Bull. Korean Math. Soc. **41** (2004), No. 3, pp. 435–450

h-STABILITY FOR NONLINEAR PERTURBED DIFFERENCE SYSTEMS

Sung Kyu Choi, Nam Jip Koo and Se Mok Song

Abstract. We show that two concepts of h-stability and h-stability in variation for nonlinear difference systems are equivalent by using the concept of n_∞-summable similarity of their associated variational systems. Also, we study h-stability for perturbed nonlinear system $y(n+1) = f(n, y(n)) + g(n, y(n), Sy(n))$ of nonlinear difference system $x(n+1) = f(n, x(n))$ using the comparison principle and extended discrete Bihari-type inequality.

1. Introduction

Let $\mathbb{N}(n_0) = \{n_0, n_0+1, \cdots, n_0+k, \cdots\}$, where n_0 is a nonnegative integer and $\mathbb{R}^m$ the m-dimensional real Euclidean space. We consider the nonlinear difference system

$$x(n+1) = f(n, x(n)), \ x(n_0) = x_0 \tag{1.1}$$

and its perturbed system

$$y(n+1) = f(n, y(n)) + g(n, y(n), Sy(n)), \ y(n_0) = y_0 \tag{1.2}$$

where $f : \mathbb{N}(n_0) \times \mathbb{R}^m \to \mathbb{R}^m$ and $g : \mathbb{N}(n_0) \times \mathbb{R}^m \times F(\mathbb{N}(n_0), \mathbb{R}^m) \to \mathbb{R}^m$, and $S : F(\mathbb{N}(n_0), \mathbb{R}^m) \to F(\mathbb{N}(n_0), \mathbb{R}^m)$ is an operator on $F(\mathbb{N}(n_0), \mathbb{R}^m) = \{y | y : \mathbb{N}(n_0) \to \mathbb{R}^m \textit{ is a sequence}\}$, and $f(n, 0) = 0 = g(n, 0, 0)$. We assume that $f_x = \frac{\partial f}{\partial x}$ exists and is continuous and invertible on $\mathbb{N}(n_0) \times \mathbb{R}^m$. Let $x(n) = x(n, n_0, x_0)$ be the unique solution of (1.1). Also, we consider its associated variational systems

$$v(n+1) = f_x(n, 0)v(n) \tag{1.3}$$

Received July 10, 2003.
2000 Mathematics Subject Classification: 39A11.
Key words and phrases: n_∞-summable similarity, h-system, Bihari-type inequality, comparison principle.
The first author was supported by Chungnam National University Fund in 2002.

and

$$z(n+1) = f_x(n, x(n, n_0, x_0))z(n). \tag{1.4}$$

The fundamental matrix solution $\Phi(n, n_0, 0)$ of (1.3) is given by

$$\Phi(n, n_0, 0) = \frac{\partial x(n, n_0, 0)}{\partial x_0}$$

and the fundamental matrix solution $\Phi(n, n_0, x_0)$ of (1.4) is given by

$$\Phi(n, n_0, x_0) = \frac{\partial x(n, n_0, x_0)}{\partial x_0}.$$

in [10].

Equation (1.2) represents several interesting equations, namely, difference equation of Volterra type as

$$y(n+1) = A(n)y(n) + \sum_{l=n_0}^{n} B(n,l)y(l)$$

where $f(n, y(n)) = A(n)y(n)$ and $Sy(n) = \sum_{l=n_0}^{n} B(n,l)y(l)$, $A(n)$ and $B(n,l)$ are $m \times m$ matrix functions on $\mathbb{N}(n_0)$ and $\mathbb{N}(n_0) \times \mathbb{N}(n_0)$ respectively, and delay difference equation as

$$y(n+1) = f(n, y(n)) + g(n, y(n), y(n-\tau))$$

etc..

The symbol $|\cdot|$ will be used to denote any convenient vector norm in $\mathbb{R}^m$.

When we study the asymptotic stability it is not easy to work with non-exponential types of stability. Medina and Pinto [11–14] extended the study of exponential stability to a variety of reasonable systems called h-systems. They introduced the notion of h-stability for difference systems as well as for differential systems. To study the various stability notions of nonlinear difference systems, the comparison principle [10] and variation of constants formula by Agarwal [1] play a fundamental role.

Now, we recall some definitions of stability notions in [12].

Definition 1.1. System (1.1) is called an *h-system* around the null solution, or more briefly an *h-system*, if there exist a positive function $h : \mathbb{N}(n_0) \to \mathbb{R}$ and constant $c \geq 1$, such that

$$|x(n, n_0, x_0)| \leq c|x_0|h(n)h^{-1}(n_0), \;\; n \geq n_0$$

for $|x_0|$ small enough (here $h^{-1}(n) = \frac{1}{h(n)}$).

If h is a bounded function, then an h-system permits the following types of stability :

DEFINITION 1.2. The zero solution of system (1.1), or more briefly system (1.1), is said to be

(hS) *h-stable* if $c \geq 1$, δ exist as well as a positive bounded function $h : \mathbb{N}(n_0) \to \mathbb{R}$ such that

$$|x(n, n_0, x_0)| \leq c|x_0|h(n)h^{-1}(n_0), \quad n \geq n_0$$

for $|x_0| \leq \delta$,

(GhS) *globally h-stable* if system (1.1) is hS for every $x_0 \in D$, where $D \subset \mathbb{R}^m$ is a region which includes the origin,

(hSV) *h-stable in variation* if the zero solution of system (1.4) is hS,

(GhSV) *globally h-stable in variation* if the zero solution of system (1.4) is GhS.

The various notions about h-stability given by Definition 1.2 include several types of known stability properties such as uniform stability, uniform Lipschitz stability and exponential asymptotic stability. See [4–9, 11–14].

DEFINITION 1.3. A function $w : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be of the *class* $\hat{F}$ if

(F_1) $w(u)$ is nondecreasing and continuous for $u \geq 0$ and positive for $u > 0$,

(F_2) there exists a nonnegative function r(multiplier function) defined on $(0, \infty)$ such that

$$w(\alpha u) \leq r(\alpha)w(u), \qquad \text{for } \alpha > 0, u \geq 0,$$

(F_3) $\lim_{\alpha \to 0^+} \frac{r(\alpha)}{\alpha}$ exists.

Conti [3] defined two $m \times m$ matrix functions A and B on $\mathbb{R}^+$ to be t_∞- *similarity* if there is an $m \times m$ matrix function S defined on $\mathbb{R}^+$ such that $S'(t)$ is continuous, $S(t)$ and $S^{-1}(t)$ are bounded on $\mathbb{R}^+$, and

$$\int_0^\infty |S' + SB - AS|dt < \infty.$$

Now, we introduce the notion of n_∞- summable similarity which is the corresponding t_∞-similarity for the discrete case.

Let $\mathfrak{M}$ denote the set of all $m \times m$ invertible matrices $A(n)$ defined on $\mathbb{N}(n_0)$ and $\mathfrak{S}$ be the subset of $\mathfrak{M}$ consisting of those nonsingular bounded matrices $S(n)$ such that $S^{-1}(n)$ is also bounded.

DEFINITION 1.4. A matrix $A(n) \in \mathfrak{M}$ is n_∞*-summablely similar* to a matrix $B(n) \in \mathfrak{M}$ if there exists an $m \times m$ matrix $F(n)$ absolutely summable over $\mathbb{N}(n_0)$, i.e.

$$\sum_{l=n_0}^{\infty} |F(l)| < \infty$$

such that

$$S(n+1)B(n) - A(n)S(n) = F(n) \tag{1.5}$$

for some $S(n) \in \mathfrak{S}$.

For the example of n_∞- summable similarity, see [8, Example 2.6].

Medina and Pinto studied the important properties about hS for the various differential systems and the nonlinear difference systems [11–14]. We also investigated hS for nonlinear differential or difference systems [4–7].

In this paper, we show that two concepts of h-stability and h-stability in variation for nonlinear difference systems are equivalent by using the n_∞-summable similarity of their associated variational systems. Also, we study h-stability for perturbed nonlinear system of (1.1) using the comparison principle and extended discrete Bihari-type inequality.

REMARK 1.5. We can easily show that the n_∞-summable similarity is an equivalence relation by the same method of Trench in [15]. Also, if two $m \times m$ matrices A and B are n_∞-summablely similar with $F(n) = 0$, then we say that they are *kinematically similar.*

2. h-stability in variation for nonlinear difference systems

For the linear difference systems, Medina and Pinto [12] showed that

$$\text{GhSV} \iff \text{GhS} \iff \text{hS} \iff \text{hSV}.$$

Also, the associated variational system inherits the property of hS from the original nonlinear system. i.e., the zero solution $v = 0$ of (1.3) is hS when the zero solution $x = 0$ of (1.1) is hS in Theorem 2 [12]. Our purpose is to characterize the h-stability in variation via n_∞-summable similarity. To do this, we need the following lemma.

LEMMA 2.1. [8, Lemma 3.3] *Assume that $f_x(n, 0)$ is n_∞-summablely similar to $f_x(n, x(n, n_0, x_0))$ for $n \geq n_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$ and $\sum_{n=n_0}^{\infty} \frac{h(n)}{h(n+1)}|F(n)| < \infty$ with the positive function $h(n)$ defined on $\mathbb{N}(n_0)$. Then the solution $v = 0$ of (1.3) is an h-system if and only if the solution $z = 0$ of (1.4) is an h-system.*

Letting $h(n)$ be bounded on $\mathbb{N}(n_0)$, we obtain the following result [13, Theorem 3.5] as a corollary of Lemma 2.1.

COROLLARY 2.2. *Assume that*

$$\sum_{n=n_0}^{\infty} \frac{h(n)}{h(n+1)} |f_x(n, x(n, n_0, x_0)) - f_x(n, 0)| < \infty, \quad n_0 \geq 0 \tag{2.1}$$

holds for $|x_0| \leq \delta$ *with some* $\delta > 0$. *Then the solution* $v = 0$ *of* (1.3) *is hS if and only if the solution* $z = 0$ *of* (1.4) *is hS.*

Proof. Setting $F(n) = f_x(n, x(n, n_0, x_0)) - f_x(n, 0)$ and $S(n) = I$ for $n \geq n_0 \geq 0$, we can easily see that $f_x(n, x(n, n_0, x_0))$ and $f_x(n, 0)$ are n_∞-summablely similar. Thus all conditions of Lemma 2.1 are satisfied, and hence the solution $z = 0$ of (1.4) is hS. □

Medina and Pinto showed that hSV implies hS [12, Theorem 3] by using the formula

$$x(n, n_0, x_0) = [\int_0^1 \Phi(n, n_0, sx_0) ds] x_0.$$

Also, they proved the converse when the condition (2.1) holds [12, Theorem 14]. In the following theorem, we can improve Theorem 14 in [12] by assuming that $f_x(n, 0)$ is n_∞-summablely similar to $f_x(n, x(n, n_0, x_0))$, instead of the above condition (2.1).

THEOREM 2.3. *Under the same conditions of Lemma 2.1, the solution* $x = 0$ *of* (1.1) *is hS if and only if the solution* $x = 0$ *of* (1.1) *is hSV.*

Proof. If $z = 0$ of (1.4) is hS, then $x = 0$ of (1.1) is also hS by Theorem 2 [12]. Also, if $x = 0$ of (1.1) is hS, then $v = 0$ of (1.3) is also hS by Theorem 3.3 [13]. Thus $x = 0$ of (1.1) is hSV by Lemma 2.1. □

REMARK 2.4. If $h(n)$ is a positive bounded function on $\mathbb{N}(n_0)$, then $\frac{h(n)}{h(n+1)}$ is not bounded in general.

For example, letting $h(n) = \exp(-\sum_{s=n_0}^{n-1} s)$, $h(n)$ is a positive bounded function on $\mathbb{N}(n_0)$ but $\lim_{n\to\infty} \frac{h(n)}{h(n+1)} = \lim_{n\to\infty} \exp(n) = \infty$. Thus if $\frac{h(n)}{h(n+1)}$ is bounded, then the condition $\frac{h(n)}{h(n+1)}|F(n)| \in l_1(\mathbb{N}(n_0))$ in Lemma 2.1 can be replaced by $|F(n)| \in l_1(\mathbb{N}(n_0))$.

EXAMPLE 2.5. To illustrate Theorem 2.3, we consider the scalar difference equation with the initial value

$$x(n+1) = f(n, x(n)) = \frac{e^{\lambda(n)} x(n)}{\sqrt{1 + 2x^2(n)}}, x(n_0) = x_0, n \geq n_0 \tag{2.2}$$

where $\lambda(n)$ is a function defined on $\mathbb{N}(n_0)$ with $0 < c < -\lambda(n)$ for some constant c. Then the two concepts of hS and hSV of (2.2) are equivalent.

Proof. The solution $x(n)$ of Eqn (2.2) with the initial value $x(n_0) = x_0$ is given by

$$x(n, n_0, x_0) = \frac{\exp(\sum_{l=n_0}^{n-1}\lambda(l))x_0}{\sqrt{1 + 2x_0^2[1 + \sum_{l=n_0}^{n-2}\exp(2\sum_{j=n_0}^{l}\lambda(j))]}},$$

where $\sum_{l=0}^{-2}\exp(2\sum_{j=0}^{2}\lambda(j)) = -1$ and $\sum_{l=0}^{-1}\exp(2\sum_{j=0}^{2}\lambda(j)) = 0$. Then we obtain

$$\begin{aligned} |x(n, n_0, x_0)| &\leq |x_0|\exp(\sum_{l=n_0}^{n-1}\lambda(l)) \\ &= c_1|x_0|h(n)h^{-1}(n_0),\ n \geq n_0, \end{aligned}$$

where $h(n) = \exp(\sum_{l=0}^{n-1}\lambda(l))$ and $c_1 = 1$. Thus the zero solution $x = 0$ of (2.2) is hS for a function $\lambda(n)$ with $0 < c < -\lambda(n)$ for some constant c. The fundamental matrix solutions of (1.4) and (1.3) are given by, respectively,

$$\begin{aligned} \Phi(n, n_0, x_0) &= \frac{\partial x(n, n_0, x_0)}{\partial x_0} \\ &= \frac{\exp(\sum_{l=n_0}^{n-1}\lambda(l))}{\{1 + 2x_0^2[1 + \sum_{l=n_0}^{n-2}\exp(2\sum_{j=n_0}^{l}\lambda(j))]\}^{\frac{3}{2}}} \end{aligned}$$

and

$$\Phi(n, n_0, 0) = \frac{\partial x(n, n_0, 0)}{\partial x_0} = \exp(\sum_{l=n_0}^{n-1}\lambda(l)).$$

Then we have

$$\begin{aligned} |\Phi(n, n_0, x_0)| &\leq |\Phi(n, n_0, 0)| \leq \exp(\sum_{l=n_0}^{n-1}\lambda(l)) \\ &\leq h(n)h^{-1}(n_0),\ n \geq n_0. \end{aligned}$$

Thus $x = 0$ of (2.2) is hSV. Also, we easily can see that $f_x(n, 0)$ and $f_x(n, x(n, 0, x_0))$ are n_∞-summablely similar with some bounded invertible matrix

$$S(n) = \frac{1 + 2x_0^2[1 + \sum_{l=0}^{n-2}\exp(2\sum_{j=0}^{l}\lambda(j))]}{1 + 2x_0^2[1 + \sum_{l=0}^{n-3}\exp(2\sum_{j=0}^{l}\lambda(j))]}$$

whose inverse $S^{-1}(n)$ is bounded. In fact, there exists $F(n)$ absolutely summable over $\mathbb{N}(0)$, i.e.,

$$\sum_{n=0}^{\infty} |F(n)| < \infty$$

such that

$$\begin{aligned} & S(n+1)f_x(n,x(n,0,x_0)) - f_x(n,0)S(n) \\ = & \left(\frac{1+2x_0^2[1+\sum_{l=0}^{n-2}\exp(2\sum_{j=0}^{l}\lambda(j))]}{1+2x_0^2[1+\sum_{l=0}^{n-1}\exp(2\sum_{j=0}^{l}\lambda(j))]} \right)^{\frac{1}{2}} e^{\lambda(n)} \\ & -e^{\lambda(n)} \left(\frac{1+2x_0^2[1+\sum_{l=0}^{n-2}\exp(2\sum_{j=0}^{l}\lambda(j))]}{1+2x_0^2[1+\sum_{l=0}^{n-3}\exp(2\sum_{j=0}^{l}\lambda(j))]} \right) \\ = & F(n), \end{aligned}$$

since

$$\begin{aligned} \sum_{n=0}^{\infty} |F(n)| & \leq e^{-c} \sum_{n=0}^{\infty} \left| -2x_0^2 \exp\left(2\sum_{l=0}^{n-2}\lambda(l)\right)\right| \\ & \leq 2x_0^2 e^{-c} \sum_{n=0}^{\infty} e^{-2c(n-1)} \\ & \leq 2x_0^2 e^{c}(1+\frac{1}{e^{2c}-1}) < \infty, \ |x_0| < \infty. \end{aligned}$$

Since all conditions of Lemma 2.1 are satisfied, the solution $v = 0$ of (1.3) is hS if and only if the solution $z = 0$ of (1.4) is also hS. Hence the two notions hS and hSV of (1.1) are equivalent by Theorem 2.3. □

3. h-stability for perturbed difference systems

In this section, we examine the property of hS for the perturbed difference system of nonlinear difference system (1.1) using the comparison principle and Bihari-type inequalities. In our subsequent discussion we assume that for any two sequences $y(n)$ and $z(n) \in F(\mathbb{N}(n_0), \mathbb{R}^m)$, the operator S satisfies the following property : $|y(n)| \leq |z(n)|$ implies $|Sy(n)| \leq |Sz(n)|$ and $|Sy(n)| \leq |S||y(n)|$ for each finite interval $n_0 \leq n \leq l$ of $\mathbb{N}(n_0)$ and $|S| : F(\mathbb{N}(n_0), \mathbb{R}^+) \to F(\mathbb{N}(n_0), \mathbb{R}^+)$ is a nondecreasing operator.

THEOREM 3.1. *Assume that $x = 0$ of (1.1) is hSV with the nonincreasing function $h(n)$ and*

$$|g(n, y, Sy)| \leq \iota(n, |y|, |Sy|)$$

where $\iota : \mathbb{N}(n_0) \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing in r and u for each fixed $n \in \mathbb{N}(n_0)$ with $\iota(n, 0, 0) = 0$. Consider the scalar difference equation

$$u(n+1) = u(n) + c\iota(n, u(n), |S|u(n)),\ u(n_0) = u_0,\ c > 1. \tag{3.1}$$

If the zero solution $u = 0$ of (3.1) is hS, then the zero solution $y = 0$ of (1.2) is also hS whenever $u_0 = c|y_0|$.

Proof. Using the discrete analogue of Alekseev's formula in [1], the solutions of (1.1) and (1.2) with the same initial values are related by

$$\begin{aligned} & y(n, n_0, y_0) \\ &= x(n, n_0, y_0) + \sum_{l=n_0}^{n-1} \int_0^1 \Phi(n, l+1, \mu(y(l), \tau)) d\tau \cdot g(l, y(l), Sy(l)), \end{aligned}$$

where $\mu(y(n), \tau) = f(n, y(n)) + \tau g(n, y(n), Sy(n))$, $\tau \in [0, 1]$ and $\Phi(n, n_0, x_0)$ is the fundamental matrix of (1.4). Then we have

$$\begin{aligned} & |y(n, n_0, y_0)| \\ \leq\ & |x(n, n_0, y_0)| + \sum_{l=n_0}^{n-1} \int_0^1 |\Phi(n, l+1, \mu(y(l), \tau))| d\tau |g(l, y(l), Sy(l))| \\ \leq\ & c|y_0| h(n) h^{-1}(n_0) + c \sum_{l=n_0}^{n-1} h(n) h^{-1}(l+1) \iota(l, |y(l)|, |Sy(l)|) \\ \leq\ & c|y_0| + c \sum_{l=n_0}^{n-1} \iota(l, |y(l)|, |S||y(l)|), \end{aligned}$$

since $h(n)$ is nonincreasing. Thus we obtain

$$\begin{aligned} |y(n)| - c \sum_{l=n_0}^{n-1} \iota(l, |y(l)|, |Sy(l)|) \ &\leq\ c|y_0| = u_0 \\ &=\ u(n) - c \sum_{l=n_0}^{n-1} \iota(l, u(l), |S|u(l)). \end{aligned}$$

By the comparison principle in [10], we have $|y(n)| < u(n)$ for all $n \geq n_0 \geq 0$. Also we see that

$$\begin{aligned} |y(n)| < u(n) &\leq c_1 u_0 h(n) h^{-1}(n_0) \\ &= d|y_0| h(n) h^{-1}(n_0),\ d = c_1 c > 1, \end{aligned}$$

since $u = 0$ of (3.1) is hS. This completes the proof. □

REMARK 3.2. Letting $g(n, y, Sy) = g(n, y)$ and $\iota(n, u, w) = \iota(n.u)$ in Theorem 3.1, we obtain the same result as that of Theorem 10 in [6].

We need the following lemma related to the discrete Bihari-type inequality to obtain the property of hS for difference system (1.2).

LEMMA 3.3. *Suppose that all conditions of Lemma 2.8 in [9] hold and let $w(u) = u^p$, where $0 < p < \infty$. Then*

(i) $u(n) \leq d \exp[\sum_{l=n_0}^{n-1}(a(l) + b(l)\sum_{k=n_0}^{l-1} c(k))]$ *for all* $n \geq n_0$, *where* $p = 1$,

(ii) $u(n) \leq [d^q + q\sum_{l=n_0}^{n-1}(a(l) + b(l)\sum_{k=n_0}^{l-1} c(k))]^{\frac{1}{q}}$ *for* $n \leq m$, *where* $p \neq 1, q = 1 - p$ *and*

$$\begin{aligned} m &= \sup\{n \in \mathbb{N}(n_0) | W(d) + \sum_{l=n_0}^{n-1}(a(l) \\ &\quad + b(l)\sum_{k=n_0}^{l-1} c(k)) \in \ Dom\ W^{-1}\} \\ &= \sup\{n \in \mathbb{N}(n_0) | W(d) + \sum_{l=n_0}^{n-1} a(l) + b(l)\sum_{k=n_0}^{l-1} c(k)) \geq -\frac{u_0^q}{q}\}. \end{aligned}$$

Proof. Let

$$\begin{aligned} v(n) &= d + \sum_{l=n_0}^{n-1} a(l) u^p(l) \\ &\quad + \sum_{l=n_0}^{n-1} b(l) \sum_{k=n_0}^{l-1} c(k) u^p(k),\ v(n_0) = d. \end{aligned}$$

Then we have

$$\begin{aligned}\Delta v(n) &= a(n)u^p(n) + b(n)\sum_{l=n_0}^{n-1} c(l)u^p(l) \\ &\leq a(n)v^p(n) + b(n)\sum_{l=n_0}^{n-1} c(l)v^p(l) \\ &\leq [a(n) + b(n)\sum_{l=n_0}^{n-1} c(l)]v^p(n).\end{aligned}$$

If $p = 1$ then $W(u) = \ln(\frac{u}{u_0})$ and $W^{-1}(u) = u_0 \exp u$. Thus we obtain

$$\begin{aligned}v(n) &\leq u_0 \exp[\ln(\frac{d}{u_0}) + \sum_{l=n_0}^{n-1}(a(l) + b(l)\sum_{k=n_0}^{l-1} c(k))] \\ &= d\exp[\sum_{l=n_0}^{n-1}(a(l) + b(l)\sum_{k=n_0}^{l-1} c(k))]\end{aligned}$$

for all $n \geq n_0$.

If $p \neq 1$ and let $d(n) = 0, k(n) = a(n) + b(n)\sum_{l=n_0}^{n-1} c(l)$ in Lemma 15.5 in [2], then we have

$$v(n) \leq [d^q + q\sum_{l=n_0}^{n-1}(a(l) + b(l)\sum_{k=n_0}^{l-1} c(k))]^{\frac{1}{q}}$$

for $n_0 \leq n \leq m$. □

THEOREM 3.4. *Suppose that the zero solution $z = 0$ of (1.4) is hS with the positive function $h(n)$ and for any $n \geq n_0$*

$$|g(n, y, Sy)| \leq a(n)|y| + b(n)\sum_{l=n_0}^{n-1} c(l)|y(l)|$$

where $a, b, c \in F(\mathbb{N}(n_0), \mathbb{R}^+)$ and

$$M(n) = \exp c_1 \left[\sum_{l=n_0}^{n-1}[\frac{h(l)}{h(l+1)}a(l) + \frac{b(l)}{h(l+1)}\sum_{k=n_0}^{l-1} h(k)c(k)]\right] < \infty.$$

Then the zero solution $y = 0$ of (1.2) is hS.

Proof. In view of the formula $x(n, n_0, y_0) = [\int_0^1 \Phi(n, n_0, sy_0)ds]y_0$ and assumption we have

$$\begin{aligned} |y(n)| &\leq c_1 h(n)h^{-1}(n_0)|y_0| + \sum_{l=n_0}^{n-1} c_1 h(n)h^{-1}(l+1)|g(l, y(l), Sy(l))| \\ &\leq c_1 h(n)h^{-1}(n_0)|y_0| + c_1 \sum_{l=n_0}^{n-1} h(n)h^{-1}(l+1)[a(l)|y(l)| \\ &\quad +b(l) \sum_{k=n_0}^{l-1} c(k)|y(k)|]. \end{aligned}$$

Putting $u(n) = |y(n)|h^{-1}(n)$, then we obtain the following inequality from Lemma 3.3 (i) :

$$\begin{aligned} u(n) &\leq c_1 u(n_0) \\ &\quad +c_1 \sum_{l=n_0}^{n-1} [\frac{h(l)}{h(l+1)} a(l)u(l) + \frac{b(l)}{h(l+1)} \sum_{k=n_0}^{l-1} h(k)c(k)u(k)] \\ &\leq c_1 u(n_0) \exp\left[c_1 \sum_{l=n_0}^{n-1} [\frac{h(l)}{h(l+1)} a(l) + \frac{b(l)}{h(l+1)} \sum_{k=n_0}^{l-1} h(k)c(k)]\right] \\ &\leq c_1 u(n_0) M(\infty). \end{aligned}$$

Hence we obtain

$$|y(n)| \leq M|y_0|h(n)h^{-1}(n_0), \; M = c_1 M(\infty) \geq 1,$$

for all $n \geq n_0$, and the proof is complete. □

COROLLARY 3.5. *Suppose that the zero solution $z = 0$ of (1.4) is hS with the positive increasing function $h(n)$ and for any $n \geq n_0$*

$$|g(n, y, Sy)| \leq a(n)|y| + b(n) \sum_{l=n_0}^{n-1} c(l)|y(l)|$$

where $a, b, c \in l_1(\mathbb{N}(n_0))$. Then the zero solution $y = 0$ of (1.2) is also hS.

Proof. From the increasing function of $h(n)$, we obtain

$$\begin{aligned} M(n) &= \exp c_1 \Big[\sum_{l=n_0}^{n-1} \Big[\frac{h(l)}{h(l+1)} a(l) + \frac{b(l)}{h(l+1)} \sum_{k=n_0}^{l-1} h(k)c(k) \Big] \\ &\le \exp c_1 \Big(\sum_{l=n_0}^{n-1} [a(l) + b(l) \sum_{k=n_0}^{l-1} c(k)] \Big) < \infty. \end{aligned}$$

Hence the zero solution $y = 0$ of (1.2) is also hS by Theorem 3.4. □

EXAMPLE 3.6. To illustrate Theorem 3.4, we consider the scalar difference equation (2.2)

$$x(n+1) = f(n, x(n)) = \frac{e^{\lambda(n)} x(n)}{\sqrt{1 + 2x^2(n)}}, \; x(n_0) = y_0$$

and its perturbation equation with same initial value

$$\begin{aligned} y(n+1) &= f(n, y(n)) + g(n, y(n), Sy(n)) \\ &= \frac{e^{\lambda(n)} y(n)}{\sqrt{1 + 2y^2(n)}} \\ &\quad + \frac{e^{-n} y^7(n)}{1 + 4y^6(n)} \sin(ny(n)) + e^{-2n} Sy(n), \; y(n_0) = y_0, \end{aligned} \tag{3.2}$$

where an operator $S : F(\mathbb{N}(n_0), \mathbb{R}) \to F(\mathbb{N}(n_0), \mathbb{R})$ is given by

$$Sy(n) = \sum_{l=n_0}^{n-1} e^{-3l} \frac{y^7(l)}{1 + 4y^6(l)} \sin(ly(l)), \; n \ge l \ge n_0.$$

Then the zero solution $y = 0$ of (3.2) is hS with a positive function $h(n)$, provided that $\lambda(n)$ is a function defined on $\mathbb{N}(n_0)$ and for all $n \ge n_0 \ge 0$,

$$\begin{aligned} M(n) &= \exp \left[\sum_{l=n_0}^{n-1} (e^{-l} + e^{-2l} \sum_{k=n_0}^{l-1} e^{-3k}) \right] \\ &\le \exp \left[\frac{e^{-(n_0-1)}}{e-1} + \frac{e^{-5(n_0-1)}}{(e^3-1)(e^5-1)} - \frac{e^{-2(n_0-4)}}{(e^3-1)(e^5-1)} \right]. \end{aligned}$$

Proof. It follows from Example 2.5 that $x = 0$ of (2.2) is hSV with the positive function $h(n) = \exp(\sum_{l=0}^{n-1} \lambda(l))$. On the other, the perturbation g of the nonlinear equation (3.2) satisfies the following conditions of Theorem 3.4 for any $n \geq n_0$,

$$|g(n, y(n), Sy(n))| \leq e^{-n}|y(n)| + e^{-2n} \sum_{l=n_0}^{n-1} e^{-3l}|y(l)|$$

and

$$\begin{aligned} M(n) &\leq \exp[\frac{e^{-n_0} - e^{-(n-n_0)}}{1 - e^{-1}} \\ &\quad + \sum_{l=n_0}^{n-1} (\frac{e^{-3n_0} e^{-2l}}{1 - e^{-3}} - \frac{e^{3n_0}}{1 - e^{-3}} \sum_{k=n_0}^{l-1} e^{-5k})] \\ &\leq \exp[\frac{e^{-(n_0-1)}}{e - 1} + \frac{e^{-5(n_0-1)}}{(e^3 - 1)(e^5 - 1)} - \frac{e^{-2(n_0-4)}}{(e^3 - 1)(e^5 - 1)}] \\ &= M(\infty) < \infty. \end{aligned}$$

Hence the zero solution $y = 0$ of (3.2) is hS by Theorem 3.4. □

THEOREM 3.7. *Suppose that $x = 0$ of (1.1) is hS and*

$$\begin{aligned} |g(n, y, Sy)| &\leq a(n)w(|y|) + b(n)|Sy|, \\ |Sy| &\leq \sum_{l=n_0}^{n-1} c(l)w(|y|), \ n \geq n_0 \end{aligned}$$

where $w \in \hat{F}$ with corresponding multiplier function r and

$$\begin{aligned} \hat{a}(n) &= \frac{h(n_0)a(n)}{h(n+1)|y_0|} r(\frac{h(n)|y_0|}{h(n_0)}) \\ \hat{b}(n) &= \frac{h(n_0)b(n)}{h(n+1)|y_0|}, \ \hat{c}(n) = r(\frac{h(n)|y_0|}{h(n_0)})c(n) \\ K &= \max_{n \in \mathbb{N}(n_0)} W^{-1}[W(d) + \sum_{l=n_0}^{n-1} (\hat{a}(l) + \hat{b}(l) \sum_{k=n_0}^{l-1} \hat{c}(k))] < \infty. \end{aligned}$$

Then for y_0 sufficiently small, every solution $y(n) = y(n, n_0, y_0)$ of (1.2) satisfies

$$|y(n)| \leq K h(n) h^{-1}(n_0)|y_0|.$$

Proof. Using the assumptions of the theorem, we have the following inequality :

$$
\begin{aligned}
|y(n)| \;\leq\; & dh(n)h^{-1}(n_0)|y_0| + d\sum_{l=n_0}^{n-1} h(n)h^{-1}(l+1)[a(l)w(|y(l)|) \\
& + b(l)\sum_{k=n_0}^{l-1} c(k)w(|y(k)|)]
\end{aligned}
$$

for all $n \geq n_0$. Let $u(n) = \frac{|y(n)|h(n_0)}{h(n)|y_0|}$. Then we have

$$u(n) \leq d + \sum_{l=n_0}^{n-1} \hat{a}(l)w(u(l)) + \sum_{l=n_0}^{n-1} \hat{b}(l)\sum_{k=n_0}^{l-1} \hat{c}(k)w(u(k)),$$

where $\hat{a}(n) = \frac{h(n_0)a(n)}{h(n+1)|y_0|} r(\frac{h(n)|y_0|}{h(n_0)})$ and $\hat{b}(n) = \frac{h(n_0)b(n)}{h(n+1)|y_0|}$, and $\hat{c}(n) = r(\frac{h(n)|y_0|}{h(n_0)})c(n)$. By Lemma 2.8 in [9] we have

$$|y(n)| \leq h(n)h^{-1}(n_0)|y_0|W^{-1}[W(d) + \sum_{l=n_0}^{n-1}(\hat{a}(l) + \hat{b}(l)\sum_{k=n_0}^{l-1}\hat{c}(k))]$$

for $n \leq m$, where

$$m = \sup\{n \in \mathbb{N}(n_0) | W(d) + \sum_{l=n_0}^{n-1}[\hat{a}(l) + \hat{b}(l)\sum_{k=n_0}^{l-1}\hat{c}(k)] \in \text{Dom}W^{-1}\}.$$

□

COROLLARY 3.8. *Suppose that $x = 0$ of (1.1) is hS with a nondecreasing $h(n)$ and*

$$
\begin{aligned}
& |\Phi(n, l+1, \mu(y(l)), \tau)g(n, y, Sy)| \\
\leq\; & h(n)h^{-1}(l)[a(l)w(|y(l)|) + b(l)\sum_{k=n_0}^{l-1} c(k)w(|y(k)|)],\; n \geq n_0
\end{aligned}
$$

where $w \in \hat{F}$ with corresponding multiplier function $r(\alpha) = \alpha$ for all $\alpha > 0$. Then

$$|y(n)| \leq h(n)h^{-1}(n_0)|y_0|W^{-1}[W(d) + \sum_{l=n_0}^{n-1}(a(l) + b(l)\sum_{k=n_0}^{l-1}c(k))]$$

for all $n \geq n_0$.

Proof. Put $u(n) = \frac{|y(n)|h(n_0)}{h(n)|y_0|}$, then we have

$$u(n) \leq d + \sum_{l=n_0}^{n-1} [a(l)w(u(l)) + b(l) \sum_{k=n_0}^{l-1} c(k)w(u(k)],$$

for all $n \geq n_0$. Thus theorem is proved by Lemma 2.8 in [9]. □

References

[1] R. P. Agarwal, *Difference equations and inequalities*, 2nd ed., Marcel Dekker, New York, 2000.

[2] D. Bainov and P. Simeonov, *Integral inequalities and applications*, Kluwer Academic Publshers, 1992.

[3] R. Conti, *Sulla t_∞-similitudine tra matrici e la stabilitaá dei sistemi differenzialelineari*, Atti Accad. Naz. Lincei. Cl. Sci. Fis. Mat. Nat. **19** (1955), no. 8, 247–250.

[4] S. K. Choi, N. J. Koo and H. S. Ryu, *h-stability of differential systems via t_∞-similarity*, Bull. Korean Math. Soc. **34** (1997), 371–383.

[5] S. K. Choi, Y. H. Goo and N. J. Koo, *Lipschitz stability and exponential asymptotic stability for the nonlinear functional differential systems*, Dyn. Syst. Appl. **6** (1997), 397–410.

[6] S. K. Choi and N. J. Koo, *Variationally stable difference systems by n_∞- similarity*, J. Math. Anal. Appl. **249** (2000), 553–568.

[7] S. K. Choi, N. J. Koo and Y. H. Goo, *Variatonally stable difference systems*, J. Math. Anal. Appl. **256** (2001), 587–605.

[8] ———, *Variationally asymptotically stable difference systems*, to appear in Discrete Dyn. Nat. Soc.

[9] S. K. Choi and N. J. Koo, *Asymptotic equivalence between two linear Volterra difference systems*, Comput. Math. Appl. **47** (2004), 461–471.

[10] V. Lakshmikantham and D. Trigiante, *Theory of difference equations with applications to numerical analysis*, Academic Press, San Diego, 1988.

[11] R. Medina, *Asymptotic behavior of nonlinear difference systems*, J. Math. Anal. Appl. **219** (1998), 294–311.

[12] R. Medina and M. Pinto, *Stability of nonlinear difference equations* , Dynam. Systems. Appl. **2** (1996), 397–404.

[13] ———, *Variationally stable difference equations*, Nolinear Anal. **2** (1997), 1141–1152.

[14] M. Pinto, *Perturbations of asymptotically stable differenctial systems*, Analysis **4** (1984), 161–175.

[15] W. F. Trench, *Linear asymptotic equilibrium and uniform, exponential, and strictly stability of linear difference systems*, Comput. Math. Appl. **36** (1998), 261–267.

SUNG KYU CHOI AND NAM JIP KOO, DEPARTMENT OF MATHEMATICS, CHUNGNAM NATIONAL UNIVERSITY, DAEJEON 305-764, KOREA
E-mail: skchoi@math.cnu.ac.kr
njkoo@math.cnu.ac.kr

SE MOK SONG, DEPARTMENT OF MATHEMATICS EDUCATION, CHONGJU UNIVERSITY, CHONGJU 360-764, KOREA
E-mail: smsong@chongju.ac.kr