^{s}e^{\lambda t}dt \\ &= \frac{1}{4|\lambda|\lambda T}\int_{-T}^{T}(|\lambda-a||\varphi(s)|+|b||\varphi(-s)|)ds \\ &\quad -\frac{e^{-\lambda T}}{4|\lambda|\lambda T}\int_{-T}^{T}e^{-\lambda s}(|\lambda-a||\varphi(s)|+|b||\varphi(-s)|)ds \\ &\to 0, \ as\ T\to\infty \end{aligned}$$

and

$$\begin{aligned} I_2 &= \frac{1}{4|\lambda|T}\int_{T}^{\infty}e^{-\lambda s}(|\lambda-a||\varphi(s)|+|b||\varphi(-s)|)ds\int_{-T}^{T}e^{\lambda t}dt \\ &= \frac{1-e^{-2\lambda T}}{4|\lambda|\lambda T}\int_{T}^{\infty}e^{\lambda(T-s)}(|\lambda-a||\varphi(s)|+|b||\varphi(-s)|)ds \\ &\to 0 \ as\ T\to\infty. \end{aligned}$$

Similarly we can show $I_3, I_4 \to 0$ as $T\to\infty$.

So $x(t)$ is pseudo almost periodic solution of (2.1).

Uniqueness. If there is another pseudo almost periodic solution $x_1(t)$ for Eq.(2.1), then the difference $x(t)-x_1(t)$ should be a solution of the homogeneous equation

$$(2.1)\qquad \dot{x}(t)+ax(t)+bx(-t)=0, b\neq 0, t\in R.$$

According to the Lemma 2 of [2], we can derive

$$(2.2)\qquad x(t)-x_1(t)=C\left(\frac{\lambda-a}{b}e^{\lambda t}+e^{-\lambda t}\right), t\in R,$$

for some constant C. If $C\neq 0$, then $x(t)-x_1(t)$ will be unbounded. This is a contradiction to the boundedness of pseudo almost periodic function. So $x(t)-x_1(t)\equiv 0$, i.e $x(t)\equiv x_1(t)$.

If $g(t)$ is almost periodic, then $g(t)=h(t)$, $\varphi(t)=0$, and so $\Phi(t)=0$. In this case $x(t)=H(t)$ is the unique almost periodic solution of

Eq.(1.3). From $g(t) = \dot{x}(t) + ax(t) + bx(-t)$ and the lemma 2.3 we conclude $\bmod (g) \subset \bmod (x)$, and so $\bmod (x) = \bmod (g)$. $\square$

Proof of theorem 2.2. We know $PAP(R)$ is a Banach space (cf. [11]) with the supremum norm $||\phi|| = \sup_{t\in R} |\phi(t)|$. For any $\phi \in PAP(R)$, from Lemma 2.1 and [Theorem 1.5, 6], we see $f(t, \phi(t), \phi(-t)) \in PAP(R)$. According to the theorem 2.1 we see the equation

$$(2.3) \qquad \dot{x}(t) + ax(t) + bx(-t) = f(t, \phi(t), \phi(-t)), b \neq 0, t \in R$$

has a unique pseudo almost periodic solution, denote it by $(T\phi)(t)$. Then we define a mapping $T : PAP(R) \to PAP(R)$. Now we show T is contracted.

For $\phi(t), \psi(t) \in PAP(R)$, the equation

$$(2.4) \qquad \dot{x}(t) + ax(t) + bx(-t) = f(t, \phi(t), \phi(-t)) - f(t, \psi(t), \psi(-t)), b \neq 0, t \in R$$

has a unique pseudo almost periodic solution $(T\phi - T\psi)(t)$, and

$$\begin{aligned}
&(T\phi - T\psi)(t) \\
= \; & -\frac{1}{2\lambda}\int_t^{\infty} e^{\lambda(t-s)}((\lambda - a)\,[f(s, \phi(s), \phi(-s)) - f(s, \psi(s), \psi(-s))] \\
& +b\,[f(-s, \phi(-s), \phi(s)) - f(-s, \psi(-s), \psi(s))]\,ds \\
& +\frac{1}{2\lambda}\int_{-\infty}^{t} e^{\lambda(s-t)}((\lambda + a)\,[f(s, \phi(s), \phi(-s)) - f(s, \psi(s), \psi(-s))] \\
& -b\,[f(-s, \phi(-s), \phi(s)) - f(-s, \psi(-s), \psi(s))]\,ds.
\end{aligned}$$

So

$$\| T\phi - T\psi \| \leq \frac{|\lambda - a| + |\lambda + a| + 2|b|}{2\lambda^2} L \, \| \phi - \psi \| .$$

Since $\frac{|\lambda-a|+|\lambda+a|+2|b|}{2\lambda^2} L < 1$, T is a contraction mapping, and so T has a unique fixed point in $PAP(R)$. That is to say the equation (1.2) has a unique pseudo almost periodic solution $x(t)$.

If $f(t, x, y) \in AP(R \times R^2)$, then for any

$$\phi(t) \in AP(R), f(t, \phi(t), \phi(-t)) \in AP(R)$$

too. The subset

$$B = \{\phi(t) : \phi \in AP(R), \quad \bmod (\phi) \subset \quad \bmod (f)\}$$

of $AP(R)$ is a Banach space with supremum norm $||\phi|| = \sup_{t\in R} |\phi(t)|$ (cf. [12]). From theorem 2.1, we conclude, for any $\phi(t) \in B$, Eq.(2.4) has a unique almost periodic solution $T\phi \in B$. We can easily prove as above that T is contracted. So T has a unique fixed point $x(t) \in B$, i.e. there is a unique harmonic solution for Eq.(1.1). From $f(t, x(t), x(-t) =$

$\dot{x} + ax(t) + bx(-t)$ and lemma 2.3, we conclude $\bmod (f) \subset \bmod (x)$, so $\bmod (x) = \bmod (f)$. The proof of theorem 2.2 is completed. □

References

[1] A. N. Sharkovskii, *Functional-differential equations with a finite group of argument transformations in Asymptotic Behavior of Solutions of Functional-Differential Eqautions*, Akad. Nauk Ukrain., Inst. Mat., Kiev, (1978), 118–142.

[2] A. R. Afidabizadeh and J. Wiener, *Boundary value problems for differential equations with reflection of argument*, Int. J. Math. Math.Sci. **8** (1985), 151–163.

[3] C. P. Cupta, *Existence and uniquness theorem for boundary value problems involving reflection of the argument*, Nonlinear Anal., **11** (1987), 1075–1083.

[4] ———, *Two point boundary value problems involving reflection of the argument*, Int. J. Math. Math. Sci. **10** (1987), 361–371.

[5] A. R. Afidabizadeh, Y. K. Huang, and J. Wiener, *Bounded solutions for Differential Equations with Reflection of the Argument*, J. Math. Anal. Appl. **135** (1988), 31–37

[6] C. Zhang, *Pseudo almost periodic solutions of some differential equations*, J. Math. Anal. Appl. **181** (1994), 62–76.

[7] ———, *Pseudo almost periodic functions and their applications*, thesis, the University of Western Ontario, 1992.

[8] E. Ait Dads and O. Arino, *Exponential dichotomy and existence of pseudo almost periodic solutions of some differential equations*, Nonlinear Anal. **27** (1996), 369–386.

[9] E. Ait Dads, K. Ezzinbi and O. Arino, *Pseudo Almost Periodic Solutions for Some Differential Equations in a Banach Space*, Nonlinear Anal. **28** (1997), 1141–1155.

[10] D. Piao, *Pseudo almost periodic solutions for the systems of differential equations with piecewise constant argument* [t], Sci. China Ser. A **44** (2001), 1156–1161.

[11] D. Piao and R. Yuan, *Pseudo almost periodic solutions of differential equations with piecewise constant argument*, Chinese Ann. Math. **20B** (1999), 489–494.

[12] A. M. Fink, *Almost Periodic Differential Equations*, Lecture Notes in Math. 377, Springer-Verlag, 1974.

[13] C. Corduneanu, *Almost Periodic Functions*, Second Edition, Chelsea Publ. Comp., New York, 1989.

[14] B. M. Levitan and V. V. Zhikov, *Almost periodic differential functions and differential equations*, Combridge: Combridge University Press, 1982.

Department of Mathematics
Ocean University of China
Qingdao, 2660071, China
E-mail: davidpiao@yahoo.com