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THOMSEN CONDITIONS ON WEBS AND THEIR CORRESPONDING LOOPS

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Abstract. We introduce certain local Thomsen condition in a 3-web and prove that it is equivalent to the equation $a - (a - b) = b$ in its corresponding loop, where we denote the loop operation additively for convenience and simplicity, even though the loop is neither associative nor commutative. Also we interpret such local Thomsen condition using orthogonality of chains in a web.

1. Introduction

W. Blaschke called systems of curves on surfaces as *webs*, and suggested to develop a theory of abstract webs in 1928. This task was carried on firstly by G. Thomsen (cf. [22]), K. Reidemeister (cf. [20]) and later by G. Bol, R. H. Bruck and others (cf. [1, 2, 3]). Each 3-web can be related to an algebraic structure such as a loop (cf. Theorem 2.1). The loops gained more interest than webs and their theory was developed. Study of Bol loops and Moufang loops were in the center among many other loops, while $K-$loops were introduced most recently and studied mainly by H. Karzel, A. Kreuzer, H. Kiechle, B. Im, and their students, though A. Kreuzer proved that $K-$loops and Bruck loops are the same (cf. [6, 10, 12, 13, 15, 16, 19]).

If certain configurational theorems named after Thomsen, Reidemeister, Bol in the web are valid (cf. [20, 21, 22]), then the corresponding loop have additional properties or turn out to be groups or even commutative groups. Webs were revisited in [4, 5, 7, 17] and studied related to K-loops in [8, 9, 11]. Reflections and rotations are newly interpreted in a web's point of view in [8, 9]. The closing of web configurations characterizing

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certain classes of webs are expressed elegantly by using these reflections and rotations in webs. The local Thomsen condition $(T, 0\,; i, j)$ and local Reidemeister condition $(R, 0, i)$ have been introduced in a 3-web and their equivalent conditions are proved in a relation to the corresponding loops in [9]. However, no figures are included in [9], which makes the theory uneasy to understand.

In this paper, we show some helpful figures in association with $(R, 0, i)$ and $(T, 0\,; i, j)$, and develop another new local Thomsen condition denoted by $(T, 0\,; k)$ to obtain Theorem 3.3 as our main result.

2. Some properties of webs and their corresponding loops

Let $\mathcal{W} = (\mathcal{P}, \mathcal{G})$ be a 3−web, i.e. a nonempty set $\mathcal{P}$ of points and a set $\mathcal{G}$ of generators(or lines), where $\mathcal{G}$ is a disjoint union of three classes $\mathcal{G}_i$ $(i = 1, 2, 3)$ such that the following two conditions hold:

W1 For each point $x \in \mathcal{P}$ and $i \in \{1, 2, 3\}$, there exists exactly one generator in $\mathcal{G}_i$ containing x, where we denote such generator by $[x]_i$,

W2 Any two generators from distinct classes intersect in exactly one point, and each generator contains at least two points.

A subset $\mathcal{C} \subset P$ is called an *i-chain* if for each $Y \in \mathcal{G}_j \cup \mathcal{G}_k$ the intersection $Y \cap C$ consists of a single point. Let $\mathcal{C}_i$ be the set of all i-chains, then $\mathcal{G}_i \subset \mathcal{C}_i$. To each chain $C \in \mathcal{C}_i$ there corresponds a *reflection* $\widetilde{C}$ as in [8, 9]:

$$\widetilde{C} : \mathcal{P} \mapsto \mathcal{P}; x \mapsto \big[[x]_j \cap C\big]_k \cap \big[[x]_k \cap C\big]_j,$$

i.e. an involution of the set $\mathcal{P}$ fixing exactly the points of C and interchanging the generators of $\mathcal{G}_j$ and $\mathcal{G}_k$, i.e. $\widetilde{C} \in Aut(\mathcal{P},\, \mathcal{G}_j \cup \mathcal{G}_k)$. In [18], such $\widetilde{C}$ is considered only in the case when $C \in \mathcal{G}_i$ and called the Bol reflection with axis C. In our general case of $C \in \mathcal{C}_i$, we call $\widetilde{C}$ a (*chain*) *reflection* (of type i).

The closing of web configurations characterizing certain classes of webs are expressed elegantly by using reflections in chains or rotations in a web(cf. [8, 9]).

A web $\mathcal{W}$ is called a *Reidemeister web* if the following Reidemeister condition RE is valid(cf. [20, 21]):

RE If $a \in \mathcal{P}, b_i \in [a]_i$, and $c_{ij} = [b_i]_j \cap [b_j]_i$ for $i, j \in \{1, 2, 3\}$ with $i \neq j$, then $[c_{12}]_3 \cap [c_{23}]_1 \cap [c_{13}]_2 \neq \emptyset$.

This Reidemeister condition can be written in the following form using chain reflections (cf. [8]):

RE′ If $A, B, C, D \in \mathcal{G}_3$ with $Fix(\widetilde{A} \circ \widetilde{B} \circ \widetilde{C} \circ \widetilde{D}|_{\mathcal{G}_1}) \neq \emptyset$, then

$\widetilde{A} \circ \widetilde{B} \circ \widetilde{C} \circ \widetilde{D}|_{\mathcal{G}_1} = id_{\mathcal{G}_1}$.

A web $\mathcal{W}$ is called a *Thomsen web* if the following Thomsen condition TH is valid (cf. [22]):

TH If $p,\, q \in \mathcal{P}$, then $[[p]_1 \cap [q]_2]_3 \cap [[p]_2 \cap [q]_3]_1 \cap [[p]_3 \cap [q]_1]_2 \neq \emptyset$.

The following local Thomsen condition $(T, 0\,; i, j)$ and local Reidemeister condition $(R, 0, i)$ have been introduced in [9]:

(T, 0 ; i, j) If $x \in [0]_i$ and $y \in [0]_j$, then $[0]_k \cap \big[[x]_k \cap [y]_i\big]_j \cap \big[[y]_k \cap [x]_j\big]_i \neq \emptyset$,

(R,0,i) Let $p,\, q \in [0]_i$, $p_k = [0]_k \cap [p]_j$, $q_j = [0]_j \cap [q]_k$, $p' = [p_k]_i \cap [q]_j$ and $q' = [q_j]_i \cap [p]_k$, then $\big[[p']_k \cap [0]_j\big]_i = \big[[q']_j \cap [0]_k\big]_i$.

On the other hand, a loop is a group without associativity, hence each right inverse and left inverse need not be the same, though it contains the two-sided identity. In this paper we prefer to denote the loop operation additively as $+$ for convenience and simplicity, though the operation is neither commutative nor associative. So our loop is a groupoid $(E, +)$ such that the equation $x + y = z$ has the unique solution in E whenever two of the three elements x, y, z are given. In other words, our loop is a quasigroup with the identity 0. For each $a \in E$ we define two permutations a^+ and ^+a on a loop $(E, +)$ by $a^+(x) = a + x$ and $^+a(x) =$

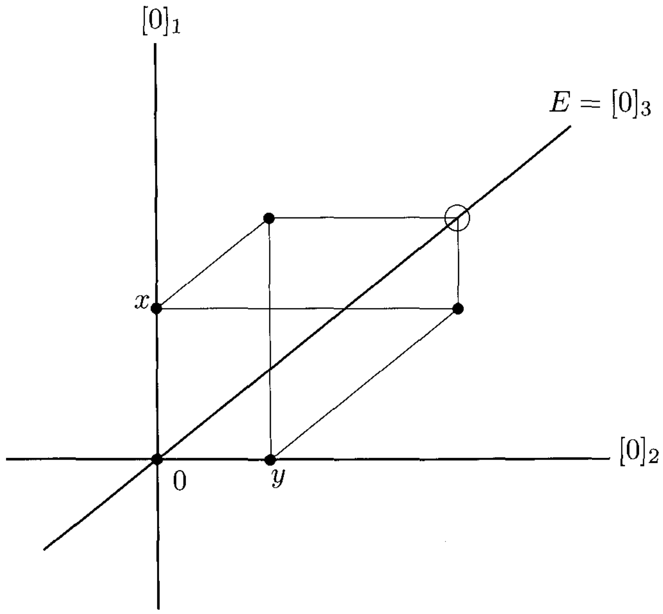

FIGURE 1. $(T, 0\,; 1, 2)$

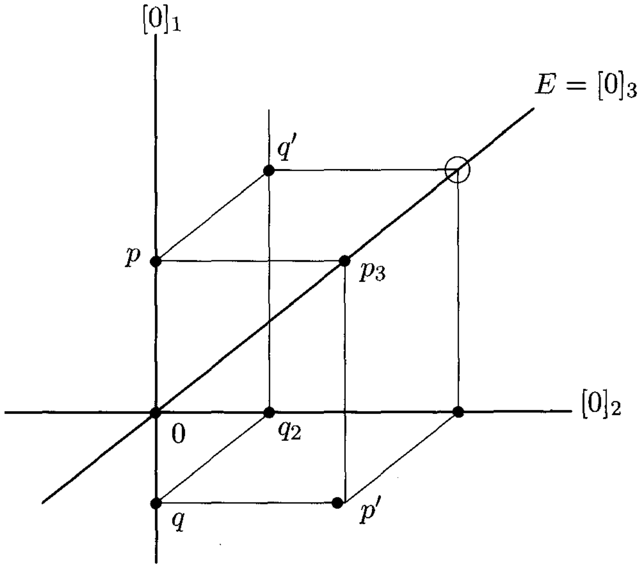

FIGURE 2. $(R, 0, 1)$

$x + a$. Then we let $-a = (a^+)^{-1}(0)$ and $\sim a = ({}^+a)^{-1}(0)$ so that $-a$ is the right inverse of a and $\sim a$ its left inverse, i.e. $\sim a + a = a + (-a) = 0$. Instead of $a + (-b)$ we write $a - b$. Also we consider permutations ν on E such that $\nu(x) = -x$, and $\delta_{a,b} = \left((a+b)^+\right)^{-1} \circ a^+ \circ b^+$ so that the equation $a + (b + c) = (a + b) + \delta_{a,b}(c)$ holds in a loop.

A loop $(E, +)$ is a *Bol-loop* if the identity $a^+ \circ b^+ \circ a^+ = (a + (b + a))^+$ holds for all $a, b \in E$. And a Bol-loop is a *Bruck-loop* if the automorphic inverse property $\nu \in Aut(E, +)$ holds and a *Moufang-loop* if ν is an antiautomorphism. On the other hand, a loop $(E, +)$ was called a *K-loop* if $\delta_{a,b} \in Aut(E, +)$, $\delta_{a,-a} = id$, $\delta_{a,b} = \delta_{a,b+a}$ and $\nu \in Aut(E, +)$ hold. And the theory on a $K-$ loop was developed independently in the beginning, even though A. Kreuzer proved that $K-$loops and Bruck loops are equivalent later on (cf. [13, 15, 16]).

For the point set $\mathcal{P}$ of a 3-web $\mathcal{W}$ we consider the following operation:
$\square_{ij} : \mathcal{P} \times \mathcal{P} \to \mathcal{P}; (x, y) \mapsto x \,\square_{ij}\, y = [x]_i \cap [y]_j$,
where $\{1, 2, 3\} = \{i, j, k\}$. By $x \,\square\, y$ we specially denote $x \,\square_{12}\, y$. Then we have the following canonical correspondence between 3-webs and loops (cf. [14] p. 81 (15.1)):

THEOREM 2.1. (*Characterization Theorem*)

(1) *Let* $\mathcal{W} = (\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)$ *be a 3-web and let* $E \in \mathcal{G}_3$ *with* $0 \in E$ *be fixed. And let* $\bar{a} = [0\Box a]_3$, $a^+ : E \to E; b \mapsto [[b]_1 \cap \bar{a}]_2 \cap E$, *and* $a + b = a^+(b)$ *for each* $a, b \in E$. *Then* $E = [0]_3$ *is a loop derived from* $\mathcal{W}$, *where* 0 *is its identity (cf. Figure 3).*

(2) *Given a loop* $(E, +)$ *with the identity element 0, let* $\mathcal{P} = E \times E$, $\mathcal{G}_1 = \{x \times E | x \in E\}$, $\mathcal{G}_2 = \{E \times x \mid x \in E\}$, *E be identified with a subset* $\{(x, x) | x \in E\} \subseteq \mathcal{P}$ *via* $x \equiv (x, x)$, *and let* $C(a^+) = \{x\Box a^+(x) | x \in E\}$. *Then* $(\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)$ *with* $\mathcal{G}_3 = \{C(a^+) | a \in E\}$ *is a 3-web. And the above construction* (1) *recovers the original loop* $(E, +)$.

Therefore, fixing a point 0 in a 3-web $\mathcal{W}$, we may call the set $E = [0]_k$ a loop-derivation $L(\mathcal{W}; 0; i, j)$ with the operation

$$a + b = \left[\left[[0]_i \cap [a]_j \right]_k \cap [b]_i \right]_j \cap E,$$

where 0 is the identity element of the loop $(E, +)$.

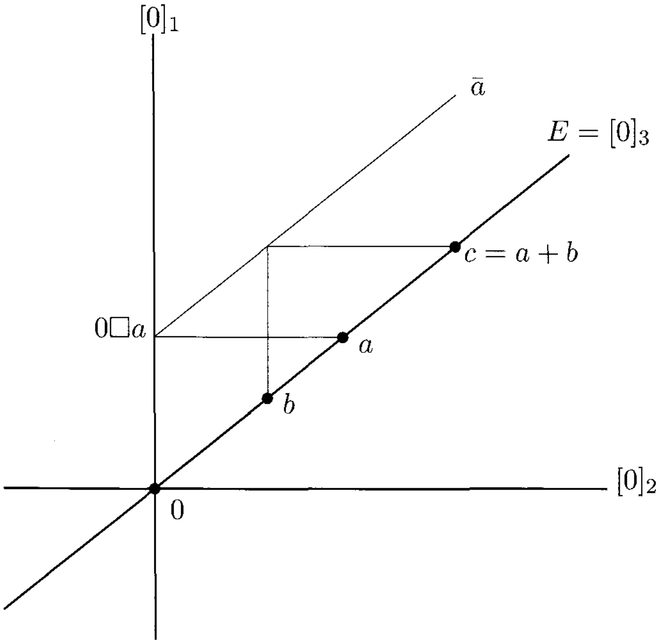

FIGURE 3. Loop operation on $E = L(\mathcal{W}; 0; 1, 2)$

Besides the reflections in chains or generators, to each point $0 \in \mathcal{P}$ and cyclic permutation $\gamma = (132)$ there corresponds in a natural way the local map called rotation $\gamma_0 : [0] = [0]_1 \cup [0]_2 \cup [0]_3 \to [0]$ defined by $\gamma_0(x) = [0]_{\gamma(i)} \cap [x]_{\gamma^{-1}(i)}$ if $x \in [0]_i$, $i \in \{1,2,3\}$. Note that $(\gamma_0)^{-1} = (\gamma^{-1})_0$ and that γ_0 induces on the set $\{[0]_1, [0]_2, [0]_3\}$ the permutation γ, while $(\gamma_0)^2$ induces γ^{-1} and $(\gamma_0)^3$ the identity. If there is an automorphism ω of the web $\mathcal{W}$ with $Fix(\omega) = \{0\}$ such that the restriction $\omega|_{[0]}$ coincides with one of the maps $(\gamma_0)^3, (\gamma_0)^2$ or γ_0, then ω is unique by (2.8) of [9], and ω is called the *extension* and we say that the point 0 is *n-extendable* for $n \in \{2,3,6\}$ if $\omega|_{[0]} = (\gamma_0)^{6/n}$. If $\omega|_{[0]} = \gamma_0$, then $\omega^2|_{[0]} = (\gamma_0)^2$ and $\omega^3|_{[0]} = (\gamma_0)^3$, i.e. if $0 \in \mathcal{P}$ is 6-extendable, then also 2- and 3-extendable, and if 0 is 2- and 3-extendable, then also 6-extendable.

From theorems (3.2), (3.6) and (3.8) of [9], we extract the following theorem and include Figures 4 and 5 relative to local Thomsen and Reidemeister conditions, which have been missing in [9].

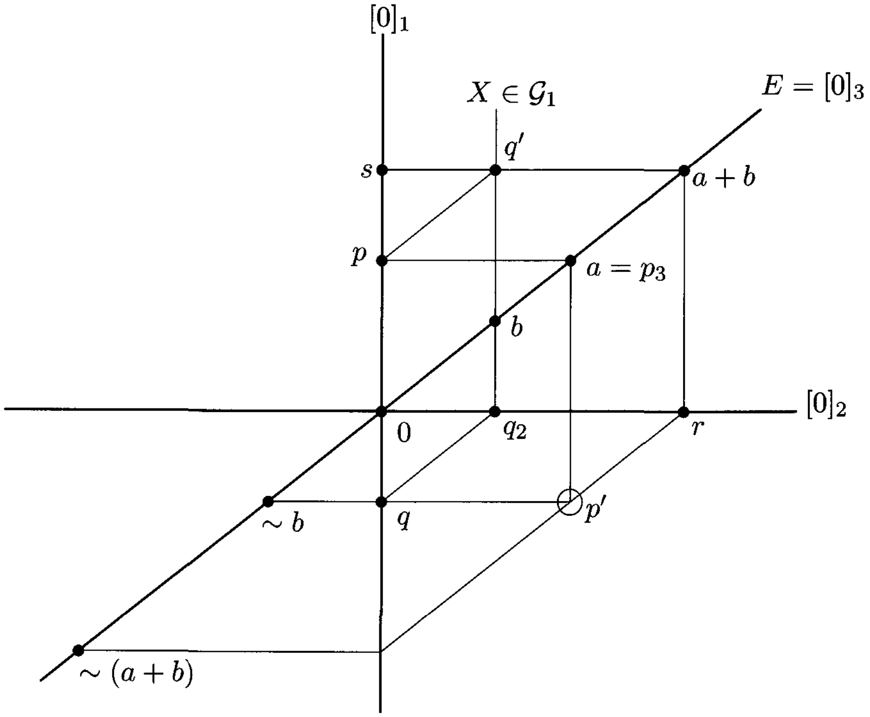

FIGURE 4. (4) ⇔ (5) ⇔ (6) of Theorem 2.2

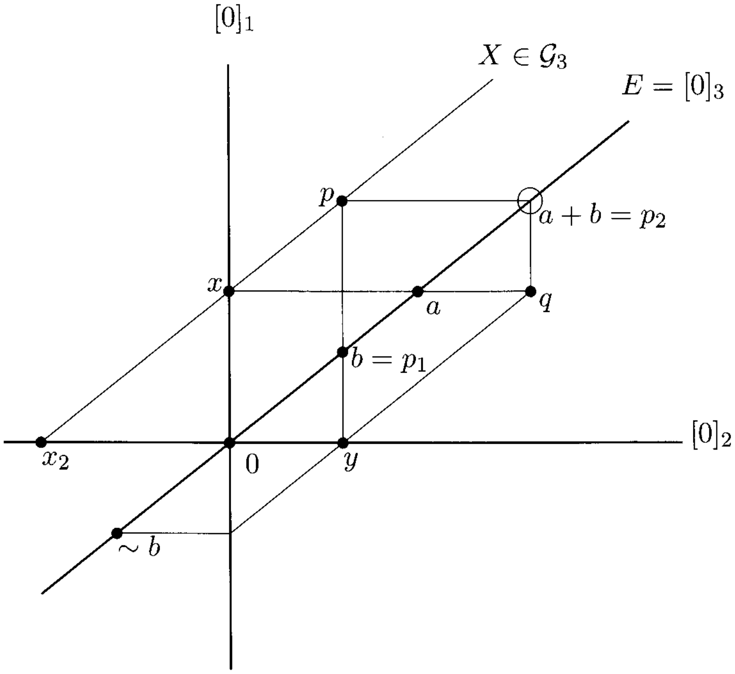

FIGURE 5. (7) ⇔ (8) ⇔ (9) of Theorem 2.2

THEOREM 2.2. *For a point 0 of a web $\mathcal{W}$ the following statements* (1), (2) *and* (3), *respectively* (4), (5) *and* (6), *respectively* (7), (8) *and* (9) *are equivalent:*

(1) *The bend-configuration BE(0; id) closes, i.e. for all* $p \in \mathcal{P}$,
$$[\,[\,[\,[p]_1\cap[0]_3]_2\cap[0]_1]_3\cap[0]_2]_1 \cap [\,[\,[\,[p]_2\cap[0]_1]_3\cap[0]_2]_1\cap[0]_3]_2 \cap [\,[\,[\,[p]_3\cap[0]_2]_1\cap[0]_3]_2\cap[0]_1]_3 \neq \emptyset$$ *(cf.* [8] *(6.3)),*

(2) $\nu \in Aut(E,+)$ *in a loop derivation* $E = L(\mathcal{W};0;i,j)$,

(3) *0 is a 2-extendable point,*

(4) $(R,0,i)$ *is satisfied,*

(5) $\sim(a+b)+a=\sim b$ *for all* $a,b\in(E,+)=L(\mathcal{W};0\,;i,j)$,

(6) *0 is a 3-extendable point,*

(7) $(T,0\,;i,j)$ *is satisfied,*

(8) $(E,+)=L(\mathcal{W};0\,;i,j)$ *is a crossed inverse loop, i.e.* $\sim b+(a+b)=a$, *i.e.* $a+(b-a)=b$ *for all* $a,b\in E$,

(9) *0 is a 6-extendable point.*

By the above theorem 2.2 and remarks, we obtain the following theorem. However, it can also be proved directly as follows:

THEOREM 2.3. *In a loop* $(E,+)$ *the following two properties* (1) *and* (2) *are equivalent to* (3)*:*

(1) $\sim(a+b)+a=\sim b$ *for all* $a,b\in E$,

(2) $\nu\in Aut(E,+)$,

(3) $b+(a-b)=a$ *for all* $a,b\in E$.

Proof. We can express (1) by ${}^{+}a\circ\nu^{-1}\circ a^{+}=\nu^{-1}$, (2) by $(\nu a)^{+}=\nu\circ a^{+}\circ\nu^{-1}$ and (3) by ${}^{+}(\nu a)=(a^{+})^{-1}$. So if (1) and (2) are valid, then ${}^{+}(\nu a)\overset{(1)}{=}\nu^{-1}\circ\big((\nu a)^{+}\big)^{-1}\circ\nu\overset{(2)}{=}\nu^{-1}\circ\big(\nu\circ(a^{+})^{-1}\circ\nu^{-1}\big)\circ\nu=(a^{+})^{-1}$, hence we have (3). Conversely, let (3) be valid, which can be also expressed by $\sim b+(a+b)=a$, since $-(\sim b)=b$. Then $\sim(a+b)+a\overset{(3)}{=}\sim(a+b)+\big(\sim b+(a+b)\big)\overset{(3)}{=}\sim b$ and this is (1), and $\sim a+(\sim b)\overset{(1)}{=}\sim a+\big(\sim(a+b)+a\big)\overset{(3)}{=}\sim(a+b)$, hence $\nu^{-1}\in Aut(E,+)$ and so $\nu\in Aut(E,+)$. □

Related to reflections and rotations in a web, there naturally corresponds various identities on the corresponding loop derivations. Now from the identities of the form $(a\pm b)\pm a=\pm b$, $a\pm(\pm b\pm a)=\pm b$, $(a\pm b)\pm b=\pm a$ and $a\pm(\pm a\pm b)=\pm b$ we see that only the following 5 different cases occur:

(i) $(a+b)-a=b$, i.e. $a+(b-a)=b$ as shown in (8) of the above theorem 2.2,

(ii) $a-(b+a)=-b$, i.e. $\sim(a+b)+a=\sim b$ as shown in (5) of the above theorem 2.2,

(iii) $(a-b)+b=a$, i.e. $(a+b)-b=a$ as shown in (2.4) of [9],

(iv) $a+(-a+b)=b$, i.e. $\delta_{a,-a}=id$ as shown in (2.3) of [9],

(v) $a-(a+b)=-b$, i.e. $a-(a-b)=b$,

where note that $-a=\sim a$ in the above (iii), (iv) and (v). In fact, for the case of the above (ii), if we look at the line E of Figure 4 carefully without changing the point a and switch the point $\sim(a+b)$ to b, then the old $\sim b$ turn into $b+a$. And b, respectively $a+b$ becomes $-(b+a)$, respectively $-b$. So we have the equivalence of two identities $a-(b+a)=-b$ and $\sim(a+b)+a=\sim b$ of (ii). The rest are direct. However, the case of (v) will be studied in the next section in the web's point of view.

3. Local Thomsen conditions and orthogonality in a web

Let $\mathcal{W}=(\mathcal{P},\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3)$ be a 3-web. Then two chains $A,B\in\mathcal{C}_i$ are called *orthogonal* and denoted by $A\perp B$ if $A\neq B$ and $\widetilde{A}(B)=B$. We set $A^{\perp}=\{X\in\mathcal{C}_i\mid X\perp A\}$ as in [8, 9].

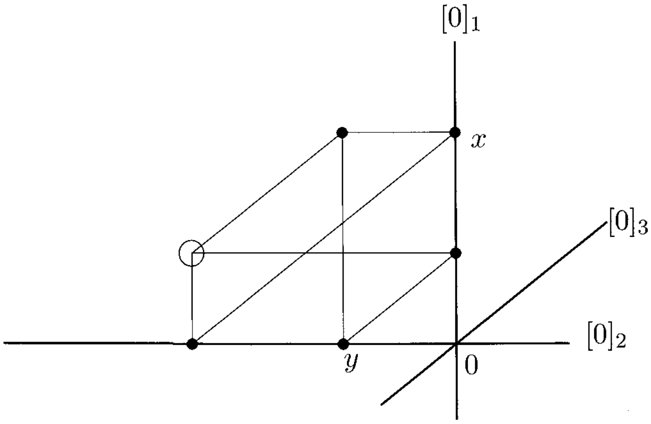

FIGURE 6. $(T, 0\,; 3)$

Now we introduce *another new local Thomsen condition* $(T, 0\,; k)$ different from $(T, 0\,; i, j)$, where $\{1, 2, 3\} = \{i, j, k\}$, as follows:

(T, 0 ; k) If $x \in [0]_i$ and $y \in [0]_j$, then the generator $\big[[x]_j \cap [y]_i\big]_k$ contains the point $\big[[x]_k \cap [0]_j\big]_i \cap \big[[y]_k \cap [0]_i\big]_j$, i.e.

$$[x\Box_{ji}y]_k \cap [x\Box_{kj}0]_i \cap [y\Box_{ki}0]_j \neq \emptyset.$$

In our web $\mathcal{W}$, we consider now the orbits $[p]^i = \{\widetilde{X}(p) \,|\, X \in \mathcal{G}_i\}$ of a point $p \in \mathcal{P}$ with respect to the generators of $\mathcal{G}_i$, $i \in \{1,\ 2,\ 3\}$. Then by definition we see the following:

THEOREM 3.1. *Each orbit $[p]^i$ is an i–chain for all $p \in \mathcal{P}$, $i \in \{1, 2, 3\}$, i.e. $[p]^i \in \mathcal{C}_i$ in a web $\mathcal{W}$.*

THEOREM 3.2. *Let $D \in \mathcal{C}_k$ with $0 \in D$, $\mathcal{G}_k \subset D^{\perp}$ and let $(E, +) = L(\mathcal{W}; 0; i, j)$, where $\{i, j, k\} = \{1, 2, 3\}$. Then we have:*

(1) *For all $d \in D$, $[d]^k = D = [0]^k = \{(-x)\Box_{ij}\, x \,|\, x \in E\}$,*

(2) *$\widetilde{\mathcal{G}_k}|_D$ acts regularly on D,*

(3) *For all a, $b \in D$, let $A = [0\Box_{ij}\, a]_k$ and let $a + b = \widetilde{A} \circ \widetilde{E}(b)$. Then $(D, +)$ is a loop isomorphic to $(E, +)$,*

(4) *$D \cap E = \{x \in E \mid x + x = 0\}$.*

Proof. (1) Let $A \in \mathcal{G}_k$, hence $A \perp D$, then $\widetilde{A}(d) \in \widetilde{A}(D) = D$ implying $[d]^k \subset D$. By the above theorem, $[d]^k \in \mathcal{C}_k$ and so $[d]^k = D$. Moreover, for $Z \in \mathcal{G}_k$ let $z_i = Z \cap [0]_i$ and $x = [z_i]_j \cap E$. Then $-x = [z_j]_i \cap E$, $0 = z_i\Box_{ij}\, z_j$ and so $\widetilde{Z}(0) = z_j\Box_{ij}\, z_i = (-x)\Box_{ij}\, x$.

(2) If $a, b \in D$, $C = [a\square_{ij}\ b]_k$, hence $C \perp D$, then $\widetilde{D}(a\square_{ij}\ b) = b\square_{ij}\ a \in \widetilde{D}(C) = C$ and $\widetilde{C}(a) = b$, and C is the only element of $\mathcal{G}_k$ with $\widetilde{C}(a) = b$.
(3) Let $\psi : D \to E$; $x \mapsto [x]_j \cap E$. Then $a+b = \widetilde{A} \circ \widetilde{E}(b) = \big[\,[\,[b]_j \cap E]_i \cap A\big]_j \cap D = \big[\,[\psi(b)]_i \cap A\big]_j \cap D$, $A = [0\square_{ij}\ \psi(a)]_k$ and so $\psi(a) + \psi(b) = [A \cap [\psi(b)]_i]_j \cap E = \psi(a+b)$.
(4) Let $x \in E$ then by the above (1) we have $(-x)\square_{ij}\ x \in D$. So $(-x)\square_{ij}\ x \in E$ if and only if $-x = x$ if and only if $x + x = 0$. $\square$

THEOREM 3.3. *In a 3-web $\mathcal{W} = (\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)$ the following three statements are equivalent with $\{1,2,3\} = \{i,j,k\}$:*
(1) *$(T, 0\,; k)$ is satisfied,*
(2) *$\mathcal{G}_k \subset ([0]^k)^{\perp} \cup \{[0]^k\}$,*
(3) *$a - (a - b) = b$ for all $a, b \in (E, +) = L(\mathcal{W}; 0\,; i, j)$.*

Proof. In order to get the clear picture we set $i = 1$, $j = 2$ and $k = 3$, from which we do not lose any generality of proof.

(1) $\Longrightarrow$ (2) If $(T, 0\,; 3)$ is satisfied, then let A, $B \in \mathcal{G}_3$ and $a = \widetilde{A}(0) \in [0]^3$. We have to show that $\widetilde{B}(a) \in [0]^3$, i.e. there exists a $C \in \mathcal{G}_3$ such that $\widetilde{B}(a) = \widetilde{C}(0)$, i.e. C has to go through the points $y = \widetilde{B}(a)\square 0$ and

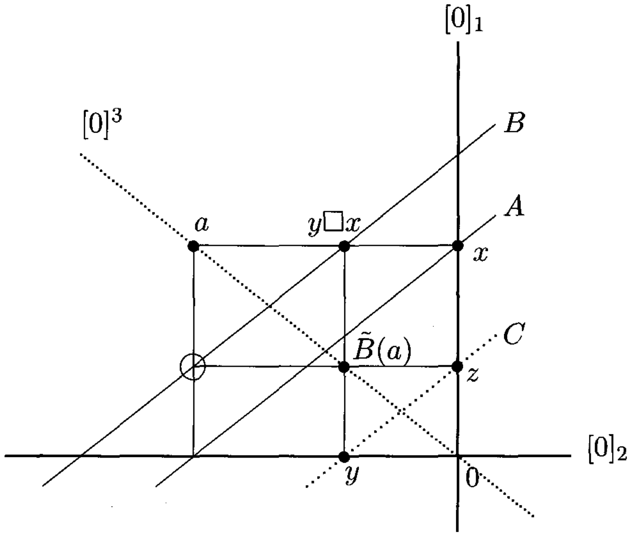

FIGURE 7. (1) $\Rightarrow$ (2) of Theorem 3.3

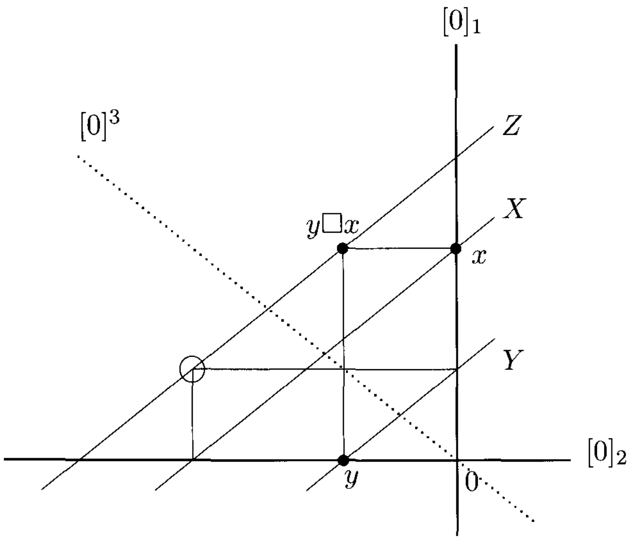

FIGURE 8. (2) $\Rightarrow$ (1) of Theorem 3.3

$z = 0\Box\widetilde{B}(a)$. Let $x = 0\Box a$. Then $[y\Box x]_3 = B$ and $\big[[x]_3 \cap [0]_2\big]_1 = [a]_1$. So $(T, 0\,; 3)$ implies $B \cap [a]_1 \cap \big[[y]_3 \cap [0]_1\big]_2 \neq \phi$ and this is equivalent to $z \in [y]_3$, hence we are done by taking C as such $[y]_3$.

(2) $\Longrightarrow$ (1) Let $\mathcal{G}_3 \subset ([0]^3)^\perp$ and let $x \in [0]_1$, $y \in [0]_2$, $X = [x]_3$, $Y = [y]_3$ and $Z = [y\Box x]_3$. Then $\widetilde{Y} \circ \widetilde{Z} \circ \widetilde{X}([0]_1) = \widetilde{Y} \circ \widetilde{Z}([x]_2) = \widetilde{Y}([y]_1) = [0]_2$ and $\widetilde{Y} \circ \widetilde{Z} \circ \widetilde{X}([0]^3) = [0]^3$. Now $\{0\} = [0]^3 \cap [0]_1 = [0]^3 \cap [0]_2$, hence $\widetilde{Y} \circ \widetilde{Z} \circ \widetilde{X}(\{0\}) = \widetilde{Y} \circ \widetilde{Z} \circ \widetilde{X}([0]^3) \cap \widetilde{Y} \circ \widetilde{Z} \circ \widetilde{X}([0]_1) = [0]^3 \cap [0]_2 = \{0\}$ implies $\widetilde{Y} \circ \widetilde{Z} \circ \widetilde{X}([0]_2) = [0]_1$, i.e. $\widetilde{Z} \circ \widetilde{X}([0]_2) = \widetilde{Y}([0]_1)$. Since $\big[[x]_3 \cap [0]_2\big]_1 = \widetilde{X}([0]_2)$, we obtain $\widetilde{Z}([[x]_3 \cap [0]_2]_1) = \widetilde{Y}([0]_1) = \big[[y]_3 \cap [0]_1\big]_2$, i.e. $\big[[x]_3 \cap [0]_2\big]_1 \cap \big[[y]_3 \cap [0]_1\big]_2 \in Z = [y\Box x]_3 = [[x]_2 \cap [y]_1]_3$ saying that $(T, 0\,; 3)$ is satisfied.

(1) $\Longleftrightarrow$ (3) Let $\psi : [0]_1 \times [0]_2 \to E \times E$ with $\psi(x, y) = \big([[[x]_2 \cap [y]_1]_3 \cap [0]_1]_2 \cap E,\ [[y]_3 \cap [0]_1]_2 \cap E\big)$. Then ψ is a bijection and if a, $b \in E$, then $\psi^{-1}(a, b) = \big([[[a]_2 \cap [0]_1]_3 \cap [[[b]_2 \cap [0]_1]_3 \cap [0]_2]_1]_2 \cap [0]_1,\ [[b]_2 \cap [0]_1]_3 \cap [0]_2\big)$. Now let a, $b \in E$ be given and let $(x, y) = \psi^{-1}(a, b)$. Then Figure 9 shows that $(T, 0\,; 3)$ is closed for x, y if and only if $a - (a - b) = b$. $\Box$

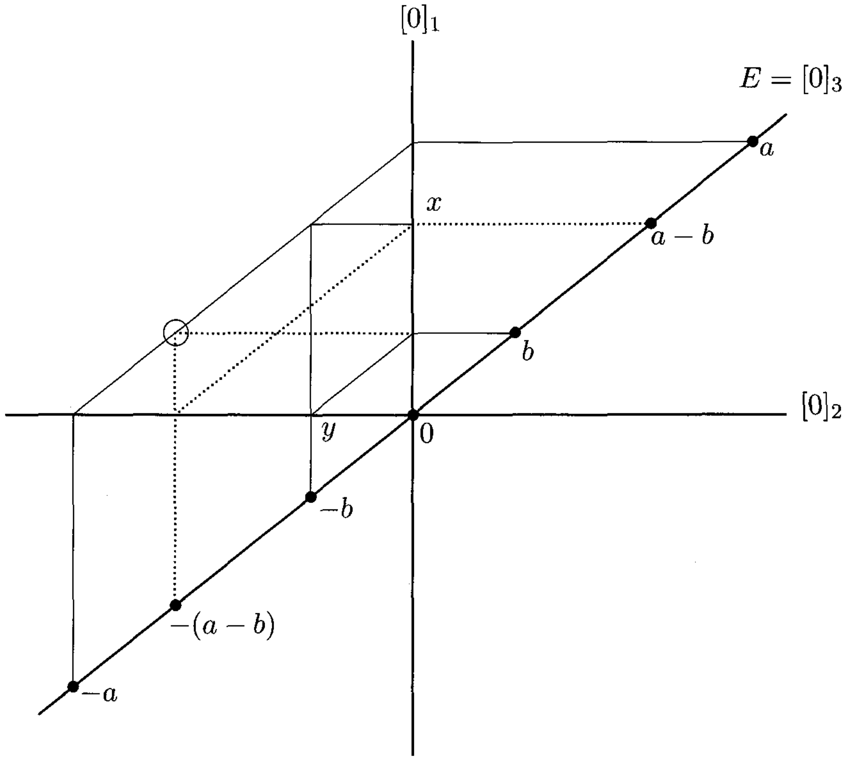

FIGURE 9. (1) $\Leftrightarrow$ (3) of Theorem 3.3

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