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RELATIVE PROJECTIVITY AND RELATED RESULTS

H. Ansari-Toroghy

Abstract. Let R be a commutative Noetherian ring and let M be an Artinian R-module. Let $M'' \subseteq M'$ be submodules of M. Suppose F is an R-module which is projective relative to M. Then it is shown that

$$Att_R(Hom_A(F, M') :_{Hom_A(F,M)} I^n), \ n \in N$$

and

$$Att_R(Hom_A(F, M') :_{Hom_A(F,M)} I^n \\ /Hom_A(F, M'') :_{Hom_A(F,M)} I^n), n \in N$$

are ultimately constant.

1. Introduction

Throughout of this paper R will denote a commutative ring (with a nonzero identity).

Let M be an R-module. A prime ideal P of R is said to be an associated prime of M if there exists an element $x \in M$ such that $P = (0 :_R Rx)$ (see [2, Chapter 4]). The set of associated primes of M is denoted by $Ass_R(M)$.

We shall follow Macdonald's terminology (see [4]) concerning secondary representation. So whenever an R-module L has a secondary representation, then the set of attached primes of L is denoted by $Att_R(L)$.

Let I be an ideal of R and suppose that F and M are respectively a flat and an Artinian R-modules. Let $M'' \subseteq M'$ be submodules of M and let $T = Hom_R(F, -)$. In [3] it has shown that the sequences of sets $Att_R(T(M') :_{T(M)} I^n)$ and $Att_R(T(M') :_{T(M)} I^n/T(M'') :_{T(M)} I^n)$, $n \in N$, are ultimately constant.

Let I, M, M', and M'' be as in the above paragraph and let F be an R-module (not necessarily flat). In this paper we will generalize the above mentioned results in the case that F is projective relative to M.

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We recall that F is projective relative to M (or F is M-projective) if and only if for any homomorphic image N of M, the homomorphism $Hom_R(F, M) \to Hom_R(F, N)$ is epic (see [1]).

2. Auxiliary results

REMARK 2.1. (a) If F is an R-module which is M-projective and X a submodule or a homomorphic image of M, then F is X-projective.

(b) If $(M_i)_{i\in I}$ is a finite family of R-modules and F is projective relative to M_i for each $i \in I$, then F is projective relative to $\oplus_{i\in I} M_i$.

(c) If F is a M-projective and

$$0 \to K \to M \to L \to 0$$

is an exact sequence of R-modules and R-homomorphisms with middle term M, then

$$0 \to Hom_R(F, K) \to Hom_R(F, M) \to Hom_R(F, L) \to 0$$

is also an exact sequence.

LEMMA 2.2. *Let R be a commutative Noetherian ring and let M be an Artinian R-module. Suppose F is an R-module which is projective relative to M and Let $E = \oplus_{P\in\Omega} E(R/P)$, where Ω is a finite subset of $Max(R)$ with $Ass_R(M) \subseteq \Omega$. Set $T(-) = Hom_R(-, E)$. Then $T(F)$ is injective relative to $T(M)$. (We recall that an R-module X is injective relative to an R-module H if and only if for any submodule L of H, $Hom_R(H, X) \to Hom_R(L, X)$ is epic (see [1]).)*

Proof. Let

$$0 \to L \to T(M)$$

be an exact sequence of R-modules and R-homomorphisms. Hence

$$T(T(M)) \to T(L) \to 0$$

is an exact sequence. But $T(T(M)) \cong M$ by [3, Lemma 3.1]. It follows that F is projective relative to $T(T(M))$. Hence

$$Hom_R(F, T(T(M)) \to Hom_R(F, T(L)) \to 0$$

is an exact sequence. It turns out that

$$Hom_R(T(M), T(F)) \to Hom_R(L, T(F)) \to 0$$

is an exact sequence. Hence $T(F)$ is injective relative to $T(M)$ and the proof is complete. □

LEMMA 2.3. *Let M_1 and M_2 are R-modules. Suppose F is an R-module which is projective relative to both M_1 and M_2. Then*

$$Hom_R(F, M_1 + M_2) = Hom_R(F, M_1) + Hom_R(F, M_2).$$

Proof. Put $T = Hom_R(F, -)$. It is clear that

$$T(M_1) + T(M_2) \subseteq T(M_1 + M_2).$$

To see the reverse inclusion let $f \in T(M_1 + M_2)$. Since F is projective relative to $M_1 \oplus M_2$ by 2.1, as in [3] (1.5), we can find $f_i \in T(M_i)$, $i = 1, 2$, such that $f = f_1 + f_2$. So the proof is complete. □

LEMMA 2.4. *Let I be a finitely generated ideal of R and let M be an R-module. Suppose F is an R-module which is projective relative to M. Put $T = Hom_R(F, -)$. Then $T(IM) = IT(M)$.*

Proof. Let $x \in R$. Since F is projective relative to M, from the exact sequence

$$M \xrightarrow{x} xM \to 0,$$

we get the exact sequence

$$T(M) \xrightarrow{x} T(xM) \to 0.$$

Hence $T(xM) = xT(M)$. Now the result follows from this and Lemma 2.3. □

LEMMA 2.5. *Let S be another commutative ring and let $\phi : R \to S$ be a ring homomorphism. Let M be an S module. Suppose F is an R-module which is projective relative to M when regarded as R-module. Then as S-modules, $F \otimes_R S$ is projective relative to M.*

Proof. Let

$$M \to N \to 0$$

be an exact sequence of S-modules and S-homomorphisms. Hence, as R-modules and R-homomorphisms,

$$Hom_R(F, M) \to Hom_R(F, N) \to 0,$$

is an exact sequence. But as S-modules,

$$Hom_S(F \otimes_R S, M) \cong Hom_R(F, M).$$

It follows that

$$Hom_S(F \otimes_R S, M) \to Hom_S(F \otimes_R S, N) \to 0.$$

is an exact sequence. So the proof is complete. □

3. Main results

Throughout this section A will denote a commutative Noetherian ring with a non-zero identity.

We recall the concept of coassociated primes from [9].

DEFINITION. An A-module L is cocyclic if $L \subseteq E(A/P)$ for some maximal ideal P of A. Also a prime ideal P of A is said to be a coassociated prime of M if there exists a cocyclic homomorphic image L of M such that $P = (0 :_A L)$. The set of coassociated primes of M is denoted by $Coass_A(M)$.

LEMMA 3.1. *Let M be an Artinian A-module and suppose F is an A-module which is projective relative to M. Then we have the following.*

(i) *$Hom_A(F, M) \neq 0$ if only if there exists $P \in Att_A(M)$ such that $P \subseteq Q$ for some $Q \in Coass_A(F)$.*

(ii) *If M is a P-secondary A-module and $Hom_A(F, M) \neq 0$, then $Hom_A(F, M)$ is a P-secondary A-module.*

Proof. (i) ($\Rightarrow$) We can use the same technique used in [3] (3.2). To see the converse, let $P \in Att_A(M)$ and let $P \subseteq Q$ for some $Q \in Coass_A(F)$. Then there exist maximal ideals P_1 and P_2 such that $P \in Ass_A(Hom_A(M, E(A/P_1)))$ and $Q \in Ass_A(Hom_A(M, E(A/P_2)))$ by [9, 1.7]. Set $\Omega = Ass_A(M) \cup \{P_1, P_2\}$ and $D(-) = Hom_A(-, \oplus_{P \in \Omega} E(A/P))$. It implies that $P \in Ass_A(D(M))$ and $Q \in Ass_A(D(F))$. Since $D(F)$ is injective relative to $D(M)$ by Lemma 2.2, from the exact sequence

$$0 \to A/P \to D(M)$$

we obtain the exact sequence

$$Hom_A(D(M), D(F)) \to Hom_A(A/P, D(F)) \to 0.$$

Further,

$$0 \to A/Q \to D(F)$$

is an exact sequence so that $Hom_A(A/Q, D(F)) \neq 0$. Also $D(F)$ is injective relative to A/P by Remark 2.1. Hence from the exact sequence

$$A/P \to A/Q \to 0,$$

we get the exact sequence

$$0 \to Hom_A(A/Q, D(F)) \to Hom_A(A/P, D(F)).$$

Hence we have $Hom_A(A/P, D(F)) \neq 0$. This in turn implies that

$$Hom_A(D(M), D(F)) \neq 0.$$

But by [3, 3.1], $M \cong D(D(M))$. Hence we have

$$Hom_A(F, M) \cong Hom_A(D(M), D(F)) \neq 0.$$

(ii) Let $r \in R$. Then $M \xrightarrow{r} M$ is nilpotent or surjective. Hence $Hom_R(F, M) \xrightarrow{r} Hom_R(F, M)$ is either nilpotent or surjective. Therefore, $Hom_R(M, E)$ is a P-secondary module and the proof is complete. □

COROLLARY 3.2. (See [3, 3.2].) *Let M be an Artinian A-module and suppose F is a flat A-module. Then we have the following.*

(i) *$Hom_A(F, M) \neq 0$ if only if there exists $P \in Att_A(M)$ such that $P \subseteq Q$ for some $Q \in Coass_A(F)$.*

(ii) *If M is a P-secondary A-module and $Hom_A(F, M) \neq 0$, then $Hom_A(F, M)$ is a P-secondary A-module.*

Proof. By [6, Theorem 5], $T = Hom_A(F, -)$ is an exact functor over the category of Artinian A-modules. It implies that F is projective relative to M. Hence the result follows from Lemma 3.1. □

THEOREM 3.3. *Let M be an Artinian A-module, and suppose that F is an A-module which is projective relative to M. Then $Hom_A(F, M)$ has a secondary representation and we have*

$$\begin{aligned} &Att_A(Hom_A(F, M)) \\ &= \{P \in Att_A(M) : P \subseteq Q \quad \text{for some } Q \in Coass_A(F)\}. \end{aligned}$$

Proof. Let $M = \sum_{i=1}^{r} M_i$ be a minimal secondary representation for M, where each M_i is P_i-secondary. Suppose for $i = 1, 2, \ldots, t$, there exists $P_i \subseteq Q_i$ for some $Q_i \in Coass_A(F)$, while this doesn't hold for $i = t+1, t+2, \ldots, r$. Let $T(-) = Hom_A(F, -)$. Then by Lemma 3.1, for $i = 1, 2, \ldots, t$, $T(M_i) \neq 0$ and it is P_i-secondary. Hence by Lemma 2.3, we have

$$T(M) = Hom_A(F, M) = \sum_{i=1}^{t} T(M_i).$$

So $Hom_A(F, M)$ has a secondary representation. We claim that the above decomposition is in fact a minimal one. To see this let for an integer j with $1 \leq j \leq t$,

$$T(M_j) \subseteq \sum_{\substack{i=1 \\ i \neq j}}^{t} T(M_i).$$

It implies that

$$T(M) = Hom_A(F, M) = \sum_{\substack{i=1 \\ i \neq j}}^{r} T(M_i) = T(\sum_{\substack{i=1 \\ i \neq j}}^{r} M_i).$$

Now from

$$0 \to \sum_{\substack{i=1 \\ i \neq j}}^{r} M_i \to M \to M/\sum_{\substack{i=1 \\ i \neq j}}^{r} M_i \to 0,$$

by using Lemma 2.1, we obtain the exact sequence

$$0 \to T(\sum_{\substack{i=1 \\ i \neq j}}^{r} M_i) \to T(M) \to T(M/\sum_{\substack{i=1 \\ i \neq j}}^{r} M_i) \to 0.$$

It implies that

$$T(M/\sum_{\substack{i=1 \\ i \neq j}}^{r} M_i) = 0.$$

But

$$M/\sum_{\substack{i=1 \\ i \neq j}}^{r} M_i \cong M_j/(M_j \cap \sum_{\substack{i=1 \\ i \neq j}}^{r} M_i)$$

is P_j-secondary. It turns out that

$$Att_A(M/\sum_{\substack{i=1 \\ i \neq j}}^{r} M_i) = \{P_j\}.$$

Hence by Lemma 3.1, we have $T(\sum_{\substack{i=1 \\ i \neq j}}^{r} M_i) \neq 0$ which is a contradiction. This completes the proof. □

REMARK 3.4. Theorem 3.3 extends [3, Theorems 3.3, 3.5, and 3.6] because as we mentioned in the proof of 3.2, every flat A-module is projective relative to every Artinian A-module.

COROLLARY 3.5. *Let M be a finitely generated A-module and suppose F is an A-module which is projective relative to M. Let P be a maximal ideal of A such that $P \in Coass_A(F)$ and let $i \geq 0$. Then $H^i_P(M)$, and $Hom_A(F, H^i_P(M))$ have the same set of attached prime ideals, so that the following are equivalent.*

(i) $H^i_P(M) \neq 0$

(ii) $Hom_A(F, H^i_P(M)) \neq 0$.

Proof. It is straightforward to see that $H^i_P(M)$ is an Artinian A-module. Further since $Ass_A(H^i_P(M)) \subseteq \{P\}$, every $P' \in Att_A(H^i_P(M))$ is contained in P. Hence by using Theorem 3.3, we have

$$Att_A(Hom_A(F, H^i_P(M))) = Att_A(H^i_P(M)).$$

The result follows from this and the proof is complete. □

Let M be an R-module, and let M' be a submodule of M. Then for an ideal I of R it follows that

$$Hom_R(L, M') :_{Hom_R(L,M)} I = Hom_R(L, (M' :_M I)).$$

THEOREM 3.6. *Let I be an ideal of R and let M be an Artinian R-module. Suppose F is an R-module which is projective relative to M. Let $M'' \subseteq M'$ be submodules of M. Further let $T(-) = Hom_A(F, -)$. Then the sequences of sets*

$$Att_R(T(M') :_{T(M)} I^n), \; n \in N$$

and

$$Att_R(T(M') :_{T(M)} I^n / T(M'') :_{T(M)} I^n), \; n \in N,$$

are ultimately constant.

Proof. By [7] (1.9), there exists a ring homomorphism $\phi : R \to S$ such that S is a commutative Noetherian ring and M has an S-module structure which is compatible with the R-module structure. Also by Lemma 2.5, $F \otimes_R S$ is projective relative to M as an S-module. Now Let $U(-) = Hom_S(F \otimes_R S, -)$. Then

$$T(M') :_{T(M)} I^n \cong U(M') :_{U(M)} (IS)^n$$

and

$$T(M') :_{T(M)} I^n / T(M'') :_{T(M)} I^n$$
$$\cong U(M') :_{U(M)} (IS)^n / U(M'') :_{U(M)} (IS)^n.$$

Hence by [5] (4.1), we can assume that R is a commutative Noetherian ring. Now by Theorem 3.3,

$$Att_R(T(M') :_{T(M)} I^n)$$
$$= \{P \in Att_R(M' :_M I^n) : P \subseteq Q \text{ for some } Q \in Coass_R(F)\}$$

and

$$Att_R(T(M') :_{T(M)} I^n / T(M'') :_{T(M)} I^n)$$
$$= \{P \in Att_R(M' :_M I^n / M'' :_M I^n) : P \subseteq Q \text{ for some } Q \in Coass_R(F)\}.$$

Hence the result follows from [8, 5.4 and 5.5]. □

COROLLARY 3.7. *Let I, F, M, and T be as in 3.6. Then the sequences of sets $Att_R(0 :_{T(M)} I^n)$ and $Att_R(0 :_{T(M)} I^n/0 :_{T(M)} I^{n-1})$, $n \in N$ are ultimately constant. Further if C and D denote respectively their ultimate constant values, then*

$$C - D \subseteq Att_R(T(M)).$$

Proof. We can apply the same technique used in proof of [3, 3.6]. So we omit the proof. □

EXAMPLE 3.8. (i) Let F and M be respectively a flat and an Artinian A-modules. Then by Remark 3.4, $Hom_A(F, M)$ has a secondary representation and

$$\begin{aligned} &Att_A(Hom_A(F, M)) \\ &= \{P \in Att_A(M) : P \subseteq Q \text{ for some } Q \in Coass_A(F)\}. \end{aligned}$$

(ii) Suppose M is a non-zero divisible abelian group which is not a flat Z module and let p be a prime number (here Z denotes the ring of integers). Then by [1, page 190, Ex. 10], and Remark 2.1, M is projective relative to the Artinian Z-module Z/p^nZ. Hence $Hom_Z(M, Z/p^nZ)$ has a secondary representation and its attached primes can be specified by 3.3.

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, GUILAN UNIVERSITY, P. O. BOX 1914, RASHT, IRAN
E-mail: ansari@guilan.ac.ir