POLYNOMIAL GROWTH HARMONIC MAPS
ON COMPLETE RIEMANNIAN MANIFOLDS

YONG HAH LEE

ABSTRACT. In this paper, we give a sharp estimate on the cardinality of the set generating the convex hull containing the image of harmonic maps with polynomial growth rate on a certain class of manifolds into a Cartan-Hadamard manifold with sectional curvature bounded by two negative constants. We also describe the asymptotic behavior of harmonic maps on a complete Riemannian manifold into a regular ball in terms of massive subsets, in the case when the space of bounded harmonic functions on the manifold is finite dimensional.

1. Introduction

In 1975, Yau [32] proved the Liouville property on a complete Riemannian manifold with nonnegative Ricci curvature, i.e., every positive harmonic function on such a manifold must be constant. Later, Cheng gave a generalization of the result of Yau for harmonic maps. In [3], he proved that if a harmonic map from a complete Riemannian manifold with nonnegative Ricci curvature to a Cartan-Hadamard manifold is contained a bounded set, then the map is constant. Recently, in a series of papers [5]-[11], Colding and Minicozzi II proved that the space of harmonic functions of polynomial growth of degree at most d on a complete Riemannian manifold

$$\mathcal{H}^d(M) = \{f : \Delta f = 0, |f|(x) = O(r^d(x)) \text{ as } r(x) \to \infty\}$$

must be finite dimensional for any $d \geq 0$, where $r(x)$ denotes the distance of any point x from a fixed point o in M, if the manifold M satisfies the volume doubling condition and the Poincaré inequality as follows:

(**V**) there exists a constant $\nu > 0$ such that for any $x \in M$ and $0 < s \leq r$,

$$V_x(r) \leq \left(\frac{r}{s}\right)^\nu V_x(s),$$

where $V_x(r)$ denotes the volume of the geodesic ball $B_r(x)$;

(**P**) there exists a constant $C > 0$ such that for any $x \in M$ and $r > 0$,

$$\int_{B_r(x)} f^2 \leq Cr^2 \int_{B_r(x)} |\nabla f|^2,$$

where $f \in C^\infty(B_r(x))$ satisfying $\int_{B_r(x)} f = 0$.

Note that these properties are satisfied on a complete Riemannian manifold M with nonnegative Ricci curvature. On the other hand, Li and Wang in [26] gave the convex hull property of harmonic maps into a Cartan-Hadamard manifold as follows:

THEOREM 1.1. *Let M be a complete Riemannian manifold such that the dimension of the space of bounded harmonic functions on M is l. Let $u : M \to N$ be a harmonic map from M into a Cartan-Hadamard manifold N, and $A = \overline{u(M)} \cap N(\infty)$, where $N(\infty)$ denotes the geometric boundary of N and $\overline{B}$ denotes the closure of a set B in $N \cup N(\infty)$. If either u is bounded, or N is two dimensional visibility manifold, or the sectional curvature satisfies $-b^2 \leq K_N \leq -a^2 < 0$, then there exists a set of points $\{q_j\}_{j=1}^k$ in $\overline{u(M)} \cap N$ with $k \leq l$ such that*

$$u(M) \subset \overline{\mathcal{C}(A \cup \{q_j\}_{j=1}^k)},$$

where $\mathcal{C}(D)$ denotes the convex hull of a set D.

Furthermore, if the maximal number of mutually disjoint d-massive subsets of M, explained later, is l_d and $u : M \to N$ satisfies that for some point $p \in N$

(1.1) $d_N(u(x), p) = O(r(x)^d)$ *as* $r(x) \to \infty$,

then there exists a set of points $\{\overline{q}_i\}_{i=1}^{k_d} = \overline{u(M)} \cap N(\infty)$ with $k_d \leq l_d - l$ such that

$$u(M) \subset \overline{\mathcal{C}(\{\overline{q}_i\}_{i=1}^{k_d} \cup \{q_j\}_{j=1}^k)}.$$

They also pointed out that the number of mutually disjoint d-massive subsets is bounded by the dimension of the space of harmonic functions of polynomial growth of degree at most d. Therefore, one can estimate the cardinality of the finite set A generating the convex hull containing the image of harmonic maps of polynomial growth of degree at most

d by estimating the dimension of the space of harmonic functions of polynomial growth of degree at most d.

In this paper, we consider some specific cases to get a sharp estimate of the cardinality of such a finite set generating the convex hull containing the image of harmonic maps of polynomial growth of degree at most d. One is the case of a connected sum of complete Riemannian manifolds, each of which satisfying the weak volume doubling condition and the mean value property as follows: Let M be a complete Riemannian manifold.

(**W**) there exist constants $C > 0$ and $\nu > 0$ such that for any $x \in M$ and sufficiently large $0 < s \leq r$,

$$V_x(r) - V_x(s) \leq C\big\{ \big(\frac{r}{s}\big)^\nu - 1 \big\} V_x(s);$$

(**M**) there exists a constant $\lambda > 0$ such that for any $x \in M$ and $r > 0$, any nonnegative subharmonic function f on M

$$f(x) \leq \frac{\lambda}{V_x(r)} \int_{B_r(x)} f.$$

Note that the condition (**W**) is weaker than (**V**), and if a manifold satisfies the conditions (**W**) and (**P**), then the mean value property (**M**) also holds on the manifold. (See [16] or [28]). Therefore, any complete Riemannian manifold with nonnegative Ricci curvature still satisfies the condition (**W**) and (**M**).

THEOREM 1.2. *Let M be a connected sum of complete Riemannian manifolds M_i, $i = 1, 2, \cdots, l$, each of which satisfies (**W**) and (**M**). Suppose that N is two dimensional visibility manifold, or a Cartan-Hadamard manifold with the sectional curvature satisfying $-b^2 \leq K_N \leq -a^2 < 0$. Let $u : M \to N$ be a harmonic map satisfying (1.1) for some $d \geq 0$. Then there exist sets of points $\{q_j\}_{j=1}^k$ in $\overline{u(M)} \cap N$ with $k \leq l$ and $\{\overline{q}_i\}_{i=1}^{k_d}$ in $\overline{u(M)} \cap N(\infty)$ with $k_d \leq C(1 + \sum_{i=1}^l d^{\nu_i - 1}) - l$ such that*

$$u(M) \subset \overline{\mathcal{C}(\{\overline{q}_i\}_{i=1}^{k_d} \cup \{q_j\}_{j=1}^k)},$$

*where ν_i denotes the order in (**W**) corresponding to each M_i*

Another case is a complete Riemannian manifold with nonnegative Ricci curvature outside a compact set and finite first Betti number.

THEOREM 1.3. *Let M be a complete n-dimensional Riemannian manifold with nonnegative Ricci curvature outside a compact set and finite first Betti number. Suppose that N is two dimensional visibility*

*manifold, or a Cartan-Hadamard manifold with the sectional curvature
satisfying* $-b^2 \leq K_N \leq -a^2 < 0$. *Let* $u : M \to N$ *be a harmonic
map satisfying (1.1) for some* $d \geq 0$. *Then there exist sets of points*
$\{q_j\}_{j=1}^k$ *in* $\overline{u(M)} \cap N$ *with* $k \leq l$ *and* $\{\bar{q}_i\}_{i=1}^{k_d}$ *in* $\overline{u(M)} \cap N(\infty)$ *with*
$k_d \leq C(1 + \sum_{i=1}^l d^{\nu_i-1}) - l \leq C(1 + ld^{n-1}) - l$ *such that*

$$u(M) \subset \overline{\mathcal{C}(\{\bar{q}_i\}_{i=1}^{k_d} \cup \{q_j\}_{j=1}^k)},$$

where l *is the number of ends of* M *and* ν_i $(\leq n)$ *denotes the order in the
volume doubling condition corresponding to each end* $E_i, i = 1, 2, \cdots, l$,
of M.

We also prove that if the dimension of the space of bounded harmonic
functions on a complete Riemannian manifold is l, then every harmonic
map on the manifold into a complete Riemannian manifold, whose image
lies inside a regular ball, is uniquely determined by l points in the closure
of the image. In particular, we give a description of the asymptotic
behavior of such harmonic maps in terms of massive subsets. This result
is a generalization of Sung, Tam and Wang [30].

Finally, we treat more general cases related to the rough isometry
between complete Riemannian manifolds. To be precise, we obtain the
same result in the case when the domain manifold is roughly isometric
to the complete Riemannian manifold with nonnegative Ricci curvature
outside a compact set and finite first Betti number.

2. Bounded harmonic maps

We consider the case of bounded harmonic maps on a complete Rie-
mannian manifold into a regular ball of another complete Riemannian
manifold. We say that a ball $\mathcal{B}_R(p)$ in a complete Riemannian manifold
N is regular if $R < \min\{\pi/(2\sqrt{\kappa})$, injectivity radius of N at $p\}$, where
$\kappa \geq 0$ is an upper bound for the sectional curvature of N. In particular,
each bounded ball in a Cartan-Hadamard manifold is the case. Actu-
ally, the asymptotic behavior of such bounded harmonic maps is closely
related to that of bounded harmonic functions. (For example, see [22]
and [30]).

We concentrate on the case that the space of bounded harmonic func-
tions on a complete Riemannian manifold is finite dimensional. In [15],
Grigor'yan proved that the maximal number of mutually disjoint massive
subsets of a complete Riemannian manifold is equal to the dimension of
the space of all bounded harmonic functions. A proper open subset Ω of

a complete Riemannian manifold M is called a massive subset if there exists a continuous function u on M such that

$$\begin{cases} \Delta u &= 0 \text{ in } \Omega; \\ u &\equiv 0 \text{ on } M \setminus \Omega; \\ \sup_\Omega u &= 1. \end{cases}$$

Such a function u is called an inner potential of the massive set Ω.

Recall the observation in [15]: Let M be a complete Riemannian manifold and $\dim \mathcal{HB}(M) = l$ for some $l \in \mathbf{N}$, where $\mathcal{HB}(M)$ denotes the space of bounded harmonic functions on M. Then by [15], there exist mutually disjoint massive subsets $\Omega_1, \Omega_2, \cdots, \Omega_l$ of M, and each Ω_j has an inner potential u_j. Fix a point o in M. For each $j = 1, 2, \cdots, l$ and $r > 0$, define a continuous function $h_{j,r}$ on $B_r(o)$ such that

$$\begin{cases} \Delta h_{j,r} &= 0 \quad \text{in } B_r(o); \\ h_{j,r} &= u_j \text{ on } \partial B_r(o). \end{cases}$$

Then by the maximum principle, $\{h_{j,r}\}_{r>0}$ is a nondecreasing sequence. In particular, its limit function h_j is harmonic in M and $0 \le u_j \le h_j \le 1$.

Since Ω_j's are mutually disjoint, $\sum_{j=1}^l u_j \le 1$ on M. This implies that $\sum_{j=1}^l h_{j,r} \le 1$ on $B_r(o)$, hence $\sum_{j=1}^l h_j \le 1$ on M. We now choose a sequence $\{\Omega_j^n : n \in \mathbf{N}\}$ for each $j = 1, 2, \cdots, l$ such that $\Omega_j^n = \{x \in \Omega_j : h_j(x) > 1 - 1/n\}$. Then $\Omega_j^{n+1} \subset \Omega_j^n$ and

$$\lim_{n \to \infty, x \in \Omega_j^n} h_i(x) = \delta_{ij}.$$

One can easily check that $h_1, h_2, \cdots, h_l$ are linearly independent. Hence every bounded harmonic function on M can be represented by a linear combination of $h_1, h_2, \cdots, h_l$.

As a consequence of the above consideration, we can describe the asymptotic behavior of bounded harmonic functions in terms of massive subsets as follows:

LEMMA 2.1. *Suppose that* $\dim \mathcal{HB}(M) = l$ *for some* $l \in \mathbf{N}$. *Then for each bounded harmonic function* f, *there exist real numbers* $a_1, a_2, \cdots, a_l$ *such that for each* $j = 1, 2, \cdots, l$,

$$\lim_{n \to \infty, x \in \Omega_j^n} f(x) = \lim_{n \to \infty, x \in \Omega_j^n} \sum_{i=1}^l a_i h_i(x) = a_j,$$

where h_i, Ω_j *and* Ω_j^n *are given above.*

In [30], Sung, Tam and Wang proved that if every bounded harmonic function on a complete Riemannian manifold is asymptotically

constant at the infinity of each nonparabolic end, then every harmonic maps whose image is contained in a regular ball is also asymptotically constant at the infinity of each end. On the other hand, the nonparabolicity of an end implies that the end is a massive subset. Furthermore, if every bounded harmonic function defined on an end is asymptotically constant at the infinity, then the end contains at most one massive set. In this viewpoint, our setting is more general one than to assume the asymptotically constant property of bounded harmonic functions on each nonparabolic end.

Modifying the argument of [30], we have a similar result as in [30], but with a different observation which is derived from the finiteness of the number of mutually disjoint massive subsets:

LEMMA 2.2. *Suppose that* $\dim \mathcal{HB}(M) = l$ *for some* $l \in \mathbf{N}$. *Let* Ω_j *and* Ω_j^n, $j = 1, 2, \cdots, l$, *be given as in Lemma 2.1. Suppose that* $g_1, g_2, \cdots, g_k$ *are any finite set of bounded superharmonic functions on* Ω_j *for some* $j = 1, 2, \cdots, l$, *with* $\liminf_{n \to \infty, x \in \Omega_j^n} g_i(x) = 0$ *for all* $i = 1, 2, \cdots, k$. *Then*

$$\liminf_{n \to \infty, x \in \Omega_j^n} g(x) = 0,$$

where $g = \max\{g_1, g_2, \cdots, g_k\}$.

Proof. For the sake of convenience, let us denote by Ω a fixed one of the mutually disjoint massive subsets $\Omega_1, \Omega_2, \cdots, \Omega_l$ of M. Suppose that the lemma is not true. Then there is $\epsilon > 0$ such that

$$\liminf_{n \to \infty, x \in \Omega^n} g(x) \geq \epsilon.$$

From this, we can choose $n \in \mathbf{N}$ such that

$$\inf_{x \in \Omega^n} g(x) \geq \epsilon$$

and

$$\inf_{x \in \Omega^n} g_i(x) \geq -\epsilon/2k$$

for all $i = 1, 2, \cdots, k$. By the definition of g, we get

$$\sum_{i=1}^{k} g_i(x) \geq \epsilon - (k-1)\epsilon/2k > \epsilon/2$$

for all $x \in \Omega^n$.

We now define a function $f_{i,r}$ on $\overline{B_r(o) \cap \Omega^n}$ by $\Delta f_{i,r} = 0$ in $B_r(o) \cap \Omega^n$ and $f_{i,r} = g_i$ on $\partial(B_r(o) \cap \Omega^n)$. Since g_i is bounded on Ω, there is a subsequence of $f_{i,r}$ converging uniformly on any compact subset of Ω^n

and its limit function f_i is harmonic in Ω^n. Since $\sum_{i=1}^k f_{i,r} = \sum_{i=1}^k g_i \geq \epsilon/2$ on $\partial(B_r(o) \cap \Omega^n)$, we get $\sum_{i=1}^k f_{i,r} \geq \epsilon/2$ on $B_r(o) \cap \Omega^n$. Thus

$$(2.1) \qquad \sum_{i=1}^k f_i \geq \epsilon/2 \ \text{on} \ \Omega^n.$$

On the other hand, since g_i is superharmonic and f_i is harmonic in Ω^n, $f_i \leq g_i$ on Ω^n. In particular,

$$\liminf_{n \to \infty, x \in \Omega^n} f_i(x) \leq \liminf_{n \to \infty, x \in \Omega^n} g_i(x) = 0.$$

We claim that

$$\liminf_{n \to \infty, x \in \Omega^n} f_i(x) = \limsup_{n \to \infty, x \in \Omega^n} f_i(x).$$

Then we get

$$\lim_{n \to \infty, x \in \Omega^n} f_i(x) = a_i$$

for some nonpositive constant a_i. However, this contradicts (2.1), hence we get the consequence.

Suppose that the claim is not true. Then we may assume that there exist a small $\epsilon > 0$ and an integer $n \in \mathbf{N}$ such that

$$\inf_{x \in \Omega^n} f_i(x) + \epsilon < \sup_{x \in \Omega^n} f_i(x) - \epsilon.$$

In particular, both $\{x \in \Omega^n : f_i(x) < \inf_{x \in \Omega^n} f_i + \epsilon\}$ and $\{x \in \Omega^n : f_i(x) > \sup_{x \in \Omega^n} f_i - \epsilon\}$ are massive subsets of Ω^n. Furthermore, each of them does not intersect each other. Thus we have at least $l+1$ mutually disjoint massive subsets of M. By [15], this contradicts the hypothesis of the lemma. Hence we get the claim. $\qquad \square$

Using Lemma 2.2, we can also describe the asymptotic behavior of harmonic maps in terms of massive subsets as follows:

LEMMA 2.3. *Suppose that* $\dim \mathcal{HB}(M) = l$ *for some* $l \in \mathbf{N}$. *Let* Ω_j *and* Ω_j^n, $j = 1, 2, \cdots, l$, *be given as in Lemma 2.1. Then for any harmonic map* u *on* M *into a regular ball* $\mathcal{B}_R(p)$, *there exist points* $q_1, q_2, \cdots, q_l$ *in* $\overline{\mathcal{B}}_R(p)$ *such that for each* $j = 1, 2, \cdots, l$,

$$\lim_{n \to \infty, x \in \Omega_j^n} u(x) = q_j.$$

Proof. Let us denote by Ω a fixed one of $\Omega_1, \Omega_2, \cdots, \Omega_l$. Put $K = \overline{u(\Omega)}$. By Jäger and Kaul [18], for all $q \in K$,

$$\Psi_q(s) = \frac{1 - \cos(\sqrt{\kappa} d_N(q, s))}{\sqrt{\kappa} \cos(\sqrt{\kappa} d_N(p, s))}$$

is convex in $\mathcal{B}_R(p)$, where d_N denotes the distance function of N. Therefore, $\Psi_q(u(x))$ is a bounded subharmonic function in Ω by Gordon [14]. Put $c_q = \limsup_{n\to\infty, x\in\Omega^n} \Psi_q(u(x))$, then $g_q(x) = c_q - \Psi_q(u(x))$ is a bounded superharmonic function in Ω and

$$\liminf_{n\to\infty, x\in\Omega^n} g_q(x) = 0.$$

By Lemma 2.2, for any finite points $q_1, q_2, \cdots, q_k$ and any $\epsilon > 0$,

$$(2.2) \qquad \bigcap_{i=1}^{k}\{x \in \Omega : g_{q_i}(x) \le \epsilon\} \ne \emptyset.$$

Put $A(q,\epsilon) = \{s \in K : c_q - \Psi_q(s) \le \epsilon\}$. Then by (2.2) and the definition of K and g_q, the intersection of finitely many $A(q,\epsilon)$ is nonempty. By the compactness of K, there exists a point $q_0 \in A(q,\epsilon)$ for all $q \in K$ and $\epsilon > 0$. In particular, $c_{q_0} - \Psi_{q_0}(q_0) \le \epsilon$ for any $\epsilon > 0$, hence $c_{q_0} \le 0$. However, since $c_{q_0} \ge 0$, we get $c_{q_0} = 0$, i.e.,

$$\limsup_{n\to\infty, x\in\Omega^n} \Psi_{q_0}(u(x)) = 0.$$

Since $0 \le \Psi_{q_0}(u(x)) \le \sup_{x\in\Omega^n} \Psi_{q_0}(u(x))$,

$$\lim_{n\to\infty, x\in\Omega^n} \Psi_{q_0}(u(x)) = 0.$$

This implies that

$$\lim_{n\to\infty, x\in\Omega^n} u(x) = q_0.$$

$\square$

We are now ready to prove an existence and uniqueness theorem on bounded harmonic maps as follows:

THEOREM 2.4. *Let M be a complete Riemannian manifold with* $\dim \mathcal{HB}(M) = l$ *for some* $l \in \mathbf{N}$. *Let h_i, Ω_j and Ω_j^n, $i, j = 1, 2, \cdots, l$, be given as in Lemma 2.1. Then for given points $q_1, q_2, \cdots, q_l$ in a regular ball $\overline{\mathcal{B}}_R(p)$, there exists a harmonic map u on M into $\mathcal{B}_R(p)$ such that for each $j = 1, 2, \cdots, l$,*

$$(2.3) \qquad \lim_{n\to\infty, x\in\Omega_j^n} u(x) = q_j.$$

Furthermore, the solution u satisfying (2.3) is unique, i.e., if $\tilde{u}$ is another harmonic map on M into $\mathcal{B}_R(p)$ such that

$$\lim_{n\to\infty, x\in\Omega_j^n} \tilde{u}(x) = \lim_{n\to\infty, x\in\Omega_j^n} u(x),$$

then $\tilde{u} \equiv u$ on M.

Proof. We can coordinatize $\overline{\mathcal{B}}_R(p)$ by means of geodesic normal coordinates centered at p. So, a map $\phi : M \to \overline{\mathcal{B}}_R(p)$ can be viewed as being an $\mathbf{R}^m$-valued map with respect to the normal coordinates centered at p as follows:

$$\phi = (\phi_1, \phi_2, \cdots, \phi_m) : M \to B_R(0) \subset \mathbf{R}^m.$$

For each $j = 1, 2, \cdots, l$, let $q_j = (a_{j1}, a_{j2}, \cdots, a_{jm})$ in the normal coordinates.

Define a map $f = (f_1, f_2, \cdots, f_m) : M \to B_R(0)$ by $f_i = \sum_{j=1}^{l} a_{ji} h_j$. Of course, each f_i is a bounded harmonic function on M. Choose a sequence $\{v_r\}$ of solutions such that $\Delta v_r = 0$ in $B_r(o)$ and $v_r = \sum_{i=1}^{m} f_i^2$ on $\partial B_r(o)$.

By Hildebrandt, Kaul and Widman [17], for each $r > 0$, there exists a harmonic map u_r from $B_r(o)$ into $\mathcal{B}_R(p)$ such that $u_r = f$ on $\partial B_r(o)$. The a priori estimate of Giaquinta and Hildebrandt [13] implies that for sufficiently large r_0 and some $\beta \in (0,1)$, $|u_r|_{C^{2,\beta}(B_{r_0}(o))}$ is bounded by a constant depending only on $M, \mathcal{B}_R(p)$ and h, where $r \geq r_0$. Hence by the Ascoli theorem, there exists a subsequence $\{u_{r_k}\}$ of $\{u_r\}$ converging uniformly on any compact subset of M. In particular, the limit map $u : M \to \overline{\mathcal{B}}_R(p)$ is also a harmonic map. On the other hand, by Lemma 3.1 in [1], there exists a constant $C > 0$ depending only on the geometry of $\overline{\mathcal{B}}_R(p)$ such that

$$(2.4) \qquad d_N(u_r(x), f(x)) \leq C\left(v_r(x) - \sum_{i=1}^{m} f_i^2(x)\right)$$

for all $x \in B_r(o)$.

Put $v = \sum_{j=1}^{l} \sum_{i=1}^{m} a_{ji}^2 h_j$. We claim that $\sum_{i=1}^{m} f_i^2 \leq v$ on M. Otherwise, $w = \sum_{i=1}^{m} f_i^2 - v$ is a bounded subharmonic function and $\sup_M w > 0$. Since for each $j = 1, 2, \cdots, l$

$$\lim_{n \to \infty, x \in \Omega_j^n} w(x) = 0,$$

there exist a sufficiently small $\epsilon > 0$ and an integer $n \in \mathbf{N}$ such that a massive subset $\{x \in M : w(x) > \sup_M w - \epsilon\}$ does not intersect $\cup_{j=1}^{l} \Omega_j^n$. However, this is impossible since the maximal number of mutually disjoint massive subsets of M is l by [14]. Therefore, we get the claim.

Since $v_r = \sum_{i=1}^{m} f_i^2 \leq v$ on $\partial B_r(o)$, by the maximum principle, we have

$$(2.5) \qquad v_r \leq v \text{ on } B_r(o)$$

for all $r > 0$. By a diagonal sequence argument, (2.4) and (2.5),

$$(2.6) \qquad d_N(u(x), f(x)) \leq C(v(x) - \sum_{i=1}^{m} f_i^2(x))$$

for all $x \in M$.

On the other hand, since

$$\lim_{n\to\infty, x\in\Omega_j^n} (v(x) - \sum_{i=1}^{m} f_i^2(x)) = 0,$$

by (2.6) we get

$$\lim_{n\to\infty, x\in\Omega_j^n} d_N(u(x), f(x)) = 0.$$

Hence by Lemma 2.1, we have

$$\lim_{n\to\infty, x\in\Omega_j^n} u(x) = \lim_{n\to\infty, x\in\Omega_j^n} f(x) = q_j$$

for each $j = 1, 2, \cdots, l$.

Next, we prove the uniqueness. By Kendall [20], there is a continuous nonnegative bounded convex function Ψ on $B_R(0) \times B_R(0)$ and $\Psi(x, y) = 0$ if and only if $x = y$. Suppose that $\tilde{u}$ and u are harmonic maps on M into $\mathcal{B}_R(p)$ such that

$$\lim_{n\to\infty, x\in\Omega_j^n} \tilde{u}(x) = \lim_{n\to\infty, x\in\Omega_j^n} u(x)$$

for each $j = 1, 2, \cdots, l$. Put $g(x) = \Psi(\tilde{u}(x), u(x))$, then g is a nonnegative bounded subharmonic function in M such that

$$\lim_{n\to\infty, x\in\Omega_j^n} g(x) = 0$$

for each $j = 1, 2, \cdots, l$.

Suppose that $\sup_M g > 0$. Then there exist a sufficiently small $\epsilon > 0$ and an integer $n \in \mathbf{N}$ such that a massive subset $\{x \in M : g(x) > \sup_M g - \epsilon\}$ does not intersect $\sum_{j=1}^{l} \Omega_j^n$. This contradicts that the maximal number of mutually disjoint massive subsets of M is l. Therefore, we have $g \equiv 0$, hence $\tilde{u} \equiv u$ on M.   □

It has to be emphasized that the only assumption imposed on the domain manifold in our result is the finite dimensionality of the space of bounded harmonic functions on the manifold. Therefore, in the case that M is either a complete Riemannian manifold with nonnegative Ricci curvature outside a compact set and finite first Betti number or a connected sum of complete Riemannian manifolds, each of which satisfies the conditions (**W**) and (**M**), we have the same consequence. As the

simplest situation, if $\dim \mathcal{HB}(M) = 1$, then any harmonic map on M into a regular ball must be constant. For example, the class being roughly isometric to a complete Riemannian manifold with nonnegative Ricci curvature are the case.

On the other hand, Cheng, Tam and Wan [4] proved that if every harmonic function on a complete Riemannian manifold M with finite Dirichlet integral is bounded, then every harmonic map with finite energy from M into a Cartan-Hadamard manifold must be bounded. In particular, by [29], if the space $\mathcal{HD}(M)$ of harmonic functions with finite Dirichlet integral on M is finite dimensional, then every harmonic function with finite Dirichlet integral on M is bounded. Therefore, using [15], in the case when $\dim \mathcal{HD}(M) = l$ for some $l \in \mathbf{N}$, the maximal number of mutually disjoint D-massive subsets of M is l. Here, that a set Ω is a D-massive subset of M means that Ω is a massive subset and has an inner potential u with finite Dirichlet integral. With these results, similarly arguing as in the case of harmonic maps into a regular ball, we get the following consequence:

THEOREM 2.5. *Let M be a complete Riemannian manifold with $\dim \mathcal{HD}(M) = l$ for some $l \in \mathbf{N}$. Let Ω_j, $j = 1, 2, \cdots, l$, be D-massive subsets of M and Ω_j^n be similarly constructed as in Lemma 2.1. Then for any harmonic map u with finite energy on M into a Cartan-Hadamard manifold N, there exist points $q_1, q_2, \cdots, q_l$ in N such that for each $j = 1, 2, \cdots, l$,*

$$(2.7) \qquad \lim_{n \to \infty, x \in \Omega_j^n} u(x) = q_j.$$

Furthermore, for any given points $q_1, q_2, \cdots, q_l$ in N, there exists a unique harmonic map u with finite energy on M into N satisfying (2.7).

3. Polynomial growth harmonic maps

We first consider a connected sum of complete Riemannian manifolds satisfying the conditions (**W**) and (**M**).

LEMMA 3.1. *Let M be a complete Riemannian manifold satisfying (**W**), and o be a fixed point in M. For any $0 < \alpha < 1/4$ and $r > 0$, let $\{x_1, x_2, \cdots, x_{m(\alpha)}\}$ be a maximal set of points in $\partial B_r(o)$ such that $d(x_i, x_j) \geq 2\alpha r$ for $i \neq j$, then $m(\alpha) \leq C\alpha^{-\nu+1}$, where C is independent of α and r.*

Proof. Fix $0 < \alpha < 1/4$ and $r > 0$. Since $B_{\alpha r}(x_i)$'s are mutually disjoint, by applying (**W**), we get

$$
\begin{aligned}
m(\alpha)V_o(r) &\leq \sum_{i=1}^{m(\alpha)} V_{x_i}(2r) \\
&\leq C\left(\frac{2}{\alpha}\right)^\nu \sum_{i=1}^{m(\alpha)} V_{x_i}(\alpha r) \\
&\leq C\left(\frac{2}{\alpha}\right)^\nu \left(V_o((1+\alpha)r) - V_o((1-\alpha)r)\right) \\
&\leq C\left(\frac{2}{\alpha}\right)^\nu \left\{ \left(\frac{1+\alpha}{1-\alpha}\right)^\nu - 1\right\} V_o((1-\alpha)r) \\
&\leq C\left(\frac{2}{\alpha}\right)^\nu \alpha V_o(r).
\end{aligned}
$$

This implies the consequence.   □

Let $M = M_1 \sharp M_2 \sharp \cdots \sharp M_l$ be a connected sum of complete Riemannian manifolds $M_i, i = 1, 2, \cdots, l$, each of which satisfies the conditions (**W**) and (**M**). Then there exist $r_0 > 0$ and fixed points $o_i \in M_i, i = 1, 2, \cdots, l$, such that for any $r \geq 2r_0$,

$$
\cup_{i=1}^l (M_i \setminus B_r(o_i)) \subset M \setminus B_{r_0}(o),
$$

where o is a fixed point in M. Applying Lemma 3.1, for any $0 < \alpha < 1/4$, $r \geq 2r_0$ and each $i = 1, 2, \cdots, l$, there exists a maximal set $\{x_1^i, x_2^i, \cdots, x_{m_i(\alpha)}^i\}$ of points in $\partial B_r(o_i) \cap M_i$ such that $d(x_j^i, x_k^i) \geq \alpha r/4$ if $j \neq k$ and $m_i(\alpha) \leq C\alpha^{-\nu_i+1}$, where ν_i denotes the order in (**W**) corresponding to each M_i.

Suppose that $\mathcal{D}$ is a rank k vector bundle over M with a metric. We now define a positive semidefinite symmetric bilinear form S_r on the space of sections $\Gamma(\mathcal{D})$ of $\mathcal{D}$ by

$$
(3.1) \quad S_r(u,v) = \frac{1}{\text{vol}A_r} \int_{A_r} \langle u, v \rangle + \sum_{i=1}^l \sum_{j=1}^{m_i(\alpha)} \frac{1}{V_{x_j^i}(\alpha r)} \int_{B_{\alpha r}(x_j^i)} \langle u, v \rangle,
$$

for $u, v \in \Gamma(\mathcal{D})$, where $A_r = \cup_{i=1}^l (M \setminus (M_i \setminus B_r(o_i)))$.

Modifying the argument in [24], we get the following two lemmas similar to those of [24]. (See also [31]).

LEMMA 3.2. *Let $\mathcal{K}$ be an N-dimensional subspace of $\Gamma(\mathcal{D})$ such that $\Delta|u|^2 > 0$ for all $u \in \mathcal{K}$. Then there is a constant $C > 0$ such that for*

any $0 < \alpha < 1/4$ and an orthonormal basis $\{u_1, u_2, \cdots, u_N\}$ for $\mathcal{K}$ with respect to $S_{(1+\alpha)r}$,

$$\sum_{i=1}^{N} S_r(u_i, u_i) \leq C(1 + \sum_{i=1}^{l} m_i(\alpha))$$

for all sufficiently large $r \geq 2r_0$.

LEMMA 3.3. Let $\mathcal{K}$ be an N-dimensional subspace of $\Gamma(\mathcal{D})$ with polynomial growth of degree at most d. Then for any $0 < \alpha < 1/4$, there exists $r \geq 2r_0$ such that if $\{u_1, u_2, \cdots, u_N\}$ is an orthonormal basis for $\mathcal{K}$ with respect to $S_{(1+\alpha)r}$, then

$$\sum_{i=1}^{N} S_r(u_i, u_i) \geq N(1 + \alpha)^{-(2d+1)}.$$

To treat polynomial growth harmonic maps, we need to introduce the concept of d-massive sets, which is introduced by Li and Wang [26], as follows:

DEFINITION 3.4. A subset Ω of M is called a d-massive subset if there exists a nonzero nonnegative subharmonic function u on M such that $u \equiv 0$ on $M \setminus \Omega$ and $u(x) = O(r^d(x))$ as $r(x) \to \infty$, where $r(x) = d(x, o)$. Such a function u is called a d-potential function of Ω.

We are now ready to prove Theorem 1.2:

Proof of Theorem 1.2. Let $\Omega_1, \Omega_2, \cdots, \Omega_N$ be disjoint d-massive subsets of M, and $u_1, u_2, \cdots, u_N$ be the corresponding potential functions. Then obviously, $u_1, u_2, \cdots, u_N$ are linearly independent. Applying Lemma 3.1, Lemma 3.2 and Lemma 3.3 with $\alpha = (4d)^{-1}$, there is a constant $C > 0$ such that

$$N \leq C(1 + \sum_{i=1}^{l} m_i(1/4d)) \leq C(1 + \sum_{i=1}^{l} d^{\nu_i - 1}),$$

where C is independent of d. This implies that the number of mutually disjoint d-massive subsets of M is bounded by $C(1 + \sum_{i=1}^{l} d^{\nu_i - 1})$.

Since each M_i, $i = 1, 2, \cdots, l$, satisfies the conditions (**W**) and (**M**), it has at most one massive subset. Hence the number of mutually disjoint massive subsets of M is bounded by l. By applying Theorem 1.1 and above results, we have Theorem 1.2. $\square$

We next consider the case of a complete Riemannian manifold with nonnegative Ricci curvature outside a compact set and finite first Betti number. Modifying [2], we have the following relative volume comparison:

LEMMA 3.5. *Let M be a complete Riemannian manifold with Ricci curvature satisfying $\mathrm{Ric}_M(x) \geq -(n-1)K/(1+r(x))^2$, where K is a positive constant and $r(x)$ denotes the distance from x to a fixed point o in M. Then for any $0 < \alpha < 1/4$ and $r > r_0$,*

$$V_o((1+\alpha)r) - V_o((1-\alpha)r) \leq C\alpha(V_o(r) - V_o(r_0)).$$

Proof. Let g be the solution of the linear equation $g'' = -Kg/(1+t)^2$ with initial condition $g(0) = 0$ and $g'(0) = 1$. Then

$$g(t) = \frac{1}{\beta_1 - \beta_2}\left((1+t)^{\beta_1} - (1+t)^{\beta_2}\right),$$

where $\beta_1 = (1 + (1+4K)^{1/2})/2$ and $\beta_2 = (1 - (1+4K)^{1/2})/2$. By the relative volume comparison in [2], for sufficiently small $\delta > 0$, we have

$$\frac{V_o((1+\alpha)r) - V_o((1-\alpha)r)}{V_o((1-\alpha)r) - V_o(r_0)}$$

$$\leq \frac{\int_{(1-\alpha)r}^{(1+\alpha)r}\left((1+t)^{\beta_1} - (1+t)^{\beta_2}\right)^{n-1}dt}{\int_{r_0}^{(1-\alpha)r}\left((1+t)^{\beta_1} - (1+t)^{\beta_2}\right)^{n-1}dt}$$

$$\leq \frac{\left((1+\alpha)^{\beta_1(n-1)} - (1-\alpha)^{\beta_1(n-1)}\right)r^{\beta_1(n-1)} + O(r^{\beta_1(n-1)-\delta})}{(1-\alpha)^{\beta_1(n-1)}r^{\beta_1(n-1)} + O(r^{\beta_1(n-1)-\delta})}$$

$$\leq C\alpha,$$

where C depends only on $\beta_1(n-1)$. $\square$

We now define ends of a complete Riemannian manifold M: We denote by $\sharp(r)$ the number of unbounded components of $M \setminus B_r(o)$, where o is a fixed point in M. Then $\sharp(r)$ is nondecreasing in $r > 0$ and we can define the number $\lim_{r\to\infty}\sharp(r) = l$, where l may be infinity. If l is finite, then we can choose $r_0 > 0$ such that $\sharp(r) = l$ for all $r \geq r_0$, and there exist mutually disjoint unbounded components $E_1, E_2, \cdots, E_l$ of $M \setminus B_{r_0}(o)$. We call each E_i an end of M for $i = 1, 2, \cdots, l$.

In [27], Liu proved that if M is a complete Riemannian manifold with nonnegative Ricci curvature outside a compact set, then M has only finitely many ends. He also proved that there exist an integer $m > 0$ and points $x_1, x_2, \cdots, x_m$ in $\partial B_r(o)$ such that for sufficiently large $r > 0$,

$$\partial B_r(o) \subset \cup_{i=1}^m B_{r/4}(x_i),$$

where o is a fixed point in M. Hence, for each end E of M, $\partial C_{E,r}$ is also covered by finitely many geodesic balls of radius $r/4$ with centers in $\partial C_{E,r}$. On the other hand, Li and Tam [25] proved that if M also has finite first Betti number, then each end E of M satisfies the volume comparison property (**C**) as follows:

(**C**) There is a constant $C > 0$ such that for any $r > 0$ large enough and any $x \in \partial C_{E,r}$,

$$\mathrm{vol} A^E_{r,r_0} \leq C V_x(r/4),$$

where $C_{E,r}$ denotes the unbounded component of $E \setminus B_r(o)$ and A^E_{r,r_0} denotes $(B_r(o) \setminus B_{r_0}(o)) \cap E$.

Note that if M is a complete Riemannian manifold with nonnegative Ricci curvature outside a compact set, then there is $r_0 > 0$ such that for each ball $B_r(x) \in M \setminus B_{r_0}(o)$, the conditions (**W**) and (**M**) is valid. Furthermore, it is sufficient to prove Lemma 3.2. Applying the argument employed in the proof of Theorem 1.2 to this case, we get Theorem 1.3 as follows:

Proof of Theorem 1.3. For this case, we redefine the inner product in (3.1) as follows: For each end E_i and any $r > r_0$, there exists a maximal set $\{x^i_1, x^i_2, \cdots, x^i_{m_i(\alpha)}\}$ of points in $\partial C_{E_i,r}$ such that $d(x^i_j, x^i_k) \geq \alpha r$ if $j \neq k$. We define a positive semidefinite bilinear form S_r by

$$S_r(u, v) = \frac{1}{\mathrm{vol} A_r} \int_{A_r} \langle u, v \rangle + \sum_{i=1}^l \sum_{j=1}^{m_i(\alpha)} \frac{1}{V_{x^i_j}(\alpha r)} \int_{B_{\alpha r}(x^i_j)} \langle u, v \rangle,$$

where $A_r = \cup_{i=1}^l (M \setminus C_{E_i,r})$.

For an end E of M, choose a maximal set $F_0 = \{x_i : j = 1, 2, \cdots, m_0\}$ of points in $\partial C_{E,r}$ such that $d(x_i, x_j) \geq r/4$ for any $i \neq j$. Then by the condition (**C**),

(3.2) $$\mathrm{vol} A^E_{r,r_0} \leq C V_{x_i}(r/4).$$

By adding some points to F_0, choose a maximal set $F_\alpha = \{y_i : i = 1, 2, \cdots, m(\alpha)\}$ of points in $\partial C_{E,r}$ such that $d(y_i, y_j) \geq \alpha r$ and $F_0 \subset F_\alpha$. Then for each $y_j \in F_\alpha$, there exists a point $x_{i_j} \in F_0$ such that $B_{r/4}(x_{i_j}) \subset B_{r/2}(y_j)$. This implies, by the condition (**W**), that

$$V_{x_{i_j}}(r/4) \leq C \alpha^{-\nu} V_{y_j}(\alpha r/2).$$

Since $\cup_{y_j \in F_\alpha} B_{\alpha r/2}(y_j) \subset A^E_{(1+\alpha)r,(1-\alpha)r}$, by (3.2) and Lemma 3.5,

$$
\begin{aligned}
m(\alpha)\mathrm{vol}A^E_{r,r_0} &\leq C\alpha^{-\nu}V_{y_j}(\alpha r/2) \\
&\leq C\alpha^{-\nu}\mathrm{vol}A^E_{(1+\alpha)r,(1-\alpha)r} \\
&\leq C\alpha^{-\nu+1}\mathrm{vol}A^E_{r,r_0}.
\end{aligned}
$$

Therefore, for each $i = 1, 2, \cdots, l$, we have $m_i(1/4d) \leq Cd^{\nu_i-1}$. In particular, the space of bounded harmonic functions on M is finite dimensional and its dimension is bounded by the number of nonparabolic ends of M. (See [25] for the proof). Since each $\nu_i \leq n$, similarly arguing as in the proof of Theorem 1.2, we get Theorem 1.3 $\square$

COROLLARY 3.6. *Let M be a connected sum of complete n-dimensional Riemannian manifolds $M_1, M_2, \cdots, M_l$ with nonnegative Ricci curvature. If either u is bounded, or N is two dimensional visibility manifold, or the sectional curvature satisfies $-b^2 \leq K_N \leq -a^2 < 0$, then there exist sets of points $\{q_j\}_{j=1}^k$ in $\overline{u(M)} \cap N$ with $k \leq l$ and $\{\bar{q}_i\}_{i=1}^{k_d}$ in $\overline{u(M)} \cap N(\infty)$ with $k_d \leq C(1 + ld^{n-1}) - l$ such that*

$$
u(M) \subset \overline{\mathcal{C}(\{\bar{q}_i\}_{i=1}^{k_d} \cup \{q_j\}_{j=1}^k)}.
$$

Finally, let us consider the class being roughly isometric to previous cases. A map, not necessarily continuous, $\varphi : X \to Y$ is called a rough isometry between two metric spaces X and Y if φ satisfies the following condition: (See [19] for the detail).

(**R**) for some $\tau > 0$, the τ-neighborhood of the image $\varphi(X)$ covers Y; there exist constants $a \geq 1$ and $b \geq 0$ such that

$$
a^{-1}d(x_1, x_2) - b \leq d(\varphi(x_1), \varphi(x_2)) \leq ad(x_1, x_2) + b
$$

for all $x_1, x_2 \in X$, where d denotes the distances of X and Y induced from their metrics, respectively.

Throughout this paper, when we say that a map $\varphi : M \to N$ is a rough isometry between complete Riemannian manifolds M and N, we assume that Ricci curvature of each manifold is bounded below by a constant and each manifold has the positive injectivity radius.

First of all, the number of ends is a rough isometric invariant and, in addition, each rough isometry between manifolds can be reduced to a rough isometry between ends. (See [21]). It is easy to prove the rough isometric invariance of the volume doubling condition $(\mathbf{V})_0$ on each end as follows: For each end E of M.

(**V**)$_0$ there exist constants $C > 0$ and $\nu > 0$ such that for any $B_r(x) \subset E$ and sufficiently large $0 < s \leq r$,

$$V_x(r) \leq C\left(\frac{r}{s}\right)^\nu V_x(s).$$

Slightly modifying the argument in [12], one can prove that the mean value property (**M**) is valid on each end being roughly isometric to any end satisfying the volume doubling condition (**V**)$_0$ and the Poincaré inequality (**P**)$_0$ defined below: For each end E of M,

(**P**)$_0$ there exist a constant $C > 0$ and an integer $k \in \mathbf{N}$ such that for any $B_r(x) \subset E$,

$$\int_{B_{r/k}(x)} f^2 \leq Cr^2 \int_{B_r(x)} |\nabla f|^2,$$

where $f \in C^\infty(B_r(x))$ with $\int_{B_{r/k}(x)} f = 0$.

Furthermore, a rough isometry between ends preserves the covering number as follows: (See [21] for the proof).

LEMMA 3.7. *Let $\varphi : E \to E'$ be a rough isometry between ends E and E'. If for any $0 < \alpha < 1/4$ and all $r > r_0$, there exist an integer $m = m(\alpha)$ and points $x_1, x_2, \cdots, x_m$ in $\partial C_{E,r}$ such that $\partial C_{E,r} \subset \cup_{i=1}^{m} B_{\alpha r/2}(x_i)$, then there exists a sequence $\{H_r\}$ of compact hypersurfaces in E' such that $d(\partial E', H_r) \to \infty$ as $r \to \infty$, $H_r \subset \cup_{i=1}^{m} B_{3a^2\alpha r}(\varphi(x_i))$. In particular, each H_r divides E' into a bounded subset K_r and the unbounded component U_r of $E' \setminus H_r$.*

Let M be a complete Riemannian manifold being roughly isometric to a complete Riemannian manifold with nonnegative Ricci curvature outside a compact set and finite first Betti number. Then M has finitely many ends E_i, $1, 2, \cdots, l$, and by Lemma 3.7, each end E_i satisfies the following: For a sufficiently small $0 < \alpha < 1/4$, there exist an integer $m_i = m_i(\alpha)$ and points $x_1^i, x_2^i, \cdots, x_{m_i}^i$ such that $d(\partial E_i, H_r^i) \to \infty$ as $r \to \infty$, $H_r^i \subset \cup_{j=1}^{m_i} B_{\tilde{\alpha} r}(\varphi(x_j^i))$ and $\cup_{j=1}^{m_i} B_{\tilde{\alpha} r}(\varphi(x_j^i))$ is connected, where H_r^i is a compact hypersurface in E_i dividing E_i into a bounded subset K_r^i and the unbounded component U_r^i of $E^i \setminus H_r^i$, and $\tilde{\alpha} = 3a^2\alpha$. In this case, we redefine the inner product in (3.1) as follows:

$$\tilde{S}_r(u, v) = \frac{1}{\text{vol} A_r} \int_{A_r} \langle u, v \rangle + \sum_{i=1}^{l} \sum_{j=1}^{m_i} \frac{1}{V_{\varphi(x_j^i)}(\tilde{\alpha} r)} \int_{B_{\tilde{\alpha} r}(\varphi(x_j^i))} \langle u, v \rangle,$$

where $A_r = \cup_{i=1}^l K_r^i$. For this new bilinear form $\tilde{S}_r$, we also have Lemma 3.2 and Lemma 3.3. (See [23] for the proof). Therefore, similarly arguing as in the proof of Theorem 1.3, we have the following result:

THEOREM 3.8. *Let M be a complete Riemannian manifold being roughly isometric to a complete n-dimensional Riemannian manifold with nonnegative Ricci curvature outside a compact set and finite first Betti number. Suppose that N is two dimensional visibility manifold, or a Cartan-Hadamard manifold with the sectional curvature satisfying $-b^2 \leq K_N \leq -a^2 < 0$. Let $u : M \to N$ be a harmonic map satisfying (1.1) for some $d \geq 0$. Then there exist sets of points $\{q_j\}_{j=1}^k$ in $\overline{u(M)} \cap N$ with $k \leq l$ and $\{\overline{q}_i\}_{i=1}^{k_d}$ in $\overline{u(M)} \cap N(\infty)$ with $k_d \leq C(1 + \sum_{i=1}^l d^{\nu_i - 1}) - l \leq C(1 + ld^{n-1}) - l$ such that*

$$u(M) \subset \overline{\mathcal{C}(\{\overline{q}_i\}_{i=1}^{k_d} \cup \{q_j\}_{j=1}^k)},$$

where l is the number of ends of M and ν_i ($\leq n$) denotes the order in the volume doubling condition corresponding to each end $E_i, i = 1, 2, \cdots, l$, of M.

COROLLARY 3.9. *Let M be a complete Riemannian manifold being roughly isometric to a connected sum of complete n-dimensional Riemannian manifolds M_i, $i = 1, 2, \cdots, l$, with nonnegative Ricci curvature. Suppose that N is two dimensional visibility manifold, or a Cartan-Hadamard manifold with the sectional curvature satisfying $-b^2 \leq K_N \leq -a^2 < 0$. Let $u : M \to N$ be a harmonic map satisfying (1.1) for some $d \geq 0$. Then there exist sets of points $\{q_j\}_{j=1}^k$ in $\overline{u(M)} \cap N$ with $k \leq l$ and $\{\overline{q}_i\}_{i=1}^{k_d}$ in $\overline{u(M)} \cap N(\infty)$ with $k_d \leq C(1 + ld^{n-1}) - l$ such that*

$$u(M) \subset \overline{\mathcal{C}(\{\overline{q}_i\}_{i=1}^{k_d} \cup \{q_j\}_{j=1}^k)}.$$

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DEPARTMENT OF MATHEMATICS EDUCATION, EWHA WOMANS UNIVERSITY, SEOUL 120-750, KOREA
E-mail: yonghah@ewha.ac.kr