Commun. Korean Math. Soc. **19** (2004), No. 3, pp. 577–584

EDGE PROPERTIES OF THE 4-VALENT MULTI 3-GON GRAPHS

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ABSTRACT. In a 4-valent multi 3-gon graph, every cut-through curve forms a simple closed circuit. Hence it is a weak arrangement of simple curves that is defined by Branko Grünbaum. In this paper, we study the edge properties of the 4-valent multi 3-gon graphs from the point of view of arrangement, and we show that they are 3 colorable.

1. Introduction

In this section, we'd like to introduce some definitions and theorems briefly. For the definitions that are not mentioned here, see the book by J. A. Bondy and U. R. Murty [1].

J. Malkevitch defined the "coded path" for the n-valent graphs and got many fascinating results [3]. For a 4-valent plane graph, that is the main object of our study, we define two kinds of coded path which are the most interesting ones. Suppose that a path reaches to the vertex u via the edge e (see Figure 1). Then, there are three edges e_1, e_2 and e_3 which can be the next edge of the path. We will call the edges e_1, e_2 and e_3 the left edge, the middle edge, and the right edge of the edge e, respectively. If a path always chooses the middle edge as the next edge, we call such a path a **cut-through path**. On the other hand, if a path chooses the left edge and the right edge alternately, we call it a **left-right path**. A right-left path is defined similarly.

Let G be a plane graph, and let $p_n(G)$ (or simply p_n) be the number of faces with n sides in G. We call a sequence of $\{p_n\}$ a **p-vector** or a **face vector** of G. We call a face F in G that has a multiple of k sides

Received October 17, 2003.
2000 Mathematics Subject Classification: 05.
Key words and phrases: arrangement of curves, multi 3-gon graphs, vertex coloring.
This work is supported by Soongsil University research fund.

a **multi k-gonal face**. If all the faces in G are multi k-gonal faces, we call it a **multi k-gon graph**. That is, $p_k = 0,\ k \not\equiv 0 \pmod 3$. Suppose that all the faces in a graph G are multi k-gonal faces except a few faces. Such a few faces are called **exceptional faces**.

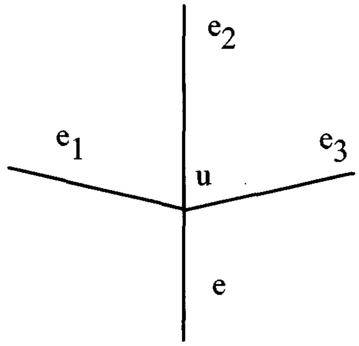


Figure 1. A cut-through path

It is a good place to state two theorems due to J. Malkevitch. For the proofs and more theorems, see J. Malkevitch [3].

THEOREM 1. *There is no 4-valent multi 3-gon graph with two adjacent exceptional faces.*

THEOREM 2. *Let G be a 4-valent multi 3-gon graph. Then every cut-through path and every left-right path form a simple closed circuit.*

J. Malkevitch also suggested a generalization of the arrangement of simple curves defined by B. Grünbaum [2, 4]. When two curves meet at a point, they either intersect or osculate. If two curves intersect at a point, we call it an **intersection point**. If two curves osculate at a point, we call it a **kissing point** or an **osculation point**. We will use the term"point" and "vertex" interchangeably throughout this paper.

A **generalized arrangement** Λ **of** n **simple curves** $\{C_1, C_2, \ldots, C_n\}$ in the Euclidean plane E^2 is a finite family of simple closed curves with the following properties:

1. every pair of curves has exactly t intersection points($t > 0$ is even) and exactly k kissing points in common,
2. exactly two curves meet at each point(or vertex).

From this definition, it is obvious that every curve has the same number of vertices $r = (n-1)(t+k)$. Hence, we will denote this arrangement by a (n, r, t, k)-**arrangement of curves**. **A generalized weak arrangement of curves** is defined by just replacing the word "exactly" with "at most" in the first condition of the above definition.

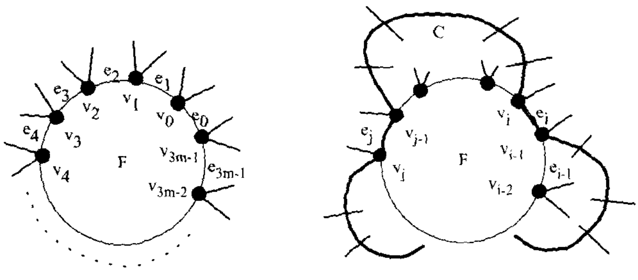
Since the number of the vertices in each curve are not the same, we will denote it by a $(n, -, t, k)$-**arrangement of curves**.

REMARK 1. Since every cut-through path forms a simple closed curve, we may view a 4-valent multi 3-gon graph as a weak arrangements of simple curves. And, it is a $(n, -, t, 0)$ arrangement of curves.

The graph of an arrangement of curves Λ, denoted by G_Λ, is a plane 4-valent graph whose vertices are common points of two curves and whose edges are the segments of curves between every pair of adjacent points(perhaps, with multiple edges). Thus a curve in the arrangement Λ is a cut-through circuit in G_Λ. From now on, when we mention a curve in a graph G, it is a curve formed by a cut-through path.

2. Main results

Let G be a 4-valent multi 3-gon graph. Then every cut-through path form a simple closed circuit. Thus we can view a 4-valent multi 3-gon graph as a generalized weak arrangement of simple curves with no osculation point. Let F be a face in G. Two sides e, e' of a face F are said to be in **multi-3 distance** if either they are the same side or there are $3k - 1$ sides between them for some positive integer k.



(a) (b)

Figure 2. The face F and the curve C

THEOREM 3. *Let G be a 4-valent multi 3-gon graph and let $e_0, e_1, \cdots, e_{3m-1}$ be sides of a face F in G labeled in counterclockwise order (see Figure 2(a)). Then, the followings hold:*

1. *If two sides e_i and e_j are in the same curve, then they are in multi-3 distance.*
2. *If $i \equiv j \pmod 3$, then either e_i and e_j are contained in the same curve or e_i and e_j are contained in two disjoint curves one for each.*

PROOF. 1. Let F be a face in G whose sides are $e_0, e_1, \cdots, e_{3m-1}$ and let v_i be the vertex that is the common endpoint of sides e_i and e_{i+1} (Figure 2 (a)).

First, we consider the case that all the sides $e_{i+1}, \ldots, e_{j-1}$ are not contained in the curve C. We assume that $0 \leq i < j \leq 3m-1$ without loss of generality(if it is necessary, relabel the edges).

Since e_i and e_j are contained in the simple curve C, two vertices v_{j-1} and v_i must be connected by the curve C as in Figure 2 (b). That is, no edges between e_i and e_j of the curve C are the edges of the face F.

Now, join two vertices v_i and v_{j-1} with an edge e' and cut out the subgraph which is surrounded by the edge e' and the part of the curve C from the vertex v_i to the vertex v_{j-1} (a thick line in Figure 3 (a)). Let G' denote such a subgraph (Figure 3 (b)) and let F' be the face that is a part of the face F in G'. Then the number of sides of F' is $(j-1)-i+1 = j-i$ in G'.

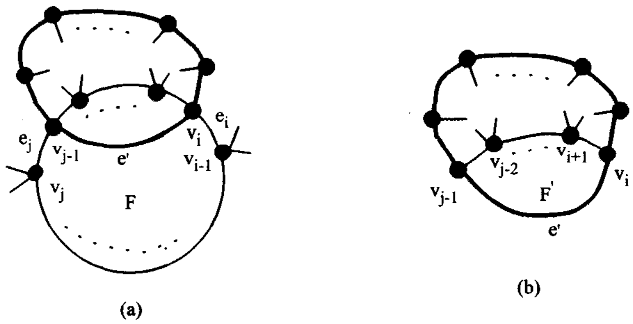


Figure 3. A subgraph G'

In the graph G', all the vertices on the boundary are 3-valent. Now, imbed G' on a half-sphere using the thick line as a great circle (Figure 4 (a)), and imbed the mirror image of the graph G' on the other side of the sphere so that two corresponding vertices on the great circle become one vertex (Figure 4 (b)). Hence we have a 4-valent graph on the sphere which is isomorphic to a 4-valent planar graph. In this graph, there are two adjacent $(j-i)$-gonal faces while all the other faces are multi 3-gon. If $j-i \not\equiv 0 \pmod 3$, then we have a 4-valent multi 3-gon graph that contains two adjacent exceptional faces, which contradicts to the Theorem 1. Therefore, $j-i \equiv 0 \pmod 3$.

For the general case, suppose that $e_{i_0}, e_{i_2}, \cdots, e_{i_k}$, $i = i_0 < i_1 < i_2 < \cdots < i_{k-1} < i_k = j$ are all the edges contained in the curve C between e_i and e_j. Then we know that $i_r \equiv i_{r+1} \pmod 3$ for all $r = 0, 1, \ldots, k-1$ by the previous result. Hence we have that $i \equiv j \pmod 3$.

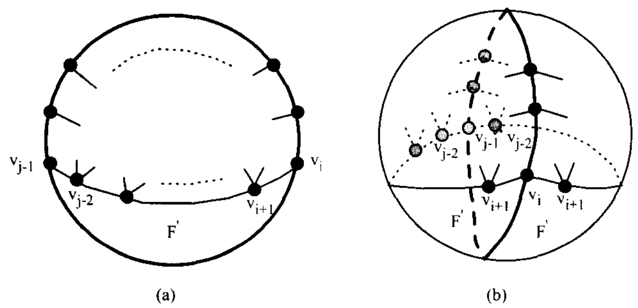


Figure 4. Imbedding the graph G'

2. Assume the contrary, that is, there are two different curves C_i and C_j containing the edges e_i and e_j, respectively and intersect each other at the vertex u. Let P_1 be a path from the vertex u to a vertex v_k on the side of the face F along the curve C_i and let P_2 be a path from the vertex v_{l-1} on the side of the face F to the vertex u along the curve C_j. We may assume that $k < l-1$ and that no other vertices inside the circuit $u,\ P_1,\ v_k, v_{k+1}, \ldots, v_{l-2}, v_{l-1}, P_2, u$ are the intersection point of two curves C_i and C_j(otherwise, choose an intersection point inside of the circuit instead of u). Then we can have a configuration shown in Figure 5 (a). Now, join two vertices v_k and v_{l-1} by the edge e' and consider the subgraph H which is inside the circuit $S = u,\ P_1,\ v_k, e', v_{l-1}, P_2, u$. Note that the degree of the vertex u is 2 while all the other vertices on the boundary of the subgraph H are 3-valent. Let F' be the face which are not the infinite face of H but contains the vertex u and let F'' be the face that is a part of the face F and is inside of the circuit S (Figure 5 (b)).

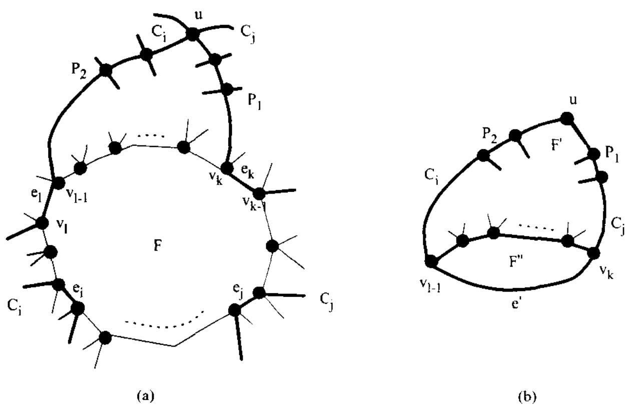


Figure 5. A subgraph H

Since the edges e_i and e_k are contained in the curve C_i, and the edges e_j and e_l are contained in the curve C_j, we have that $i \equiv k \pmod 3$ and $j \equiv l \pmod 3$ by the previous result. Since $i \equiv j \pmod 3$, we know that $k \equiv l \pmod 3$, that is, the face F'' is multi 3-gon.

Now, make two edges having the vertex u in H one edge by just removing the vertex u, then the face F' becomes a $(3n-1)$-gon for some positive integer n. Using the same method applied in the proof of 1, we have a 4-valent multi 3-gon graph on the sphere with two exceptional faces F'. This contradicts to the Theorem 1. Hence two curves C_i and C_j must be disjoint. This completes the proof. □

REMARK 2. The converse of fact 2 in Theorem 3 is not true. There are two edges sharing the same face and contained in two disjoint curves one for each, but not in multi 3 distance.

Let Λ be the family of cut-through curves $\{C_1, C_2, \ldots, C_n\}$ in a 4 valent multi 3-gon graph G. We say that two curves C_i and C_j containing two edges e_i and e_j one for each are in multi-3 distance if two edges e_i and e_j are in multi-3 distance. We define a relation in Λ as following: a curve C_i has a relation with a curve C_j if there is a sequence of curves $\{C_i = C_{i_0}, C_{i_1}, \ldots, C_{i_k}, C_{i_k} = C_j\}$ such that C_r and C_{r+1}, $r = 0, 1, \ldots, k-1$ are in multi-3 distance.

REMARK 3. Suppose that two curves share a face with edges in a multi-3 distance. If they share another face with edges e and e', then e and e' are also in multi-3 distance. Otherwise, we are able to construct a graph with two adjacent exceptional faces by applying the same method used in the proof of Theorem 3.

Hence, this relation is well defined and it is an equivalence relation so that we have three disjoint equivalence classes. Let Λ_0, Λ_1, and Λ_2 be such equivalence classes.

COROLLARY 4. *Let G be a 4-valent multi 3-gon graph. Let Λ_0, Λ_1, and Λ_2 be equivalence classes defined as above. Then every curve C never cuts through the curves in the same equivalence class containing C. Moreover, it never cuts through the curves in the same equivalence class consecutively.*

PROOF. Since every curve C is a simple curve and the curves in the same equivalence class are disjoint, it never cuts through itself and other curves in the same class. Suppose that C cuts through the curves C_1, C_2 that are in the same equivalence class consecutively (clearly, the class containing C_1 and C_2 is different from the class containing C, Figure 6). Then there is only one side between e and e', that is, e and e' are not in multi-3 distance that contradicts to the fact that two curves are in the same equivalence class. Thus it must cut through the curves in other two equivalence classes in alternating manner. □

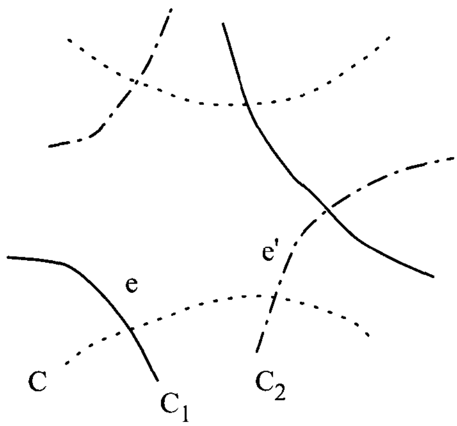


Figure 6. cut through in alternating manner

COROLLARY 5. *Let G_Λ be the graph of a weak arrangement of simple curves Λ. Suppose that there is a partition, Λ_0, Λ_1, and Λ_2 satisfying the following property:*

- *every curve C cuts through the curves in the other two equivalence classes not containing C in an alternating way. In another words, it never cuts through the curves in the same equivalence class consecutively.*

Then G is a multi 3-gon graph.

PROOF. Let F be a face in G and let $e_0, e_1, e_2, \cdots, e_{k-1}$ be sides of F labeled in counterclockwise order. Assume that e_0 is in a curve in the class Λ_0 and e_1 is in a curve in the class Λ_1. Since every curve cut through the curves in other equivalence classes in alternating manner, e_2 must be in a curve in the class Λ_2 and e_3 must be in a curve in the class Λ_0. Continuing this argument, we know that two edges in multi-3 distance are contained in the curves which are in the same equivalence class. Hence $k \equiv 0 \pmod 3$. □

Since every 4-valent multi-3 gon graph has special properties stated above, it is also easy to determine a vertex coloring.

COROLLARY 6. *Every 4-valent multi-3 gon graph is 3-colorable.*

PROOF. Every vertex is an intersection point of two curves each of which contained in two different equivalence classes, respectively. Label each vertex with 1, 2 and 3 if it is an intersection point of the curves in the classes $\{\Lambda_0, \Lambda_1\}$, $\{\Lambda_0, \Lambda_2\}$ and $\{\Lambda_1, \Lambda_2\}$, respectively. Now assign the same color to vertices of the same label. Then it is a proper coloring because every curve never cut through the curves in the same equivalence class consecutively, that is, the label of the vertex is different from 4 adjacent vertices. Hence it is 3-colorable. □

For further study, we may ask following questions:

- Is there a way to determine the maximum or the minimum number of cut-through curves in a 4-valent multi 3-gon graphs?
- For the 4-valent graphs in which many 4-gons and many multi 3-gons reside, is it possible to find some properties similar to the previous facts?
- What can we say for the left-right path in a 4-valent multi 3-gon graph?

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