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# POSTNIKOV SECTIONS AND GROUPS OF SELF PAIR HOMOTOPY EQUIVALENCES

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ABSTRACT. In this paper, we apply the concept of the group $\mathcal{E}(X, A)$ of self pair homotopy equivalences of a CW-pair $(X, A)$ to the Postnikov system. By using a short exact sequence related to the group of self pair homotopy equivalences, we obtain the following result: for any Postnikov section $X_n$ of a CW-complex $X$, the group $\mathcal{E}(X_n, X)$ of self pair homotopy equivalences on the pair $(X_n, X)$ is isomorphic to the group $\mathcal{E}(X)$ of self homotopy equivalences on $X$. As a corollary, we have, $\mathcal{E}(K(\pi, n), M(\pi, n)) \equiv \mathcal{E}(M(\pi, n))$ for each $n \geq 1$, where $K(\pi, n)$ is an Eilenberg-Mclane space and $M(\pi, n)$ is a Moore space.

## 1. Introduction

If $X$ is a based topological space, let $\mathcal{E}(X)$ denote the set of homotopy classes of self homotopy equivalences of $X$. Then $\mathcal{E}(X)$ is a group with group operation given by the composition of homotopy classes. The group $\mathcal{E}(X)$ is a fundamental object in the homotopy theory and has been studied extensively by several authors; for instances, M. Arkowitz [1], K. Maruyama [6], J. Rutter [7], N. Sawashita [8] and A. Sieradski [9], et al..

Let $\mathcal{E}(X, A)$ denote the set of pair homotopy classes of self pair homotopy equivalences of a CW-pair $(X, A)$. Then it is a group, a homotopy invariant and this concept is a generalization of that of the group $\mathcal{E}(Y)$ for a CW-complex $Y$. Moreover, for a CW-pair $(X, A)$, there exists a exact sequence

$$(1) \qquad 1 \to \mathcal{E}(X, A; id_A) \to \mathcal{E}(X, A) \to \mathcal{E}(A),$$

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where $\mathcal{E}(X, A; id_A)$ is the subgroup of $\mathcal{E}(X, A)$ which consists of the pair homotopy classes of the self pair homotopy equivalences such that the restriction to $A$ is the identity on $A$ ([5]). In this paper, we show that for a pair $(X_n, X)$, the sequence (1) becomes a split short exact sequence, where $X_n$ be the $n$-th Postnikov section of a CW-complex $X$. We also show that $\mathcal{E}(X_n, X; id_X)$ is trivial. By the exactness, we obtain the following main results:

THEOREM. *Let $X$ be a CW-complex and $\{X_n\}$ the Postnikov system of $X$. Then for any section $X_n$, the group $\mathcal{E}(X_n, X)$ is isomorphic to $\mathcal{E}(X)$.*

COROLLARY. *For each $n \geq 1$, $\mathcal{E}(K(\pi, n), M(\pi, n))$ is isomorphic to $\mathcal{E}(M(\pi, n))$, where $K(\pi, n)$ is an Eilenberg-Mclane space and $M(\pi, n)$ is a Moore space.*

## 2. The groups of self pair homotopy equivalences and certain exact sequences

In this section, we will introduce some definitions and some theorems in [5] with brief proofs, which are needed to develop our assertion.

In the category of pairs, the "objects" are maps $(X_1, *) \to (X_2, *)$ and "morphism" from $\alpha : X_1 \to X_2$ to $\beta : Y_1 \to Y_2$ is a pair of maps $(f_1, f_2)$ such that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\alpha} & X_2 \\ f_1 \downarrow & & \downarrow f_2 \\ Y_1 & \xrightarrow[\beta]{} & Y_2 \end{array}$$

is commutative, i.e., $\beta f_1 = f_2 \alpha$. A homotopy of $(f_1, f_2)$ is just a pair of homotopies $(f_{1t}, f_{2t})$ such that $\beta f_{1t} = f_{2t}\alpha$. This category reduces to the category of ordinary pairs of spaces (with base point) if we restrict ourselves to maps $\alpha$ which are inclusions. If $(f_1, f_2)$ is homotopic to $(g_1, g_2)$ by the homotopy $(f_{1t}, f_{2t})$, we denote by

$$(f_{1t}, f_{2t}) : (f_1, f_2) \simeq (g_1, g_2).$$

We denote by $[f_1, f_2]$ the homotopy class of the morphism $(f_1, f_2) : \alpha \to \beta$ and by $\Pi(\alpha, \beta)$ the set of all homotopy classes from $\alpha$ to $\beta$. $(f_1, f_2)$ is called a *homotopy equivalent morphism*, or simply a *homotopy equivalence* if there is a morphism $(g_1, g_2)$ such that $(g_1, g_2) \circ (f_1, f_2) \simeq (id_{X_1}, id_{X_2})$ and $(f_1, f_2) \circ (g_1, g_2) \simeq (id_{Y_1}, id_{Y_2})$. Such morphism $(g_1, g_2)$ is called a *homotopy inverse* of $(f_1, f_2)$. Furthermore, $(f_1, f_2)$ is called a *self homotopy equivalent morphism*, or simply a *self homotopy equivalence* if $\alpha = \beta$ and a *self pair homotopy equivalent morphism*, or simply a *self pair homotopy equivalence* if $\alpha = \beta = i : A \to X$ is the inclusion.

DEFINITION 2.1. For a given object $\alpha$, we define the subset $\mathcal{E}(\alpha)$ of $\Pi(\alpha, \alpha)$ by

$$\mathcal{E}(\alpha) = \{[f_1, f_2] \in \Pi(\alpha, \alpha) \mid (f_1, f_2) \textit{is a homotopy equivalence}\}.$$

Especially, for a CW-pair $(X, A)$, if $\alpha = i : A \to X$ is the inclusion, we denote $\mathcal{E}(i)$ by $\mathcal{E}(X, A)$. If $(f_1, f_2)$ is a morphism from the inclusion $i$ to itself, then $f_1|_A = f_2$. Thus we can consider the morphism $(f_1, f_2)$ as the pair map $f_1 : (X, A) \to (X, A)$. So the group $\mathcal{E}(X, A)$ is just the *group of pair homotopy equivalences*, i.e.,

$$\mathcal{E}(X, A) = \{[f] | f : (X, A) \to (X, A) \textit{is a pair homotopy equivalence}\}.$$

We also define the subset $\mathcal{E}(X, A; id_A)$ by

$$\mathcal{E}(X, A; id_A) = \{[id_A, f] \in \mathcal{E}(X, A) \mid id_A \text{ is the identity on A}\}.$$

These sets are all groups, homotopy invariants in the category of pairs and generalizations of several concepts of the group of self homotopy equivalences.

THEOREM 2.2. *Let $\alpha : X_1 \to X_2$ be an object in the category of pairs. Then the set $\mathcal{E}(\alpha)$ has a group structure induced by the composition of morphisms.*

*Proof.* Let $[f_1, f_2]$ and $[g_1, g_2]$ be elements of $\mathcal{E}(\alpha)$. Then

$$[f_1, f_2] \circ [g_1, g_2] = [f_1 g_1, f_2 g_2] \in \mathcal{E}(\alpha),$$

since $(f_1 g_1, f_2 g_2)$ is a self homotopy equivalent morphism on $\alpha$. For each $[f_1, f_2] \in \mathcal{E}(\alpha)$, let $(h_1, h_2)$ be a homotopy inverse morphism of $(f_1, f_2)$. Then $[h_1, h_2]$ is the inverse element of $[f_1, f_2]$. Moreover, $[id_{X_1}, id_{X_2}]$ is the identity element of $\mathcal{E}(\alpha)$. □

THEOREM 2.3. *If $\alpha$ and $\beta$ have same homotopy type, then $\mathcal{E}(\alpha)$ and $\mathcal{E}(\beta)$ are isomorphic.*

*Proof.* Suppose that $\alpha : X_1 \to X_2$ and $\beta : Y_1 \to Y_2$ have the same homotopy type by a homotopy equivalent morphism $(e_1, e_2) : \alpha \to \beta$ with the homotopy inverse morphism $(e'_1, e'_2) : \beta \to \alpha$. Define $\Psi : \mathcal{E}(\alpha) \to \mathcal{E}(\beta)$ by

$$\Psi[f_1, f_2] = [(e_1, e_2) \circ (f_1, f_2) \circ (e'_1, e'_2)].$$

Then $\Psi$ is an isomorphism. □

REMARK. Let $X$ be a CW-complex and $\alpha : * \to X$ the constant map. Then we have $\mathcal{E}(\alpha) = \mathcal{E}(X)$. Similarly, if $\alpha : X \to *$ is a constant map, then we have $\mathcal{E}(\alpha) = \mathcal{E}(X)$. Moreover, for the identity map $id_X : X \to X$, we have $\mathcal{E}(id_X) = \mathcal{E}(X)$.

Now we fit three groups $\mathcal{E}(X, A; id_A)$, $\mathcal{E}(X, A)$ and $\mathcal{E}(A)$ together into an exact sequence.

THEOREM 2.4. *For a CW-pair $(X, A)$, there exists an exact sequence as follows:*

$$1 \to \mathcal{E}(X, A; id_A) \to \mathcal{E}(X, A) \to \mathcal{E}(A). \tag{2}$$

*Proof.* Let $\Phi : \mathcal{E}(X, A; id_A) \to \mathcal{E}(X, A)$ be the inclusion. Then it is trivial that $\Phi$ is a monomorphism. Define $\Psi : \mathcal{E}(X, A) \to \mathcal{E}(A)$ by

$$\Psi[f_1, f_2] = [f_1]$$

for $[f_1, f_2] \in \mathcal{E}(X, A)$. Then $\Psi$ is well-defined. Let $[f_1, f_2] = [g_1, g_2] \in \mathcal{E}(X, A)$. Then there exists a homotopy $(F|_A, F) : i \times id_I \to i$ such that $(F|_A, F) : (f_1, f_2) \simeq (g_1, g_2)$, where $i : A \to X$ is the inclusion and $id_I$ is the identity on the unit interval $[0, 1]$ . Since $F|_A : f_1 \simeq g_1$, we have

$$\Psi[f_1, f_2] = [f_1] = [g_1] = \Psi[g_1, g_2].$$

Furthermore, $\Psi$ is a homomorphism, since the group operations of $\mathcal{E}(X, A)$ and $\mathcal{E}(A)$ are induced by the composition of maps.

Now we show the exactness at $\mathcal{E}(X, A)$. The image of $\Phi$ is contained in the kernel of $\Psi$, since

$$\Psi\Phi[id_A, f] = \Psi[id_A, f] = [id_A] \in \mathcal{E}(A).$$

Thus it remains for us to show that the kernel of $\Psi$ is contained in the image of $\Phi$. That is, each element $[f_1, f_2] \in \mathcal{E}(X, A)$ such that $[f_1] = [id_A] \in \mathcal{E}(A)$ belongs to $\mathcal{E}(X, A; id_A)$. Let $[f_1, f_2]$ be such an element. Since $f_1 \simeq id_A$ relative to $*$ in $A$, there exists a homotopy $H : A \times I \to A$ such that $H(a, 0) = f_1(a)$, $H(a, 1) = a$ and $H(*, t) = *$. Then the map $f_2 \sqcup H : X \times 0 \sqcup A \times I \to X$ defined by $(f_2 \sqcup H)|_{X \times 0} = f_2$ and $(f_2 \sqcup H)|_{A \times I} = iH$ has an extension $F : X \times I \to X$. Let $\overline{f} = F(\cdot,\ 1)$. Then, for each $a \in A$, we have

$$\overline{f}(a) = F(a, 1) = H(a, 1) = a.$$

So $(id_A, \overline{f})$ is a morphism from $i$ to itself, where $i : A \to X$ is the inclusion . But $(f_1, f_2)$ is homotopic to $(id_A, \overline{f})$ by the homotopy $(H, F)$ in the category of pairs. Therefore, $[f_1, f_2] = [id_A, \overline{f}] \in \mathcal{E}(X, A; id_A)$. □

DEFINITION 2.5. The CW-pair $(X, A)$ is called *the self-homotopy equivalence extendable pair* if for every homotopy equivalence $f : A \to A$, there exists a homotopy equivalence $\overline{f} : X \to X$ such that $(f, \overline{f}) : i \to i$ is a self homotopy equivalent morphism in the category of pairs, where $i : A \to X$ is the inclusion. In this case, $\overline{f}$ is called a *homotopy equivalence extension* of $f$.

The following proposition gives a homotopical property of homotopy equivalence extensions.

PROPOSITION 2.6. *Let $(X, A)$ be a homotopy equivalence extendable pair, and $f$ and $g$ self homotopy equivalences on $A$. If $f$ and $g$ are homotopic relative to $*$, then there are homotopy equivalence extensions $\overline{f}$ and $\overline{g}$ of $f$ and $g$ respectively such that $(f, \overline{f})$ and $(g, \overline{g})$ are homotopic in the category of pairs.*

*Proof.* Let $H : A \times I \to A$ be a homotopy between $f$ and $g$. Then we have $H(a, 0) = f(a)$, $H(a, 1) = g(a)$ and $H(*, t) = *$. Since $(X, A)$ is a homotopy equivalence extendable pair, there exists a homotopy equivalence extension $\overline{f} : X \to X$ of $f$. Define $\overline{f} \sqcup iH : X \times 0 \sqcup A \times I \to X$ by $(\overline{f} \sqcup iH)|_{X \times 0} = \overline{f}$ and $(\overline{f} \sqcup iH)|_{A \times I} = iH$, where $i : A \to X$ is the inclusion. Then it is well-defined, since $\overline{f}(a) = f(a) = H(a, 0)$, for each $a \in A$. Since the inclusion $i : A \to X$ is a cofibration, the map $(\overline{f} \sqcup H)$ has an extension $\overline{H} : X \times I \to X$. Define $\overline{g} : X \to X$ by $\overline{g}(x) = \overline{H}(x, 1)$. Then $\overline{g}(a) = \overline{H}(a, 1) = H(a, 1) = g(a)$. So $(g, \overline{g})$ is a morphism. Since $\overline{g}$ is homotopic to $\overline{f}$ by the homotopy $\overline{H}$, $\overline{g}$ is a self homotopy equivalence. Furthermore, we have $(\overline{H}, H) : (\overline{f}, f) \simeq (g, \overline{g})$, since $\overline{H} \circ i = i \circ H$, where $i : A \to X$ is the inclusion. Therefore, $\overline{g}$ is a homotopy equivalence extension of $g$. □

## 3. Proof of the main theorem

Let $X$ be a CW-complex, $X_n$ be the $n$-th Postnikov section of $X$ and $i_n : X \to X_n$ the inclusion. It is a well-known fact that $X_n$ can be obtained by attaching $(i+1)$-cells $(i > n)$ to $X$, so that $X_n$ kills the homotopy groups $\pi_i(X)$ for $i > n$. Thus for every $n \geq 1$, $X_n$ has the following properties:

(a) $(X_n, X)$ is a relative CW-complex with cells in dimensions $\geq n+2$;
(b) $\pi_i(X_n) = 0$ if $i > n$;
(c) $i_n^\sharp : \pi_i(X_n) \to \pi_i(X)$ is an isomorphism if $i \leq n$.

Now we introduce the following proposition needed in this section.

PROPOSITION 3.1. ([3], p131) *Suppose $S$ is a set of integers and $(Y, X)$ is a relative CW-complex such that if $e_\alpha \subset Y - X$ is a cell,* dim

$e_\alpha \in S$. *Suppose that* $\pi_{i-1}(Z,*) = 0$ *for any* $i \in S$. *Then any map* $f : X \to Z$ *admits an extension* $\overline{f} : Y \to Z$*:*

$$\begin{array}{ccc} Y & & \\ i\uparrow & \searrow \overline{f} & \\ X & \underset{f}{\to} & Z \end{array}$$

PROPOSITION 3.2. *For each Postnikov section* $X_n$*,* $n \geq 1$*, the CW-pair* $(X_n, X)$ *is a homotopy equivalence extendable pair.*

*Proof.* Let $f : X \to X$ be a self map. Consider the map $i_n f : X \to X_n$. Since $X_n - X$ has cells in dimensions $\geq n+2$ and $\pi_{i+1}(X_n) = 0$ for any $i \geq n$, $i_n f$ has an extension $\overline{f} : X_n \to X_n$ by the Proposition 3.1:

$$\begin{array}{ccc} X_n & \xrightarrow{\overline{f}} & X_n \\ i_n \uparrow & & \uparrow i_n \\ X & \underset{f}{\to} & X \end{array}$$

Thus $(f, \overline{f}) : i_n \to i_n$ is a morphism in the category of pairs. Let us show that $\overline{f}$ is a self homotopy equivalence. Since $f$ is a self homotopy equivalence, there exists a homotopy inverse $g$ and a homotopy $H : X \times I \to X$ such that $H(x,0) = (f \circ g)(x)$, $H(x,1) = x$ and $H(*,t) = *$. Let $\overline{g}$ be an extension of $g$ constructed in the above manner. Define a map

$$\overline{f} \circ \overline{g} \sqcup i_n H \sqcup id_{X_n} : X_n \times 0 \sqcup X \times I \sqcup X_n \times 1 \to X_n$$

by $\overline{f} \circ \overline{g} \sqcup i_n H \sqcup id_{X_n}|_{X_n \times 0} = \overline{f} \circ \overline{g}$, $\overline{f} \circ \overline{g} \sqcup i_n H \sqcup id_{X_n}|_{X \times I} = i_n H$ and $\overline{f} \circ \overline{g} \sqcup i_n H \sqcup id_{X_n}|_{X_n \times 1} = id_{X_n}$. Since $X_n \times I - (X_n \times 0 \sqcup X \times I \sqcup X_n \times 1)$ has cells of the form $e^i_\alpha \times e^1$, where $e^i_\alpha \subset X_n - X$ and $e^1 = I - \{0,1\}$. But $X_n - X$ has cells in dimensions $\geq n+2$. So $X^n \times I - (X_n \times 0 \sqcup X \times I \sqcup X_n \times 1)$ has cells in dimensions $\geq n+3$. Since $\pi_i(X_n) = 0$ for $i > n$, the map $\overline{f} \circ \overline{g} \sqcup i_n H \sqcup id_{X_n}$ has an extension $\overline{H} : X_n \times I \to X_n$ by Proposition 3.1. The extension $\overline{H}$ is a homotopy between $\overline{f} \circ \overline{g}$ and $id_{X_n}$ relative to $*$ in $X_n$. Similarly, $\overline{g} \circ \overline{f}$ is homotopic to $id_{X_n}$ relative to $*$ in $X_n$.

Thus $\overline{g}$ is a homotopy inverse of $\overline{f}$ and $\overline{f}$ is an equivalence extension of $f$. □

REMARK 3.3. In the proof of the above proposition, we have $\overline{H}(i_n \times id_I) = i_n H$ since $\overline{H}$ is an extension of $i_n H$. This means that $(H, \overline{H})$ is a homotopy between $(f, \overline{f}) \circ (g, \overline{g})$ and $(id_X, id_{X_n})$ in the category of pairs. So we have $[f, \overline{f}] \in \mathcal{E}(X_n, X)$.

THEOREM 3.4. *Let $X_n$ be the n-th Postnikov section of $X$ for each $n \geq 1$. Then we have the following split short exact sequence:*

$$1 \to \mathcal{E}(X_n, X; id_X) \xrightarrow{\Phi} \mathcal{E}(X_n, X) \xrightarrow{\Psi} \mathcal{E}(X) \to 1. \tag{3}$$

*where $\Phi$ is the inclusion and $\Psi$ is a homomorphism defined by $\Psi[f, \overline{f}] = [f]$.*

*Proof.* By Theorem 2.4, we have the following exact sequence;

$$1 \to \mathcal{E}(X_n, X; id_X) \xrightarrow{\Phi} \mathcal{E}(X_n, X) \xrightarrow{\Psi} \mathcal{E}(X). \tag{4}$$

Thus it is sufficient to show that there is a homomorphism $J : \mathcal{E}(X) \to \mathcal{E}(X_n, X)$ such that $\Psi \circ J = id_{\mathcal{E}(X)}$. Let $[f]$ be an element in $\mathcal{E}(X)$. Then there is a homotopy equivalence extension $\overline{f}$ of $f$ by Proposition 3.2. Define $J : \mathcal{E}(X) \to \mathcal{E}(X_n, X)$ by $J[f] = [f, \overline{f}]$.

Let us show that $J$ is well-defined. By Proposition 2.6 and Remark 3.3, it is sufficient to show that if $\overline{f_0}$ and $\overline{f_1}$ are any two homotopy equivalence extensions of $f$, then $(f, \overline{f_0})$ and $(f, \overline{f_1})$ are homotopic in the category of pairs. Define a map

$$\overline{f_0} \sqcup i_n f \sqcup \overline{f_1} : X_n \times 0 \sqcup X \times I \sqcup X_n \times 1 \to X_n$$

by $\overline{f_0} \sqcup i_n f \sqcup \overline{f_1}|_{X_n \times 0} = \overline{f_0}$, $\overline{f_0} \sqcup i_n f \sqcup \overline{f_1}|_{X \times I} = i_n f$ and $\overline{f_0} \sqcup i_n f \sqcup \overline{f_1}|_{X_n \times 1} = \overline{f_1}$, where $i_n$ is the inclusion from $X$ to $X_n$. By Theorem 3.1, $\overline{f_0} \sqcup i_n f \sqcup \overline{f_1}$ has an extension $\overline{H} : X_n \times I \to X_n$. Since $\overline{H}(i_n \times id_I) = i_n f$, the pair map $(f, \overline{H})$ is a homotopy between $(f, \overline{f_0})$ and $(f, \overline{f_1})$ in the category of pairs.

Moreover, $\Psi \circ J = id_{\mathcal{E}(X)}$ by definitions of $\Psi$ and $J$.

Let us show that $J$ is a homomorphism. Let $[f]$ and $[g]$ be elements in $\mathcal{E}(X)$. Since

$$J([f] \cdot [g]) = J[f \circ g] = [f \circ g, \overline{f \circ g}]$$

and

$$J[f] \cdot J[g] = [f, \overline{f}] \cdot [g, \overline{g}] = [f \circ g, \overline{f} \circ \overline{g}],$$

we have to show that $(f \circ g, \overline{f \circ g})$ is homotopic to $(f \circ g, \overline{f} \circ \overline{g})$ in the category of pairs. Let $H : X \times I \to X$ be the map given by $H(x, t) = f(g(x))$ for $(x, t) \in X \times I$. Define a map

$$\overline{f \circ g} \sqcup i_n H \sqcup \overline{f} \circ \overline{g} : X_n \times 0 \sqcup X \times I \sqcup X_n \times 1 \to X_n$$

by $(\overline{f \circ g} \sqcup i_n H \sqcup \overline{f} \circ \overline{g})|_{X_n \times 0} = \overline{f \circ g}$, $(\overline{f \circ g} \sqcup i_n H \sqcup \overline{f} \circ \overline{g})|_{X \times I} = i_n H$ and $(\overline{f \circ g} \sqcup i_n H \sqcup \overline{f} \circ \overline{g})|_{X_n \times 1} = \overline{f} \circ \overline{g}$. By Proposition 3.1, the map $\overline{f \circ g} \sqcup i_n H \sqcup \overline{f} \circ \overline{g}$ has an extension $\overline{H} : X_n \times I \to X_n$. Since $\overline{H}(i_n \times id_I) =$

$i_n H$, the pair $(H, \overline{H})$ is a homotopy between $(f \circ g, \overline{f \circ g})$ and $(f \circ g, \overline{f} \circ \overline{g})$ in the category of pairs. □

THEOREM 3.5. *Let $X_n$ be the $n$-th Postnikov section for $n \geq 1$. Then $\mathcal{E}(X_n, X)$ is isomorphic to $\mathcal{E}(X)$.*

*Proof.* By Theorem 3.4, it is sufficient to show that $\mathcal{E}(X_n, X; id_X)$ is trivial. Let us show that $\mathcal{E}(X_n, X; id_X) = \{[id_X, id_{X_n}]\}$. Let $[id_X, \overline{f}]$ be an element in $\mathcal{E}(X_n, X; id_X)$ and $H : X \times I \to X$ be the map given by $H(x, t) = x$ for $(x, t) \in X \times I$. Define

$$H' : X_n \times 0 \sqcup X \times I \sqcup X_n \times 1 \to X_n$$

by $H'|_{X_n \times 0} = \overline{f}$, $H'|_{X \times I} = i_n H$ and $H'|_{X_n \times 1} = id_{X_n}$. By Proposition 3.1, $H'$ has an extension $\overline{H} : X_n \times I \to X_n$. So we have $\overline{H}(x, 0) = \overline{f}$, $\overline{H}(x, 1) = id_{X_n}$, $\overline{H}(*, t) = *$ and $\overline{H}(i_n \times id_I) = i_n H$. Therefore, the pair $(H, \overline{H})$ is a homotopy between $(id_X, \overline{f})$ and $(id_X, id_{X_n})$. This implies $[id_X, \overline{f}] = [id_X, id_{X_n}]$. □

The Eilenberg-Maclane space $K(\pi, n)$ can be obtained from the Moore space $M(\pi, n)$ by killing homotopy groups of order $\geq n + 1$. That is, $k(\pi, n) = M(\pi, n)_n$. Thus we have following corollary:

COROLLARY 3.6. *For each $n \geq 1$, $\mathcal{E}(K(\pi, n), M(\pi, n))$ is isomorphic to $\mathcal{E}(M(\pi, n))$.*

We know that $\mathcal{E}(K(\pi, n)) = Aut(\pi)$, where $Aut(\pi)$ is the group of automorphisms on $\pi$ [1]. Moreover, it is a well known fact that if a group $\pi$ is non abelian, then $Aut(\pi)$ is not trivial. Thus for such group $\pi$, $\mathcal{E}(K(\pi, n))$ is not trivial. But $\mathcal{E}(K(\pi, n), M(\pi, n); id_{M(\pi,n)})$ is always trivial by Theorem 3.4. So $\mathcal{E}(X_n, X; id_X)$ is not isomorphic to $\mathcal{E}(X_n)$ in general.

EXAMPLE 3.7. It is well-known facts that $\mathcal{E}(\mathbb{R}P^n) \equiv \mathbb{Z}_2 \equiv \mathcal{E}(S^n)$ [1]. Since $\mathbb{R}P^2 = M(\mathbb{Z}_2, 1)$, $\mathbb{R}P^\infty = K(\mathbb{Z}_2, 1)$, $\mathbb{C}P^\infty = K(\mathbb{Z}_2, 2)$ and $S^2 = M(\mathbb{Z}_2, 2)$, we have

$$\mathcal{E}(\mathbb{R}P^\infty, \mathbb{R}P^2) \equiv \mathcal{E}(\mathbb{R}P^2) \equiv \mathbb{Z}_2$$

and

$$\mathcal{E}(\mathbb{C}P^\infty, S^2) \equiv \mathcal{E}(S^2) \equiv \mathbb{Z}_2.$$

More generally, since $S^n = M(\mathbb{Z}, n)$, we have

$$\mathcal{E}(K(\mathbb{Z}, n), M(\mathbb{Z}, n)) \equiv \mathcal{E}(K(\mathbb{Z}, n), S^n)) \equiv \mathcal{E}(S^n)) \equiv \mathbb{Z}_2.$$

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