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CENTRAL LIMIT TYPE THEOREM FOR WEIGHTED PARTICLE SYSTEMS

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ABSTRACT. We consider a system of particles with locations

$$\{X_i^n(t) : t \geq 0, i = 1, \ldots, n\}$$

in R^d, time-varying weights $\{A_i^n(t) : t \geq 0, i = 1, \ldots, n\}$ and weighted empirical measure processes $V^n(t) = \frac{1}{n}\sum_{i=1}^n A_i^n(t)\delta_{X_i^n}(t)$, where δ_x is the Dirac measure. It is known that there exists the limit of $\{V_n\}$ in the week* topology on $M(R^d)$ under suitable conditions. If $\{X_i^n, A_i^n, V^n\}$ satisfies some diffusion equations, applying Ito formula, we prove a central limit type theorem for the empirical process $\{V^n\}$, i.e., we consider the convergence of the processes $\eta_t^n \equiv \sqrt{n}(V^n - V)$. Besides, we study a characterization of its limit.

1. Introduction

We consider $\{X_i(t) : t \geq 0, i \in N\}$ as a system of particles with locations in R^d, time-varying weights $\{A_i(t) : t \geq 0, i \in N\}$ and weighted empirical measures of the form

$$V(t) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} A_i(t)\delta_{X_i(t)}, \tag{1.1}$$

where δ_x is the Dirac measure at x and the limit exists in the weak* topology on $\mathcal{M}(R^d)$, the collection of all finite signed Borel measures on R^d. Kurtz and Xiong [6] proved that under some conditions

$$\{X_i, A_i, V\}_{i\geq 1}$$

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is the unique solution of the following equations:

(1.2)

$$X_i(t) = X_i(0) + \int_0^t \sigma(X_i(s), V(s))dB_i(s) + \int_0^t c(X_i(s), V(s))ds + \int_{U\times[0,t]} a(X_i(s), V(s), u)W(du, ds)$$

and

$$\begin{aligned} A_i(t) = A_i(0) &+ \int_0^t A_i(s)\gamma^T(X_i(s), V(s))dB_i(s) \\ &+ \int_0^t A_i(s)d(X_i(s), V(s))ds \\ &+ \int_{U\times[0,t]} A_i(s)\beta(X_i(s), V(s), u)W(du, ds), \end{aligned} \tag{1.3}$$

where the $\{B_i\}$ are independent standard R^d-valued Brownian motions and W, independent of $\{B_i\}$, is Gaussian white noise with

$$E[W(A,t)W(B,t)] = \mu(A\cap B)t.$$

We assume that μ is a Borel measure on a complete separable metric space U and $\{A_i(0), X_i(0)\}$ are i.i.d. and independent of $\{B_i\}$ and W.

As one part of that paper, in [6] they proved that (1.1) is approximated by the weighted empirical measure process of the following finite interacting particle system:

(1.4)

$$X_i^n(t) = X_i(0) + \int_0^t \sigma(X_i^n(s), V^n(s))dB_i(s) + \int_0^t c(X_i^n(s), V^n(s))ds + \int_{U\times[0,t]} a(X_i^n(s), V^n(s), u)W(du, ds)$$

and

$$\begin{aligned} A_i^n(t) = A_i(0) &+ \int_0^t A_i^n(s)\gamma^T(X_i^n(s), V^n(s))dB_i(s) \\ &+ \int_0^t A_i^n(s)d(X_i^n(s), V^n(s))ds \\ &+ \int_{U\times[0,t]} A_i^n(s)\beta(X_i^n(s), V^n(s), u)W(du, ds), \end{aligned} \tag{1.5}$$

for $i = 1, 2, \dots, n$, where

$$(1.6) \qquad V^n(t) = \frac{1}{n}\sum_{i=1}^{n} A_i^n(t)\delta_{X_i^n(t)}.$$

In this paper we study a convergence of the fluctuations for the McKean-Vlasov model related with the above processes. For every integer n, let η^n be the fluctuation process defined by

$$\eta_t^n = \sqrt{n}\big(V^n(t) - V(t)\big).$$

In the following we prove that the fluctuations belong uniformly in n and t to a certain Hilbert space, a weighted Sobolev space. Weighted Sobolev embedding theory is one of the keys to obtain the tightness of $\{\eta_t^n\}$.

S. Meleard [10] considered this kind of convergence for the classical McKean-Vlasov case. Since the pioneering work by McKean [9], limits of empirical measure processes for systems of interacting diffusions have been studied by Chiang et al. [1], Graham [3], Kallianpur and Xiong [5], Kurtz and Xiong [6], [7], [8] and reference therein.

While we have prepared this paper on the tightness of $\{\eta^n\}$ in the dual space of Schwartz space $\mathcal{S}$, which is the space of infinitely differentiable functions with bounded supports, Kurtz sent us the preprint of [8]. This paper also deals with the same problem. Since the weighted Sobolev space is another interesting infinite-dimensional space, we have reconsidered the tightness of $\{\eta^n\}$ in the weighted Sobolev space. The present paper also differs from [8] in the following aspect: Kurtz and Xiong in [8] use Mitoma's theorem, which actually induces to show the tightness of real-valued processes. We show the tightness of $\{\eta^n\}$ as a distribution-valued processes using the structure of weighted Sobolev space.

In the next section we state some basic facts about the system (1.1)-(1.3) obtained in [6], [7] and [8] for the convenience of the reader. We also introduce some lemmas which are going to be used later and a brief review on weighted Sobolev space.

In section 3, we study the tightness of the process, $\{\eta^n\}$ in a weighted Sobolev space.

NOTATIONS..

-$\mathcal{M}_+(R^d)$ *is the collection of all finite and positive Borel measure on* R^d.

-For every integer j, C_b^j denote the space of bounded functions with bounded derivatives of order greater than or equal to 1 and less than j.

-The letter C and K with sub-indices denote constant numbers which can change from line to line.

2. Preliminaries

First, we define the Wasserstein metric as the following: For $\nu_1, \nu_2 \in \mathcal{M}_+(R^d)$,

$$\rho(\nu_1, \nu_2) = \sup\{|\langle \phi, \nu_1\rangle - \langle \phi, \nu_2\rangle| : \phi \in \mathbf{B}_1\},$$

where $\mathbf{B}_1 = \{\phi : |\phi(x) - \phi(y)| \le |x-y|, |\phi(x)| \le 1 \, \forall x, y \in R^d\}$. Note that the metric ρ determines the topology of weak convergence on $\mathcal{M}_+(R^d)$. For a bounded Lipshitz functions f, define

$$\|f\|_L = \sup_{x \in R^d} |f(x)| + \sup_{x,y \in R^d} \frac{|f(x) - f(y)|}{|x-y|}.$$

Let $\{f_k\}$ be a dense subset of $C_b(R^d)$ such that $\|f_i\|_L < \infty$ for each i. Define

$$\tilde{\rho}(\nu_1, \nu_2) = \sum_{i=1}^{\infty} \frac{|\langle \nu_1, f_i\rangle - \langle \nu_2, f_i\rangle|}{2^i \|f_i\|_L}.$$

Note that $\tilde{\rho}(\nu_1, \nu_2) \le \rho(\nu_1, \nu_2)$. For the equation system (1.1)-(1.3), we assume that $\sigma : R^d \times \mathcal{M}(R^d) \to R^{d \times d}$, $c : R^d \times \mathcal{M}(R^d) \to R^{d\times d}$, $\alpha : R^d \times \mathcal{M}(R^d) \times U \to R^{d\times d}$, $\gamma : R^d \times \mathcal{M}(R^d) \to R^{d \times d}$, $d : R^d \times \mathcal{M}(R^d) \to R$ and $\beta : R^d \times \mathcal{M}(R^d) \times U \to R$ satisfy the following Condition A;

CONDITION A.

(A1) There exists a constant K such that for each $x \in R^d, \nu \in M_+(R^d)$

$$|\sigma(x,\nu)|^2 + |c(x,\nu)|^2 + \int_U |\alpha(x,\nu,u)|^2 \mu(du) \\ + |\gamma(x,\nu)|^2 + |d(x,\nu)|^2 + \int_U |\beta(x,\nu,u)|^2 \mu(du) \le K^2.$$

(A2) For each $x_1, x_2 \in R^d$, $\nu_1, \nu_2 \in \mathcal{M}_+(R^d)$,

$$\begin{aligned}&|\sigma(x_1,\nu_1)-\sigma(x_2,\nu_2)|^2+|c(x_1,\nu_1)-c(x_2,\nu_2)|^2\\&+|\gamma(x_1,\nu_1)-\gamma(x_2,\nu_2)|^2+\int_U|\alpha(x_1,\nu_1,u)-\alpha(x_2,\nu_2,u)|^2\mu(du)\\&+|d(x_1,\nu_1)-d(x_2,\nu_2)|^2+\int_U|\beta(x_1,\nu_1,u)-\beta(x_2,\nu_2,u)|^2\mu(du)\\&\le K^2(|x_1-x_2|^2+\tilde{\rho}(\nu_1,\nu_2)^2).\end{aligned}$$

(A3) There exists constant $\lambda > 1$ and $K > 0$ such that for any i.i.d. sequence (ξ_i, ζ_i), $i = 1, 2, \ldots$ and $x \in R^d$ we have

$$E|\sigma(x, \frac{1}{n}\sum_{i=1}^{n}\xi_i\delta_{\zeta_i}) - \sigma(x,\mu)|^{2\lambda} \le \frac{KE\xi_1^{2\lambda}}{n^\lambda},$$

where $\mu(\cdot) = E[\xi_1 1_{\zeta_1 \in \cdot}]$. A similar inequality holds for the other coefficients.

(A4) For each $\nu \in \mathcal{M}(R^d)$, and $u \in U$

$$\sigma(\cdot,\nu), c(\cdot,\nu), \alpha(\cdot,\nu,u), \gamma(\cdot,\nu), d(\cdot,\nu), \beta(\cdot,\nu,u) \in C_b^{1+D,2D}(R^d),$$

where $D = [\frac{d}{2}] + 1$.

In the following we state some facts about the system (1.1)-(1.3).

THEOREM 2.1 [6]. *Suppose that Condition* (A1) *and* (A2) *hold and*

$$E|A_1(0)|^2 + E|X_1(0)|^2 < \infty.$$

Then the system has the unique solution.

LEMMA 2.2 [8]. *Suppose Condition* (A1) *holds. If* (X_i^n, A_i^n, V^n), *for* $i = 1, 2, \ldots, n$ *is the solution of* (1.1)-(1.3) *and for some* $p > 0$

$$E|A_1(0)|^p + E|X_1(0)|^p < \infty \tag{2.1}$$

then for every $t \ge 0$,

$$\sup_{1\le n\le\infty} E \sup_{0\le s\le t}(|A_i^n(s)|^p + |X_i^n(s)|^p) < \infty.$$

THEOREM 2.3 [8]. *Under the assumptions* (A1)-(A3), *we have*

$$E\sup_{t\leq T}\big(|X_i^n(t\wedge\tau_m^n)-X_i(t\wedge\tau_m^n)|^{2\lambda}$$
$$+(\frac{1}{n}\sum_{i=1}^{n}|A_i^n(t\wedge\tau_m^n)-A_i(t\wedge\tau_m^n)|^{\lambda})^2\big)\leq\frac{c_1(T,m)}{n^{\lambda}},$$

where $c_1(T,m)$ *is a constant,*

$$\tau_m^n=\inf\{t:\frac{1}{n}\sum_{i=1}^{n}A_i^n(t)^2>m^2\text{ or }\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}A_i(t)^2>m^2\},\tag{2.2}$$

and

$$\sup_n P(\tau_m^n\leq T)\leq\frac{1}{m^2}8e^{(K^2+K)T}EA_i(0)^2.\tag{2.3}$$

COROLLARY 2.4 [7]. *Assume* (2.1) *holds for any* $p>1$ *and* (A1)-(A3). *Then for each bounded Lipschitz function* f *and each* $t\geq 0$,

$$E[V^n(t)f-V(t)f|1_{\{t\leq\tau_m^n\}}]\leq\frac{c_2(t,m)\|f\|_L}{\sqrt{n}},\tag{2.4}$$

where $c_2(t,m)$ *is a constant.*

The rate of convergence given by (2.4) is equivalent to weak convergence. The estimate (2.4) implies the following.

COROLLARY 2.5 [7]. *Assume* (2.1) *holds for any* $p>1$ *and* (A1)-(A3). *For each* $t\geq 0$,

$$E\big[\tilde{\rho}(V^n(t),V(t))1_{\{t\leq\tau_m^n\}}\big]\leq\frac{c_2(t,m)}{\sqrt{n}}.$$

Recall $(V(t),X_i(t),A_i(t))$ in (1.1)-(1.3). If $A_i(t)\equiv 1$ for all t and i, that is weights never vary, and there is no term driven by white noise in $X_i(t)$, then the limiting empirical process like $V(t)$ is deterministic and characterized by McKean-Vlasov equation. Here, the limit $V(t)$ is still stochastic. The classical McKean-Vlasov limit satisfies the following equation (see [6] and [9]):

$$X(t)=X(0)+\int_0^t\sigma(X(s),P(s))dB(s)+\int_0^t c(X(s),P(s))ds,$$

where $P(s)$ is required to be the distribution of $X(t)$.

Our model is much more general with time varying weights and terms related to white noise. By the following Theorem 2.6 our setting (X, A, V) satisfies the following stochastic differential equation:

(2.5)
$$\begin{aligned} X(t) =& X(0) + \int_0^t \sigma(X(s), V(s))dB(s) + \int_0^t c(X(s), V(s))ds \\ &+ \int_{U\times[0,t]} \alpha(X(s), V(s), u)W(du, ds) \end{aligned}$$

and

(2.6)
$$\begin{aligned} A(t) =& A(0) + \int_0^t A(s)\gamma^T(X(s), V(s))dB(s) + \int_0^t A(s)d(X(s), V(s))ds \\ &+ \int_{U\times[0,t]} A(s)\beta(X(s), V(s), u)W(du, ds), \end{aligned}$$

where $V(t)$ is the random measure determined by

(2.7)
$$\langle \phi, V(t)\rangle = E[A(t)\phi(X(t))|\mathcal{F}_t^W],$$

where $\{\mathcal{F}_t^W\}$ is the filtration generated by W (see in detail [6]).

THEOREM 2.6 [6]. *Let (X, A, V, B, W) satisfy* (2.5)-(2.7).

Then there exists a solution $(\{X_i\}, \{A_i\}, \{B_i\}, \tilde{V}, \tilde{W})$ of (1.1)-(1.3) *such that $(X_1, A_1, \tilde{V}, B_1, \tilde{W})$ has the same distribution as (X, A, V, B, W). Conversely, if there exists a pathwise unique solution $(\{X_i\}, \{A_i\}, \{B_i\}, V, W)$ of* (1.1)-(1.3)*, then (X_1, A_1, V, B_1, W) is a solution of* (2.5)-(2.7).

THEOREM 2.7 [6]. *Let V be the weighted empirical measure for the particle system given by Theorem 2.2 and 2.3. Then V is a solution of the following SDE;*

(2.8)
$$\begin{aligned} \langle \phi, V(t)\rangle =& \langle \phi, V(0)\rangle + \int_0^t \langle d(\cdot, V(s))\phi + L(V(s))\phi, V(s)\rangle ds \\ &+ \int_{U\times[0,t]} \langle \beta(\cdot, V(s), u)\phi \\ &+ \alpha^T(\cdot, V(s), u)\nabla\phi, V(s)\rangle W(du, ds), \forall \phi \in C_b^2(R^d), \end{aligned}$$

where $\nabla\phi(x) = (\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \dots, \frac{\partial\phi}{\partial x_d})(x)$

$$(2.9) \qquad L(v)\phi(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x,v)\partial_{x_i}\partial_{x_j}\phi(x) + \sum_{i=1}^{d} b_i(x,v)\partial_{x_i}\phi(x)$$

with

$$a(x,v) = \sigma(x,v)\sigma^T(x,v) + \int_U \alpha(x,v,u)\alpha^T(x,v,u)\mu(du)$$

and

$$b(x,v) = c(x,v) + \sigma(x,v)\gamma(x,v) + \int_U \beta(x,v,u)\alpha(x,v,u)\mu(du).$$

Now we briefly review the weighted Sobolev space. Consider the space of all real-valued function g defined on R^d with partial derivatives up to order j such that

$$\|g\|_{j,\alpha} = (\sum_{k=1}^{j}\int_{R^d}\frac{|D^k g(x)|^2}{1+|x|^{2\alpha}}dx)^{\frac{1}{2}} < \infty,$$

where $|\cdot|$ denotes the Euclidean norm on R^d, and if $\bar{k} = (k_1, k_2, \dots, k_d)$, then $k = \sum_{i=1}^{d} k_i$ and $D^k g = \frac{\partial^{\bar{k}} g}{\partial x_1^{k_1}, \dots, \partial x_d^{k_d}}$.

Let $W_0^{j,\alpha}$ be the closure of the set of functions of class C^∞ with compact support for this norm. We denote by $W_0^{-j,\alpha}$ its dual space and let

$$C^{j,\alpha} = \{g : \lim_{|x|\to\infty}\frac{|D^k g(x)|}{1+|x|^\alpha} = 0, \forall k \le j\}, \ \|g\|_{C^{j,\alpha}} = \sum_{k=1}^{j}\sup_{x\in R^d}\frac{|D^k g(x)|}{1+|x|^\alpha}.$$

We have the following embedding:

$$W_0^{m+j,\alpha} \hookrightarrow C^{j,\alpha}, \quad m > \frac{2}{d}, \quad j \ge 0, \quad \alpha \ge 0$$
$$W_0^{-j,\alpha+\beta} \hookrightarrow C^{-j,\alpha}, \quad \alpha > \frac{2}{d}, \quad j \ge 0, \alpha \ge 0$$
$$W_0^{-j,\alpha+\beta} \hookrightarrow W_0^{-(m+j),\alpha}, \quad m > \frac{2}{d}, \quad j \ge 0, \quad \alpha \ge 0, \quad \beta > \frac{2}{d}.$$

We are going to use the following lemma introduced by Meleard [10].

LEMMA 2.8. *For every fixed* $x, y \in R^d$, $D_{xy}, D_x, H_x : W_0^{1+D,2D} \to R$ *defined by* $D_{xy}(\phi) = \phi(x) - \phi(y)$, $D_x(\phi) = \phi(x)$, $H_x(\phi) = \sum_{i=1}^d \frac{\partial \phi}{\partial x_j}(x)$ *are continuous and*

$$\begin{aligned}\|D_{xy}\|_{-(1+D),2D} \le & K_1|x-y|(1+|x|^{2D}+|y|^{2D})\\ \|D_x\|_{-(1+D),2D} \le & K_2(1+|x|^{2D})\\ \|H_x\|_{-(1+D),2D} \le & K_3(1+|x|^{2D}),\end{aligned}$$

where $\|x\|^\alpha = (x_1^2 + \cdots + x_d^2)^{\frac{\alpha}{2}}$, *and* K_1, K_2 *and* K_3 *are some constants.*

3. Tightness of $\{\eta^n_\cdot\}$

In this section, we are going to show that $\{\eta^n_\cdot\}$ is relatively compact in $W_0^{-2(1+D),D}$. To show the criteria of relative compactness of $\{\eta^n_\cdot\}$ in $W_0^{-2(1+D),D}$, we review the following facts in [2] and [4].

REMARK 3.0.

(1) A sequence of adaptive processes $\{Y^n_\cdot\}$ on the filtered spaces $(\Omega^n, \mathcal{F}^n, P^n)$, taking values in a Hilbert space H (with a norm $\|\cdot\|_H$) is relatively compact in $C([0,T], H)$ if both following conditions hold:
I. There exists a Hilbert space H_0 (with a norm $\|\cdot\|_{H_0}$) such that $H_0 \hookrightarrow H$ and for each $t \le T$,

$$\sup_n E^n \|Y_t^n\|_{H_0}^2 < \infty.$$

II. For each $T > 0$, there exists a family $\{\gamma_n(\delta) : 0 < \delta < 1\}$ of non-negative random variables satisfying

$$E^n[\|Y_t^n - Y_s^n\|_H^2 | \mathcal{F}_t^n] < E^n[\gamma_n(\delta)|\mathcal{F}_t^n],$$

$0 \le t \le T$, $|t-s| < \delta$; in addition,

$$\lim_{\delta \to 0} \sup_n E^n[\gamma_n(\delta)] = 0.$$

(2) Due to (2.3) for given $\epsilon > 0$, there exists large enough m such that $\sup_n P(\tau_m^n \le T) < \epsilon$. Therefore to prove the tightness of $\{\eta^n_\cdot\}$ on $[0,T]$, it is enough to prove the tightness of $\{\eta^n_{\cdot\wedge\tau_m^n}\}$ on $[0,T]$ since

$$\begin{aligned}\omega(t) &= \omega(t)1_{(\tau_m^n \le T)} + \omega(t)1_{(\tau_m^n > T)}\\ &= \omega(t)1_{(\tau_m^n \le T)} + \omega(t\wedge\tau_m^n)1_{(\tau_m^n > T)}.\end{aligned}$$

For all $t \in [0,T]$ and for all $\omega \in \{\tau_m^n \geq T\}$, we have

$$\frac{1}{n}\sum_{i=1}^{n} A_i^n(t)^2 < m^2 \text{ and } \lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k} A_i(t)^2 < m^2.$$

Similarly if (2.1) holds for any $p > 1$, from Lemma 2.2 we may assume that for all $t \in [0,T]$ and $\omega \in \{\tau_m^n \geq T\}$,

$$\frac{1}{n}\sup_{t\leq T}\sum_{i=1}^{n} A_i^n(t)^4 < c(T,m).$$

The following lemmas are the adaptations of the results of Meleard in [10] but the proof of Lemma 3.1 should be reset with this generalized processes.

LEMMA 3.1. *If* (2.1) *holds for any* $p > 1$ *with the assumptions* (A1)-(A3), *the family* $\{\eta_t^n\}_{n=1}^{\infty}$, $t \leq T$ *is bounded uniformly in* n *and* t *in* $W_0^{-(1+D),2D}$, *that is*

$$\sup_n \sup_{t\leq T} E(\|\eta_t^n\|^2_{-(1+D),2D}) < \infty. \tag{3.1}$$

Proof. Let for each $\phi \in W_0^{(1+D),2D}$ and for any $t \in [0,T]$

$$S_t^n(\phi) = \sqrt{n}\Big(\frac{1}{n}\sum_{i=1}^{n}[(A_i^n(t)\phi(X_i^n(t)) - A_i^n(t)\phi(X_i(t))]\Big)$$

$$U_t^n(\phi) = \sqrt{n}\Big(\frac{1}{n}\sum_{i=1}^{n}[(A_i^n(t) - A_i(t))\phi(X_i(t))]\Big)$$

$$T_t^n(\phi) = \sqrt{n}\Big(\frac{1}{n}\sum_{i=1}^{n}[A_i(t)\phi(X_i(t)) - \langle V_t,\phi\rangle]\Big).$$

Let $\{\phi_p\}$ be a completely orthonormal system(CONS) in $W_0^{1+D,2D}$, then

$$\sum_{p=1}^{\infty}\langle\eta_t^n,\phi_p\rangle^2 \leq C\Big(\sum_{p=1}^{\infty} S_t^n(\phi_p)^2 + \sum_{p=1}^{\infty} U_t^n(\phi_p)^2 + \sum_{p=1}^{\infty} T_t^n(\phi_p)^2\Big),$$

where C is a constant. Since $(A_i^n, X_i^n(t), X_i(t))$, $i = 1, 2, \ldots, n$ are exchangeable (see [8]), we have

$$\begin{aligned} ES_t^n(\phi_p)^2 =& \frac{1}{n} E\big(\sum_{i=1}^n A_i^n(t)(\phi_p(X_i^n(t)) - \phi_p(X_i(t))\big)^2 \\ \le& \frac{1}{n} n^2 E\big(A_1^n(t)(\phi_p(X_1^n(t)) - \phi_p(X_1(t))\big)^2 \\ \le& nEA_1^n(t)^2 E\big(\phi_p(X_i^n(t)) - \phi_p(X_i(t))\big)^2. \end{aligned}$$

Hence

$$E\sum_{p=1}^{\infty} S_t^n(\phi_p)^2 \le nEA_1^n(t)^2 E\|D_{X_1^n(t)X_1(t)}\|^2_{(1+D),2D},$$

where the mapping $D_{X_i^n(t),X_i(t)}$ is defined in Lemma 2.8. Also from Lemma 2.8

(3.2)

$$\begin{aligned} & nE[A_1^n(t)^2 \|D_{X_1^n(t)X_1(t)}\|^2_{(1+D),2D} 1_{\tau_m^n \ge T}] \\ \le& nK_1 E[A_1^n(t)^2] E[|X_1^n(t) - X_1(t)|^2 (1 + |X_1^n(t)|^{4D} + |X_1(t)^{4D}) 1_{\tau_m^n \ge T}] \\ \le& nK_2 E[A_1^n(t)^2] E[|X_1^n(t) - X_1(t)|^4 1_{\tau_m^n \ge T}]^{\frac{1}{2}} E[1 + |X_1^n(t)|^{8D} \\ & + |X_1(t)|^{8D}|]^{\frac{1}{2}} \\ \le& nK_2 E[\sup_{t \le T} A_1^n(t)^2] (\frac{c_1(T,m)}{n^2})^{\frac{1}{2}} E[\sup_{t \le T}(1 + |X_1^n(t)|^{8D} + |X_1(t)|^{8D})]^{\frac{1}{2}} \\ <& \infty \end{aligned}$$

by Lemma 2.2 and Theorem 2.3. Hence from (2.3) and (3.2)

$$\sup_n E\sum_{p=1}^{\infty} S_t^n(\phi_p)^2 < \infty.$$

Similarly,

$$\begin{aligned} EU_t^n(\phi)^2 =& \frac{1}{n} E\big(\sum_{i=1}^n (A_i^n(t) - A_i(t))\phi(X_i(t))\big)^2 \\ \le& \frac{1}{n} n^2 E[(A_1^n(t) - A_1(t))^2] E[\phi(X_1(t))^2]. \end{aligned}$$

Therefore by the definition of $D_{X_1(t)}$ in Lemma 2.8, Theorem 2.3 and (2.3) induce that

(3.3)
$$\begin{aligned}E\sum_{p=1}^{\infty}U_t^n(\phi_p)^2 &\le nE[(A_1^n(t)-A_1(t))^2]E[\|D_{X_1(t)}\|^2]\\ &\le \frac{\sqrt{c_1(T,m)}}{n}K_2'(1+|X_1(t)|^{8D})^{\frac{1}{2}}<\infty\end{aligned}$$

$$\begin{aligned}E[T_t^n(\phi)^2|\mathcal{F}_t^W] &= \frac{1}{n}\sum_{i=1}^{n}E[(A_i(t)\phi(X_i(t))-\langle V_t,\phi\rangle|\mathcal{F}_t^W]^2\\ &\le \frac{1}{n}\sum_{i=1}^{n}E[(A_i(t)\phi(X_i(t))^2|\mathcal{F}_t^W]-E\langle V_t,\phi\rangle^2\\ &\le \frac{1}{n}\sum_{i=1}^{\infty}E[(A_i(t)\phi(X_i(t))^2|\mathcal{F}_t^W].\end{aligned}$$

Therefore

(3.4)
$$\begin{aligned}E\sum_{p=1}^{\infty}T_t^n(\phi_p)^2 &= \frac{1}{n}\sum_{i=1}^{n}E[\sum_{p=1}^{\infty}(A_i(t)^2\phi_p(X_i(t))^2]\\ &\le \sum_{p=1}^{\infty}E[(A_1(t)^4]E[\phi_p(X_1(t))^4]\\ &\le K_3E[1+|X_1(t)|^{4D}]<\infty.\end{aligned}$$

Hence from (3.2)-(3.4) and Parseval's identity

$$\sup_n\sup_{t\le T}E\|\eta_t^n\|^2_{-(1+D),2D}=\sup_n\sup_{t\le T}E\sum_{p=1}^{\infty}\langle\eta_t^n,\psi_p\rangle^2<\infty.$$

□

LEMMA 3.2. *If* (2.1) *holds for any* $p>1$ *with the assumptions* (A1)-(A4), *the random operator* $L(\cdot)$ *defined by* (2.9) *is a linear continuous mapping from* $W_0^{2+2D,D}$ *into* $W_0^{1+D,2D}$ *and for all* $\phi\in W_0^{2+2D,D}$, *uniformly in* n *and* ω,

(3.5)
$$\|L(V^n(s))\phi\|_{(1+D),2D}\le K\|\phi\|_{2+2D,D}.$$

Proof. Intrinsically, if we follow the proof of Lemma 5.6 in [10] we can get (3.5). □

PROPOSITION 3.3. *If* (2.1) *holds for any* $p > 1$ *with the assumptions* (A1)-(A4),

$$\sup_n E(\sup_{t\le T} \|\eta_t^n\|^2_{-2(1+D),D}) < \infty.$$

Proof. Applying Ito's formula to (1.5) and (1.6), for every $\phi\in C_b^2(R^d)$, we have

$$\begin{aligned}A_i^n(t)\phi(X_i^n(t)) =& A_i^n(0)\phi(X_i^n(0))\\ &+\int_0^t A_i^n(s)[\phi(X_i^n(s))\gamma^T(X_i^n(s),V^n(s))\\ &+\nabla^T\phi(X_i^n(s))\sigma(X_i^n(s),V^n(s))]dB_i(s)\\ &+\int_0^t A_i^n(s)[\phi(X_i^n(s))d(X_i^n(s),V^n(s))\\ &+L(V^n(s))\phi(X_i^n(s))]ds\\ &+\int_{U\times[0,t]} A_i^n(s)[\phi(X_i^n(s))\beta(X_i^n(s),V^n(s),u)\phi\\ &+\alpha^T(X_i^n(s),V^n(s),u)\nabla\phi(X_i^n(s))]W(du,ds).\end{aligned}$$

Note that

$$\langle V^n(t),\phi\rangle = \frac{1}{n}\sum_{i=1}^n A_i^n(t)\phi(X_i^n(t)).$$

Therefore

(3.6)
$$\begin{aligned}\langle V^n(t),\phi\rangle =& \langle V^n(0),\phi\rangle\\ &+\frac{1}{n}\sum_{i=1}^n\int_0^t A_i^n(s)[\phi(X_i^n(s))\gamma^T(X_i^n(s),V^n(s))\\ &+\nabla^T\phi(X_i^n(s))\sigma(X_i^n(s),V^n(s))]dB_i(s)\\ &+\int_0^t\langle V^n(s),d(\cdot,V^n(s))\phi+L(V^n(s))\phi\rangle ds\\ &+\int_{U\times[0,t]}\langle V^n(s),\beta(\cdot,V^n(s),u)\phi\\ &+\alpha^T(\cdot,V^n(s),u)\nabla\phi\rangle W(du,ds).\end{aligned}$$

Then, by (2.8) and (3.6) we have

(3.7)

$$\begin{aligned}
\langle \eta_t^n, \phi\rangle &= \langle \eta_0^n, \phi\rangle \\
&+ \frac{1}{\sqrt{n}} \sum_{i=1}^n \int_0^t A_i^n(s)[(\phi(X_i^n(s))\gamma^T(X_i^n(s), V^n(s)) \\
&+ \nabla^T \phi(X_i^n(s))\sigma(X_i^n(s), V^n(s)))]dB_i(s) \\
&+ \sqrt{n} \int_0^t [\langle V^n(s), d(\cdot, V^n(s))\phi + L(V^n(s))\phi\rangle \\
&- \langle V(s), d(\cdot, V(s))\phi + L(V(s))\phi\rangle]ds \\
&+ \sqrt{n} \int_{U\times[0,t]} \big(\langle V^n(s), \beta(\cdot, V^n(s), u)\phi + \alpha^T(\cdot, V^n(s), u)\nabla\phi\rangle \\
&- \langle V(s), \beta(\cdot, V(s), u)\phi + \alpha(\cdot, V(s), u)\nabla\phi\rangle\big) W(du, ds).
\end{aligned}$$

Let $\{\psi_p\}$ be a CONS in $W_0^{2+2D,D}$ of functions of class C^∞ with compact support on R^d. Recall that $\eta_t^n = \sqrt{n}(V^n(t) - V(t))$

(3.8)

$$\begin{aligned}
&E\langle \eta_t^n, \psi_p\rangle^2 \\
&\le C\big(E\langle \eta_0^n, \psi_p\rangle^2 \\
&+ E(\int_0^t \langle \eta_s^n, d(\cdot, V^n(s)\psi_p + L(V^n(s))\psi_p\rangle ds)^2 \\
&+ nE(\int_0^t \langle V(s), d(\cdot, V^n(s))\psi_p + L(V^n(s))\psi_p\rangle \\
&- \langle V(s), d(\cdot, V(s))\psi_p + L(V(s))\psi_p\rangle)ds)^2 \\
&+ E|\frac{1}{\sqrt{n}} \sum_{i=1}^n \int_0^t A_i^n(s)\psi_p(X_i^n(s))\gamma(X_i^n(s), V^n(s)) \\
&+ \nabla^T\psi_p(X_i^n(s))\sigma(X_i^n(s), V^n(s))dB_i(s)|^2 \\
&+ E(\int_{U\times[0,t]} \langle \eta_s^n, \beta(\cdot, V^n(s), u)\psi_p + \alpha^T(\cdot, V^n(s), u)\nabla\psi_p\rangle W(du, ds))^2 \\
&+ nE(\int_{U\times[0,t]} (\langle V(s), \beta(\cdot, V^n(s), u)\psi_p + \alpha^T(\cdot, V^n(s), u)\nabla\psi_p\rangle \\
&- \langle V(s), \beta(\cdot, V(s), u)\psi_p + \alpha^T(\cdot, V(s), u)\nabla\psi_p\rangle)W(du, ds))^2\big).
\end{aligned}$$

To handle the martingale parts of (3.7) let

(3.9)

$$
\begin{aligned}
&M_t^n(\psi_p)\\
=&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_0^t A_i^n(s)\psi_p(X_i^n(s))(\gamma(X_i^n(s),V^n(s))\\
&+\nabla^T\psi_p(X_i^n(s))\sigma(X_i^n(s),V^n(s))dB_i(s)\\
&+\int_{U\times[0,t]}\langle\eta_s^n,\beta(\cdot,V^n(s),u)\psi_p+\alpha^T(\cdot,V^n(s),u)\nabla\psi_p\rangle W(du,ds)\\
&+\sqrt{n}E\int_{U\times[0,t]}(\langle V(s),\beta(\cdot,V^n(s),u)\psi_p+\alpha^T(\cdot,V^n(s),u)\nabla\psi_p\rangle\\
&-\langle V(s),\beta(\cdot,V(s),u)\psi_p+\alpha^T(\cdot,V(s),u)\nabla\psi_p\rangle)W(du,ds)\\
\equiv&\bar{M}_t^n(\psi_p)+\tilde{M}_{1,t}^n(\psi_p)+\tilde{M}_{2,t}^n(\psi_p).
\end{aligned}
$$

By Schwartz's and Doob's inequality, we get

$$
\begin{aligned}
&E\sum_{p=1}^{\infty}\sup_{t\le T}\langle\eta_t^n,\psi_p\rangle^2\\
&\le C\big(E\|\eta_0^n\|^2_{-2(1+d),D}\\
&+E\int_0^T\sum_{p=1}^{\infty}\langle\eta_s^n,d(\cdot,V^n(s)\psi_p+L(V^n(s))\psi_p\rangle^2ds \qquad (3.10)\\
&+nE\int_0^T\sum_{p=1}^{\infty}\langle V(s),d(\cdot,V^n(s))\psi_p+L(V^n(s))\psi_p\\
&-d(\cdot,V(s))\psi_p-L(V(s))\psi_p\rangle^2ds \qquad (3.11)\\
&+\sum_{p=1}^{\infty}E\sup_{t\le T}M_t^n(\psi_p)^2\big). \qquad (3.12)
\end{aligned}
$$

To estimate (3.10), let $H_s^n(\psi)=\langle\eta_s^n,L(V^n(s)\psi\rangle$ for any function $\psi\in W_0^{2+2D,D}$. Then by Lemma 3.2, uniformly in ω

$$
|\langle\eta_s^n,L(V^n(s))\psi\rangle|\le K\|\eta_s^n\|_{-(1+D),2D}\|\psi\|_{2+2D,D}. \qquad (3.13)
$$

Hence by Parseval's inequality,

(3.14)

$$\begin{aligned}\sup_n \sum_{p=1}^{\infty} \langle \eta_s^n, L(V^n(s))\psi_p\rangle^2 = & \sup_n \|H_s^n\|^2_{-2(1+D),D} \\ & \leq K^2 \sup_n \|\eta_s^n\|^2_{-(1+D),2D} < \infty.\end{aligned}$$

Similarly by Condition (A1),

$$E \sup_n \sum_{p=1}^{\infty} \langle \eta_s^n, d(\cdot, V^n(s))\psi_p\rangle^2 \leq K^2 E \sup_n \|\eta_s^n\|^2_{-(1+D),2D} < \infty. \tag{3.15}$$

Hence

$$(3.10) \leq 2K^2 E \sup_n \|\eta_s^n\|^2_{-(1+D),2D}.$$

For (3.11), let for any function $\psi \in W_0^{2(1+D),D}$

$$\tilde{H}_s^n(\psi) = \langle V(s), d(\cdot, V^n(s))\psi - d(\cdot, V(s))\psi\rangle.$$

Then

$$\begin{aligned}|\tilde{H}_s^n(\psi)|^2 =& |\langle V(s), d(\cdot, V^n(s))\psi - d(\cdot, V(s))\psi\rangle|^2 \\ \leq & K^2 \|V(s)\|^2_{-(1+D),2D} \tilde{\rho}(V^n(s), V(s))^2 \|\psi\|^2_{(1+D),2D}.\end{aligned}$$

Therefore

$$\|\tilde{H}_s^n\|^2_{-(1+D),2D} \leq K^2 \|V(s)\|^2_{-(1+D),2D} \tilde{\rho}(V^n(s), V(s))^2$$

$$E \sup_{s \leq T} n \sum_{p=1}^{\infty} |\tilde{H}_s^n(\psi_p)|^2 = nE \sup_{s \leq T} \|\tilde{H}_s^n\|^2_{-2(1+D),D}$$

$$\begin{aligned}& E \sup_{s \leq T} n \|\tilde{H}^n\|^2_{-(1+D),2D} \\ & \leq nK^2 \cdot E \sup_{s \leq T} \|V(s)\|^2_{-(1+D),2D} E \sup_{s \leq T} \tilde{\rho}(V^n(s), V(s))^2 \\ & \leq c_3(T, m),\end{aligned} \tag{3.16}$$

where $c_3(T,m)$ is a constant. Similarly, by Corollary 2.5

$$(3.17)\qquad E\sup_{s\le T} n\sum_{p=1}^{\infty}\langle V(s), L(V^n(s))\psi_p - L(V(s))\psi_p\rangle^2 < c_4(T,m),$$

where $c_4(T,m)$ is a constant. By the Remark 3.0(2), we can deduce that (3.11) $<\infty$.

Finally,

$$\begin{aligned}\sup_n E(\sup_{t\le T}\sum_{p=1}^{\infty}|M_t^n(\psi_p)|^2) &= \sup_n E(\sup_{t\le T}\|M_t^n\|^2_{-2(1+D),D})\\ &\le \sup_n E(\sup_{t\le T}\|M_t^n\|^2_{-(1+D),2D}) < \infty,\end{aligned}$$

by the following Proposition 3.4. Therefore we get

$$\sup_n E(\sup_{t\le T}\|\eta_t^n\|^2_{-2(1+D),D}) < \infty.$$

□

PROPOSITION 3.4. *If (2.1) holds for any $p>1$ with the assumptions (A1)-(A4), the process M^n is a $W_0^{-(1+D),2D}$–valued martingale and it satisfies*

$$(3.18)\qquad \sup_n E(\sup_{t\le T}\|M_t^n\|^2_{-(1+D),2D}) < \infty.$$

Proof. Let

$$(3.19)\qquad M_t^n(\phi) \equiv \bar{M}_t^n(\phi) + \tilde{M}_{1,t}^n(\phi) + \tilde{M}_{2,t}^n(\phi),$$

for any test function, ϕ as (3.6). Let $\{\phi_p\}_{p=1}^{\infty}$ be a CONS in $W_0^{1+D,2D}$ of functions of class C^∞ with compact support on R^d. First,

$$\begin{aligned}&E\sum_{p=1}^{\infty}\bar{M}_T^n(\phi_p)^2\\ &= E\sum_{p=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}\int_0^T A_i^n(s)[\phi_p(X_i^n(s))\gamma(X_i^n(s),V^n(s))\\ &\quad+\nabla^T\phi_p(X_i^n(s))\sigma(X_i^n(s),V^n(s))]^2 ds\\ &\le KE\sum_{p=1}^{\infty}\int_0^T (A_1^n(s)^2(\phi_p^2(X_1^n(s))\gamma^2(X_1^n(s),V^n(s))\\ &\quad+\nabla^T\phi_p(X_1^n(s))^2\sigma^2(X_1^n(s),V^n(s)))ds\end{aligned}$$

$$\leq K_1(E\sum_{p=1}^{\infty}\int_0^T A_1^n(s)^2\phi_p^2(X_1^n(s)) + A_1^n(s)^2\nabla\phi_p(X_1^n(s))^2 ds)$$

by Condition A1

$$\leq K_1\cdot c(T,m)(E\sum_{p=1}^{\infty}\int_0^T \phi_p^2(X_1^n(s)) + \nabla\phi_p(X_1^n(s))^2 ds) \quad \text{by Lemma 2.2}$$

$$\leq K_2 E(\sup_{s\leq T}(1+|X_i^n(s)|^{4D}) \quad \text{by Lemma 2.8,}$$

where K_1 and K_2 are some constants. Hence by Lemma 2.2

$$\sup_n \sum_{p=1}^{\infty} E\tilde{M}_T^n(\phi_p)^2 < \infty.$$

For the second part of (3.19)

$$\begin{aligned}
&E\sum_{p=1}^{\infty}\tilde{M}_{1,t}^n(\phi_p)^2\\
&= E\sum_{p=1}^{\infty}\Big(\int_{U\times[0,t]}\langle\eta_s^n, \beta(\cdot, V^n(s), u)\phi_p\\
&\quad + \alpha(\cdot, V^n(s), u)\nabla^T\phi_p\rangle W(du, ds)\Big)^2\\
&= E\sum_{p=1}^{\infty}\int_{U\times[0,t]}\langle\eta_s^n, \beta(\cdot, V^n(s), u)\phi_p\\
&\quad + \alpha(\cdot, V^n(s), u)\nabla^T\phi_p\rangle^2\mu(du)ds\\
&\leq K^2 E\sum_{p=1}^{\infty}\int_{U\times[0,t]}\langle\eta_s^n, \phi_p\rangle^2 + \langle\eta_s^n, \nabla^T\phi_p\rangle^2 ds \quad \text{by Condition (A1)}\\
&\leq K^2 E\|\eta_t^n\|^2_{(1+D),2D}.
\end{aligned}$$

To show $E\sum_{p=1}^{\infty}\tilde{M}_{2,t}^n(\phi_p)^2 < \infty$,

$$\begin{aligned}
&E\tilde{M}_{2,t}^n(\phi_p)^2\\
&= nE\int_{U\times[0,t]}\Big(\langle V(s), \beta(\cdot, V^n(s), u)\phi_p - \beta(\cdot, V(s), u)\phi_p\rangle\\
&\quad + \langle V(s), \alpha^T(\cdot, V^n(s), u)\nabla\phi_p - \alpha^T(\cdot, V(s), u)\nabla\phi_p\rangle\Big)^2 duds.
\end{aligned}$$

$$
\begin{aligned}
&E\sum_{p=1}^{\infty}\int_{U\times[0,t]}\langle V(s),\beta(\cdot,V^n(s),u)\phi_p-\beta(\cdot,V(s),u)\phi_p\rangle^2 duds\\
&\le E\sum_{p=1}^{\infty}\int_0^t\langle V(s),\phi_p\rangle^2\Big(\int_{U\times[0,t]}(\beta(\cdot,V^n(s),u)-\beta(\cdot,V(s),u))^2du\Big)ds\\
&\le E\sum_{p=1}^{\infty}\int_0^t\langle V(s),\phi_p\rangle^2K^2\tilde{\rho}(V^n(s),V(s))^2ds\\
&\le (E\|V(t)\|^4_{-(1+D),2D})^{\frac{1}{2}}\frac{c_2(t,m)}{n},
\end{aligned}
$$

where the forth moments exists by the same way used in Proposition 2.1 [8]. Continuing a similar estimation shows that

$$
E\|\tilde{M}^n_{2,t}\|^2_{1+D,2D}=E\sum_{p=1}^{\infty}\tilde{M}^n_{2,t}(\phi_p)^2<\infty.
$$

□

THEOREM 3.5. *If (2.1) holds for any $p>1$ with the assumptions (A1)-(A4), the family $\{\eta^n_\cdot\}_{n=1}^{\infty}$, is relatively compact in the space*

$$
C_{W_0^{-2(1+D),2D}}[0,T],
$$

for any given $T>0$.

Proof. Let $H_0=W_0^{-(1+D),2D}$ and $H=W_0^{-2(1+D),D}$. Then Condition I of the criteria of tightness in Remark 3.0 is satisfied with H_0 and H. From the estimation of (3.14) and (3.15)-(3.17)

$$
\begin{aligned}
&E\|\eta^n_t-\eta^n_s\|^2_{-2(1+D),D)}\\
&\le c_5(T,m)|t-s|\sup_n\sup_{s\le T}\{\|\eta^n_s\|_{-(1+D),2D}+K_3\}\\
&\le c_6(T,m)|t-s|
\end{aligned}
$$

where K_3, $c_5(T,m)$ and $c_6(T,m)$ are some constants. Hence the second condition of the criteria of tightness in Remark 3.0 is satisfied. □

REMARK 3.6. We denote $\mathcal{S}'$ as the dual space of Schwartz space $\mathcal{S}$, which is the space of infinitely differentiable functions with bounded supports. Also we define

$$\partial f(\mu)\nu = \sqrt{n}(f(\mu + n^{-\frac{1}{2}}\nu) - f(\mu)),$$

for any $f : M_+(R) \to R$ and $\mu, \nu \in M_+(R)$. Let $M.$ be a $\mathcal{S}'$-valued martingale which is independent of $W(du, ds)$ and

$$\langle M(\phi)\rangle_t = \int V_1(s)[\phi\gamma(\cdot, V(s)) + \nabla\phi\sigma(\cdot, V(s))^2]ds, \quad \text{for } \phi \in \mathcal{S},$$

where $V_1(s) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n A_i^2(t)\delta_{X_i(t)}$. Let $F_1 : R_+ \to \mathcal{L}(\mathcal{S}', \mathcal{S}')$ and $F_2 : R_+ \times U \to \mathcal{L}(\mathcal{S}', \mathcal{S}')$ be given by

$$\begin{aligned}\langle \phi, F_1(s)\nu\rangle =& \langle \phi d(\cdot, V(s)) + L(V(s))\phi, \nu\rangle \\ &+ \langle \phi\nabla d(\cdot, V(s))\nu + \nabla(L(V(s))\phi)\nu, V(s)\rangle\end{aligned}$$

and

$$\begin{aligned}\langle \phi, F_2(s)\nu\rangle =& \langle \phi\beta(\cdot, V(s), u) + \nabla^T\alpha(\cdot, V(s), u), \nu\rangle \\ &+ \langle \phi\nabla\beta(\cdot, V(s), u)\nu + \nabla^T\phi\alpha(\cdot, V(s), u)\nu, V(s)\rangle.\end{aligned}$$

Kurtz and Xiong [8] have showed that any limit of $\{\eta^n_.\}$ satisfies the following SDE on $\mathcal{S}'$:

$$\eta_t = \eta_0 + M(t) + \int F_1(s)\eta_s ds + \int_{U\times[0,t]} F_2(s, u)\eta_s W(du, ds).$$

References

[1] T. Chiang, G. Kallianpur and P. Sundar, *Propagation of chaos and McKean-Vlasov equation in duals of nuclear spaces*, Appl. Math. Optim. **24** (1991), 55–83.

[2] S. Ethier and T. Kurtz, *Markov processes: Characterization and convergence*, Wiley, 1986.

[3] C. Graham, *Nonlinear Ito-Skorohod equations and martingale problem with discrete jump sets*, Stochastic Process. Appl. **40** (1992), 69–82.

[4] A. Joffe and M. Metivier, *Weak convergence of sequences of semimartingales with application to multitype branching processes*, Adv. in Appl. Probab. **18** (1986), 20–65.

[5] G. Kallianpur and J. Xiong, *Asymptotic behavior of a system of interacting nuclear-space-valued stochastic differential equation driven by Poisson random measures*, Appl. Math. Optim. **30** (1994), 175–201.

[6] T. Kurtz and J. Xiong, *Particle representation for a class of nonlinear SPDEs*, Stochastic Process. Appl. **83** (1999), 103–126.

[7] ———, *Numerical solutions for a class of SPDEs with applications to filtering*, Stochastics in Finite and Infinite Dimension (2000), 233–258.

[8] ———, *A stochastic evolution equation arising from the fluctuation of a class of interesting particle systems*, to be submitted.

[9] H. McKean, *Propogation of chaos for a class of non-linear parabolic equations*, Lecture series in Differential Equations **7** (1967), 41–57.

[10] S. Meleard, *Asymptotic behavior of some interacting particle systems; McKean-Vlasov and Boltzman models*, Lecture notes in math. **1627** (1995), 42–95.

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