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# ALMOST KÄHLER METRICS WITH NON-POSITIVE SCALAR CURVATURE WHICH ARE EUCLIDEAN AWAY FROM A COMPACT SET

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ABSTRACT. On $\mathbb{R}^{2n}$, $n \geq 2$, with the standard symplectic structure we construct compatible almost Kähler metrics with negative scalar curvature on a polydisc which are Euclidean away from the polydisc.

## 1. Introduction

An almost Kähler structure on a smooth manifold $M$ is a triple $(g, \omega, J)$ where $\omega$ is a symplectic form, $J$ an almost complex structure, and $g$ is an $\omega$-compatible Riemannian metric, i.e. $\omega(x, y) = g(Jx, y)$ for tangent vectors $x, y$ to $M$. A manifold with an almost Kähler structure is called an almost Kähler manifold.

With Gromov's theory of pseudoholomorphic curves [3] and Taubes' recent works on Seiberg-Witten invariants [6], the study of almost Kähler manifolds is becoming a more interesting subject than ever. However, it suffered for a long time from a lack of effective approaches with concrete interesting examples.

In Riemannian geometry one of the interesting problems associated to a curvature, for example the scalar, the Ricci or the sectional curvature, is whether there exists a metric on a Euclidean space $\mathbb{R}^m$ which has the curvature negative on a ball and is Euclidean away from it. The nature of this question concerns the flexibility of the curvature under consideration. More interestingly it is empirically related to the harder problem of whether given a manifold of dimension $\geq 3$ admits a metric with the curvature negative, as shown in Lohkamp's recent works [4, 5].

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Motivated by this, in this paper we ask the above question in the almost Kähler category as an approach to almost Kähler metrics from a Riemannian viewpoint.

In the general Riemannian case it is not hard to construct a metric on $\mathbb{R}^m$ which has the scalar curvature negative on a ball and is Euclidean outside the ball. Actually, in the afore-mentioned works Lohkamp proved a stronger result in Proposition 2.1 of [4] and the much harder case of Ricci curvature in Proposition 6.1 of [5]. In these works *conformal deformation* plays a technically essential role. But in a sharp contrast we cannot use conformal deformation in almost Kähler category because a metric conformal to an almost Kähler one is not almost Kähler in general. Therefore to deal with even the scalar curvature one should find a way distinct from the general Riemannian case.

To construct the desired metrics on the even-dimensional Euclidean space $\mathbb{R}^{2n}$, $n \geq 2$, we considered almost Kähler metrics with $n$-dimensional torus group symmetry. We managed to choose metrics which are Euclidean outside a polydisc, i.e. the product of 2-dimensional discs, and have the scalar curvature non-positive inside the polydisc. But the scalar curvature is zero precisely along some thin subset of the polydisc. Thus an elaborating argument was needed to perturb the metrics near this thin subset to get the scalar curvature negative everywhere inside the polydisc. In sum we proved;

THEOREM 1. *On $\mathbb{R}^{2n}$, $n \geq 2$, with the standard symplectic structure there exist compatible almost Kähler metrics which have the scalar curvature negative on a polydisc and are Euclidean outside of it. Furthermore these metrics can be chosen to be invariant under the $n$-dimensional torus group.*

The paper is organized as follows. In section 2 we provide preliminaries for almost Kähler metrics and scalar curvature. In section 3 we construct almost Kähler metrics on $\mathbb{R}^4$ which have non-positive scalar curvature and are Euclidean outside a polydisc. But these metrics have zero scalar curvature along a thin subset inside the polydisc. In section 4 we deform the metrics of section 3 near this thin subset to have the scalar curvature negative inside the polydisc. In section 5 we explain how to get similar metrics in higher dimensions.

## 2. Preliminaries

In this section we explain definitions and formulas which will be needed in later sections.

A symmetric $(2,0)$-tensor $h$ (from now on we simply write $(2,0)$-tensor as 2-tensor) on an almost Kähler manifold can be decomposed as $h = h^+ + h^-$, where $h^+$ and $h^-$ are symmetric 2-tensors defined by $h^+(x,y) = \frac{1}{2}\{h(x,y)+h(Jx,Jy)\}$ and $h^-(x,y) = \frac{1}{2}\{h(x,y)-h(Jx,Jy)\}$ for two vectors $x$ and $y$ tangent to $M$. A symmetric 2-tensor $h$ is called *J-invariant* or *J-anti-invariant* if $h = h^+$ or $h = h^-$ respectively.

Given a symplectic form $\omega$ on a smooth manifold $M$, we denote by $\Omega_\omega$ the set of all $\omega$-compatible Riemannian metrics. It is well known that $\Omega_\omega$ is naturally an infinite dimensional Fréchet manifold. According to Blair [2], for a smooth curve $g_t$ in $\Omega_\omega$ with the corresponding curve $J_t$ of almost complex structures, $h = \frac{dg_t}{dt}|_{t=0}$ is $J_0$-anti-invariant. Conversely, for $g$ in $\Omega_\omega$ with the corresponding $J$, any $J$-anti-invariant symmetric 2-tensor $h$ is tangent to a smooth curve in $\Omega_\omega$. More precisely, $ge^{ht}$ is such a smooth curve in $\Omega_\omega$, where $ge^{ht}$ is defined by $ge^{ht}(x,y) = g(x, e^{ht}y)$. Here $h$ and so $e^h$ is understood as a $(1,1)$-tensor lifted with respect to $g$. So $ge^{ht}(x,y) = g(x,y) + g(x, \sum_{k=1}^{\infty} \frac{h^k}{k!} y)$.

We denote by $\nabla$, $R$, $r$ and $s$ the Levi-Civita connection, the Riemannain curvature tensor, the Ricci tensor and the scalar curvature of a Riemannian manifold $(M,g)$. For tangent vector fields $X, Y, Z, W$, the Riemannian curvature tensor $R$ is defined by $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$.

For a local tangent frame $\{e_i\}_{i=1,2,\dots,m}$ of $M$, $m = \dim M$, we shall adopt the usual notational convention: $R_{ijkl} = g(R(e_i,e_j)e_k,e_l)$ and $r_{ij} = r(e_i,e_j)$.

In a Riemannian manifold $(M,g)$ the differential at $g$ , in the direction of a symmetric 2-tensor $h$, of its scalar curvature $s$ is given in [1], page 63, by

$$s_g'(h) = \Delta_g(tr_g h) + \delta_g(\delta_g h) - g(r_g, h),$$

where $r_g$ is the Ricci curvature tensor of $g$, $\Delta_g$ is the Laplacian operator, $tr_g h$ is the trace of $h$ with respect to $g$, $\delta_g h$ is the divergence of $h$ which can be written in local coordinates as $(\delta h)_\lambda = -\nabla^\nu h_{\nu\lambda}$ and finally $\delta_g(\cdot)$ on a 1-form is the formal adjoint of the (usual) differential on functions.

## 3. Almost-Kähler metrics on $\mathbb{R}^4$ with non-positive scalar curvature

In this section we shall find almost Kähler metrics on $\mathbb{R}^4$ which have non-positive scalar curvature and are Euclidean outside a poydisc. But these metrics have zero scalar curvature along a thin subset inside the polydisc.

We consider a metric on $\mathbb{R}^4$ of the form

$$g_0 = f^2 dr^2 + \frac{r^2}{f^2} d\theta^2 + h^2 d\rho^2 + \frac{\rho^2}{h^2} d\sigma^2, \tag{3.1}$$

where $(r,\theta), (\rho,\sigma)$ are the polar coordinates for each summand of $\mathbb{R}^4 := \mathbb{R}^2 \times \mathbb{R}^2$ respectively and $f$, $h$ are smooth positive functions on $\mathbb{R}^4$, which are functions of $r$ and $\rho$ only. Let $e_1 = \frac{1}{f}\frac{\partial}{\partial r}$, $e_2 = \frac{f}{r}\frac{\partial}{\partial \theta}$, $e_3 = \frac{1}{h}\frac{\partial}{\partial \rho}$, $e_4 = \frac{h}{\rho}\frac{\partial}{\partial \sigma}$. A smooth almost complex structure $J_0$ on $\mathbb{R}^4$ is defined by $J_0(e_1) = e_2, J_0(e_2) = -e_1, J_0(e_3) = e_4, J_0(e_4) = -e_3$. With the standard symplectic structure $\omega_0$ on $\mathbb{R}^4$ the triple $(g_0, \omega_0, J_0)$ is an almost-Kähler structure on $\mathbb{R}^4$. The Riemannian curvature components of $g_0$ of the form $R_{ijij}$ are computed as follows;

$$\begin{aligned}
R_{1212} &= -\frac{3f_r}{rf^3} + \frac{3f_r^2}{f^4} - \frac{f_{rr}}{f^3} - \frac{f_\rho^2}{f^2h^2},\\
R_{1313} &= \frac{f_{\rho\rho}}{fh^2} - \frac{f_\rho h_\rho}{fh^3} + \frac{h_{rr}}{f^2h} - \frac{f_r h_r}{f^3 h},\\
R_{1414} &= -\frac{h_{rr}}{hf^2} + \frac{h_r f_r h + 2fh_r^2}{f^3h^2} + \frac{f_\rho}{\rho f h^2} - \frac{f_\rho h_\rho}{fh^3},\\
R_{2424} &= -\frac{h_r}{rf^2h} + \frac{h_r f_r}{f^3 h} - \frac{f_\rho}{\rho f h^2} + \frac{f_\rho h_\rho}{fh^3},\\
R_{3434} &= -\frac{3h_\rho}{\rho h^3} - \frac{h_{\rho\rho}}{h^3} + \frac{3h_\rho^2}{h^4} - \frac{h_r^2}{f^2h^2},\\
R_{2323} &= -\frac{f_{\rho\rho}}{fh^2} + \frac{f_\rho(2f_\rho h + fh_\rho)}{f^2h^3} + \frac{h_r}{rf^2h} - \frac{f_r h_r}{f^3 h},
\end{aligned}$$

where $f_r = \frac{\partial f}{\partial r}, f_{rr} = \frac{\partial^2 f}{\partial r \partial r}$, etc.. The scalar curvature is

$$\begin{aligned}
s_{g_0} &= 2(R_{2112} + R_{3113} + R_{4114} + R_{3223} + R_{4224} + R_{4334})\\
&= 2\Big(\frac{f_{rr}}{f^3} + \frac{3f_r}{rf^3} - \frac{3f_r^2}{f^4} - \frac{f_\rho^2}{h^2f^2} + \frac{h_{\rho\rho}}{h^3} + \frac{3h_\rho}{\rho h^3} - \frac{3h_\rho^2}{h^4} - \frac{h_r^2}{h^2f^2}\Big).
\end{aligned}$$

The crucial observation in order to show $s_{g_0} \leq 0$ is that $s_{g_0}$ can be expressed as follows;

$$s_{g_0} = -\{(f^{-2})_{rr} + \frac{3}{r}(f^{-2})_r + (h^{-2})_{\rho\rho} + \frac{3}{\rho}(h^{-2})_\rho\} - \frac{2f_\rho^2}{h^2f^2} - \frac{2h_r^2}{h^2f^2}.$$

Setting $F = f^{-2}$ and $H = h^{-2}$, we shall find $F$ and $H$ which satisfy

$$F_{rr} + \frac{3}{r}F_r + H_{\rho\rho} + \frac{3}{\rho}H_\rho = 0. \tag{3.2}$$

Set $F_{rr} + \frac{3}{r}F_r = \alpha(r)\beta(\rho)$ where $\alpha$, $\beta$ are smooth functions on $\mathbb{R}$ which satisfy at least that

$$\begin{aligned} \alpha(r) &= 0 \quad \text{for } r \geq 1, \\ \beta(\rho) &= 0 \quad \text{for } \rho \geq 1. \end{aligned}$$

These two functions $\alpha$ and $\beta$ will be specified more below.

Since $(r^3 F_r)_r = r^3 F_{rr} + 3r^2 F_r = r^3\alpha(r)\beta(\rho)$, we do integration with proper boundary conditions on $F$ to get

$$F(r,\rho) = \beta(\rho)\int_0^r (\frac{1}{y^3}\int_0^y x^3\alpha(x)\,dx)\,dy + 1.$$

We now specify the function $\alpha$ as follows; first consider a smooth function $k(y)$ on $\mathbb{R}$ such that

$$\begin{cases} a)\ k(y) = 0 \quad \text{for} \quad y \leq 0, y \geq 1, \\ b)\ |k'(y)|_{C_0} \leq |y^3|_{C_0}, \\ c)\ \int_0^1 \frac{k(y)}{y^3}\,dy = 0, \\ d)\ 0 < \int_0^r \frac{k(y)}{y^3}\,dy < 1 \quad \text{for any } r \text{ with } 0 < r < 1 \end{cases} \tag{3.3}$$

and then define $\alpha(y) = \frac{k'(y)}{y^3}$. Here we may choose one such function $k$ so that the derivative $\alpha'$ of $\alpha$ is zero at exactly three points in the open interval $(0,1)$ as in Fig.1. Similarly we set

$$H(r,\rho) = -\alpha(r)\int_0^\rho (\frac{1}{y^3}\int_0^y x^3\beta(x)\,dx)\,dy + 1,$$

where $\beta(y) = \frac{\tilde{k}'(y)}{y^3}$ and the function $\tilde{k}(y)$ is a smooth function on $\mathbb{R}$ satisfying (3.3). We may choose $\tilde{k}$ similarly to $k$ (or equally if one prefers) so that the derivative $\beta'$ of $\beta$ is zero at three points in $(0,1)$. Hence the graph of $\beta$ is similar to that of $\alpha$.

Then the functions $F(r,\rho)$ and $H(r,\rho)$ satisfy the equation (3.2) and

$$\begin{aligned} F,\ H &\equiv 1 \quad \text{for} \quad r \geq 1 \quad \text{or} \quad \rho \geq 1, \\ F,\ H &> 0 \quad \text{for} \quad r < 1 \quad \text{and} \quad \rho < 1. \end{aligned}$$

For such functions $F$ and $H$, we have

$$s_{g_0} = -\frac{H}{2F^2}\beta'(\rho)^2\{\int_0^r (\frac{1}{y^3}\int_0^y x^3\alpha(x)\,dx)\,dy\}^2 - \frac{F}{2H^2}\alpha'(r)^2\{\int_0^\rho (\frac{1}{y^3}\int_0^y x^3\beta(x)\,dx)\,dy\}^2.$$

From the choices of $\alpha$ and $\beta$ , the derivatives $\alpha^{'}(r)$ and $\beta^{'}(\rho)$ are zero at $r_0 = 0, r_1, r_2, r_3$ and $\rho_0 = 0, \rho_1, \rho_2, \rho_3$, where $0 \leq r_i, \rho_j < 1$, respectively. It is now simple to check

$$s_{g_0}(r,\rho) = \begin{cases} 0 & \text{if } r \geq 1 \text{ or } \rho \geq 1, \\ 0 & \text{at } (r_i, \rho_j),\ i,j = 0,1,2,3, \\ \text{negative} & \text{if } \ r,\rho \in (0,1) \ \text{ with } r \neq r_i \text{ or } \rho \neq \rho_j. \end{cases}$$

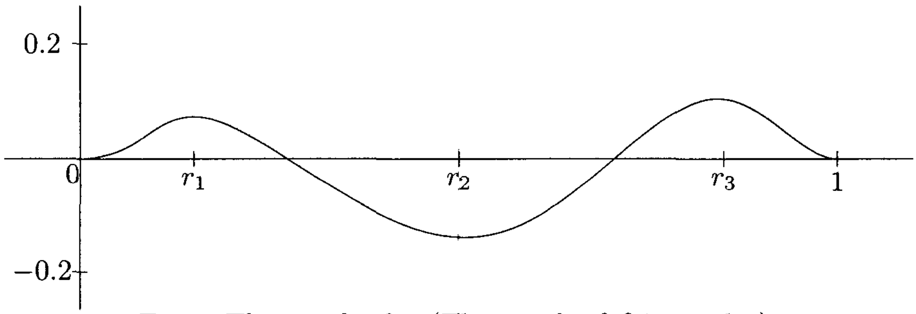


Fig.1. The graph of $\alpha$ (The graph of $\beta$ is similar).

Let $B = \{(r,\theta,\rho,\sigma)|0 \leq r,\rho < 1, 0 \leq \theta,\sigma < 2\pi\}$ which is a polydisc and let $D^{ij} = \{(r_i,\theta,\rho_j,\sigma)|0 \leq \theta,\sigma < 2\pi\}$ for each $i = 0,1,2,3$ and $j = 0,1,2,3$. The metric $g_0$ is Euclidean on $B^c$, the complement of $B$, and $s_{g_0}$ is negative on $B \setminus \cup_{i,j=0,1,2,3}D^{ij}$ but $s_{g_0} = 0$ on $\cup_{i,j=0,1,2,3}D^{ij}$. So to get a metric with the further property that $s_{g_0} < 0$ on $B$, it is natural to deform the metric $g_0$ near $\cup_{i,j=0,1,2,3}D^{ij}$. This will be done in section 4.

## 4. Almost-Kähler metrics with negative scalar curvature on a polydisc, which is Euclidean outside the polydisc

In this section we denote by $(g_0, \omega_0, J_0)$ the almost-Kähler structure constructed in section 3. Suppose that we have any symmetric 2-tensor $\eta$ which satisfies the following;

1) $\eta$ is $J_0$-anti-invariant,

2) $s'_{g_0}(\eta) = \Delta_{g_o}(tr_{g_0}\eta) + \delta_{g_0}(\delta_{g_0}\eta) - g_0(r_{g_0}, \eta) < 0$
on $D := \cup_{i,j=0,1,2,3} D^{ij}$,

3) the support of $\eta$ is contained in $B$.

Then the metric defined by $g_t = g_0 e^{\eta t}$ lies in $\Omega_{\omega_0}$ by condition 1) as explained in section 2. Since $s_{g_0}$ is negative on $B \setminus D$ and zero on $D$, by 2) and 3) we have $s_{g_t} < 0$ on $B$ for small $t$ by the Taylor series expansion argument for its scalar curvature; $s_{g_t} = s_{g_0} + s'_{g_0}(\eta)t + s''_{g_0}(\eta)\frac{t^2}{2} + \cdots$. Therefore by 3) we will have $\omega_0$-compatible almost-Kähler metrics $g_t$, which are Euclidean on $B^c$ with $s_{g_t} < 0$ on $B$ for small $t$. To find such an $\eta$, first we consider the expression of $\Delta(tr\eta) + \delta(\delta\eta)$ at a point $p \in D$ via a local coordinate system. Let $p$ be a point in a component $D^{ij}$ of $D$, $i, j = 1, 2, 3$, and choose a local coordinate $x = (x_1, x_2, x_3, x_4)$ near $p$ such that $x_1 = f(p)r$, $x_2 = \frac{r(p)}{f(p)}\theta$, $x_3 = h(p)\rho$, $x_4 = \frac{\rho(p)}{h(p)}\sigma$. Suppose that $\tilde{\eta}$ is a $J_0$-anti-invariant symmetric 2-tensor such that it is expressed in the coordinate $x$ by $\tilde{\eta} = \tilde{\eta}_{st} dx_s \otimes dx_t$ where $\tilde{\eta}_{st} = \tilde{\eta}(\frac{\partial}{\partial x_s}, \frac{\partial}{\partial x_t})$ are functions of $x_1$ and $x_3$ only. If we express $\tilde{\eta}$ as a matrix $\tilde{\eta} = (\tilde{\eta}_{st})$ , then $\tilde{\eta}$ satisfies the condition 1) if and only if it is of the following form ;

$$\tilde{\eta} = \begin{pmatrix} \tilde{\eta}_{11} & \tilde{\eta}_{12} & \tilde{\eta}_{13} & \tilde{\eta}_{14} \\ \tilde{\eta}_{12} & -a^{-2}\tilde{\eta}_{11} & a^{-1}b\tilde{\eta}_{14} & -a^{-1}b^{-1}\tilde{\eta}_{13} \\ \tilde{\eta}_{13} & a^{-1}b\tilde{\eta}_{14} & \tilde{\eta}_{33} & \tilde{\eta}_{34} \\ \tilde{\eta}_{14} & -a^{-1}b^{-1}\tilde{\eta}_{13} & \tilde{\eta}_{34} & -b^{-2}\tilde{\eta}_{33} \end{pmatrix}, \tag{4.1}$$

where $a = \frac{f^2}{f(p)^2}\frac{r(p)}{r}$ and $b = \frac{h^2}{h(p)^2}\frac{\rho(p)}{\rho}$. As $\tilde{\eta}$ is determined by six functions $\tilde{\eta}_{11}, \tilde{\eta}_{12}, \tilde{\eta}_{13}, \tilde{\eta}_{14}, \tilde{\eta}_{33}, \tilde{\eta}_{34}$, we shall only find these six $\tilde{\eta}_{st}$'s. Note that at $p$, $\tilde{\eta}_{st}$'s have the following relation:

$$\tilde{\eta}_{11}(p) = -\tilde{\eta}_{22}(p),\ \tilde{\eta}_{33}(p) = -\tilde{\eta}_{44}(p),\ \tilde{\eta}_{14}(p) = \tilde{\eta}_{23}(p),\ \tilde{\eta}_{13}(p) = -\tilde{\eta}_{24}(p).$$

We can see that at $p$

$$\Delta(tr\tilde{\eta})|_p + \delta(\delta\tilde{\eta})|_p = \tilde{\eta}_{11,11}(p) + \tilde{\eta}_{33,33}(p) + 2\tilde{\eta}_{13,13}(p) + L(p), \tag{4.2}$$

where $\tilde{\eta}_{st,kl} = \frac{\partial^2 \tilde{\eta}_{st}}{\partial x_k \partial x_l}$ and $L$ consists of lower order terms. If we choose $\tilde{\eta}_{11}(p) = -\tilde{\eta}_{22}(p)$ to be a nonzero number $c$ and all other $\tilde{\eta}_{st}(p) = 0$ then at $p$

$$\begin{aligned} g_0(r_{g_0},\tilde{\eta}) &= c(r_{11}-r_{22}) \\ &= c(R_{2112}+R_{3113}+R_{4114}-R_{1221}-R_{3223}-R_{4224}) \\ &= -2c\frac{f_{\rho\rho}}{fh^2} \quad (f_\rho = h_r = 0 \text{ at } p) \\ &= c\frac{H}{F}\beta''(\rho)\int_0^r(\frac{1}{y^3}\int_0^y x^3\alpha(x)\,dx)\,dy \\ &\neq 0 \quad (\text{Recall the graph of } \beta). \end{aligned}$$

We will take the value $c$ to be 1 or $-1$ so that $g_0(r_{g_0},\tilde{\eta}) > 0$. For these values $\tilde{\eta}_{11}(p) = -\tilde{\eta}_{22}(p) = \pm 1$, $\tilde{\eta}_{12}(p) = \tilde{\eta}_{13}(p) = \tilde{\eta}_{14}(p) = \tilde{\eta}_{34}(p) = \tilde{\eta}_{33}(p) = 0$ and any values $\frac{\partial \tilde{\eta}_{st}}{\partial x_k}(p)$, $\tilde{\eta}_{33,33}(p)$ and $\tilde{\eta}_{13,13}(p)$, we choose the value $\tilde{\eta}_{11,11}(p)$ so that (4.2) becomes zero. Then we can choose smooth functions $\tilde{\eta}_{st}(x_1,x_3)$ on $\mathbb{R}^4$, extending these values at $p$, with support in a small neighborhood $U \subset B$ of $p$. Note that we may and will choose $U$ to be a neighborhood of $D^{ij}$ because $\tilde{\eta}_{st}$'s are functions of $x_1$ and $x_3$ only.

Since $g_0(r_{g_0},\tilde{\eta})$ and the equation $\Delta(tr\tilde{\eta}) + \delta(\delta\tilde{\eta}) = 0$ are $S^1 \times S^1$-invariant, these functions $\tilde{\eta}_{st}$'s determine a $J_0$-anti-invariant symmetric 2-tensor $\tilde{\eta}$ with support contained in a neighborhood $U \subset B$ of $D^{ij}$ satisfying $\Delta(tr\tilde{\eta}) + \delta(\delta\tilde{\eta}) - g_0(r_{g_0},\tilde{\eta}) < 0$ along $D^{ij}$.

In the case that $p \in D^{ij}$ where $i = 0$ or $j = 0$, $\tilde{\tilde{\eta}} = cr^2dr^2 - \frac{cr^2}{f^4}d\theta^2 + c\rho^2 - \frac{c\rho^2}{h^4}d\sigma^2$ for some constant $c$ satisfies $\Delta(tr\tilde{\tilde{\eta}}) + \delta(\delta\tilde{\tilde{\eta}}) \to 4c$ as $r \to 0$ or $\rho \to 0$. Multiplying $\tilde{\tilde{\eta}}$ by a smooth cut off function with support in a small neighborhood of $D^{ij}$ and choosing an appropriate c we obtain a $J_0$-anti-invariant symmetric 2-tensor $\tilde{\eta}$ with support contained in a neighborhood of $D^{ij}$ satisfying $s'_{g_0}(\tilde{\eta}) < 0$ along $D^{ij}$.

So far for each component $D^{ij}$, $i,j = 0,1,2,3$ we have chosen a neighborhood $U^{ij} \subset B$ of $D^{ij}$ and a $J_0$-anti-invariant symmetric 2-tensor $\tilde{\eta}^{ij}$ on $\mathbb{R}^4$ such that

i) $\tilde{\eta}^{ij}$ satisfies $s'_{g_0}(\tilde{\eta}^{ij}) < 0$ along $D^{ij}$.

ii) The support of $\tilde{\eta}^{ij}$ is contained in $U^{ij}$.

Moreover $U^{ij}$'s could be chosen small so that

iii)$U^{ij}$'s are disjoint each other.

Then $\eta = \sum_{i,j=0}^{3}\tilde{\eta}^{ij}$ is the desired $J_0$-anti-invariant symmetric 2-tensor.

REMARK. In the above we have chosen $\tilde{\eta}_{11}(p) = -\tilde{\eta}_{22}(p) = \pm 1$ and all other $\tilde{\eta}_{st}(p) = 0$. But in fact, whatever values $\tilde{\eta}_{st}(p)$, $\frac{\partial \tilde{\eta}_{st}}{\partial x_k}(p)$, $\tilde{\eta}_{33,33}(p)$

and $\tilde{\eta}_{13,13}(p)$ we choose we can choose $\tilde{\eta}_{11,11}(p)$ so that $\Delta(tr\tilde{\eta})|_p + \delta(\delta\tilde{\eta})|_p - g_0(r_0,\tilde{\eta})|_p$ to be negative. Extending these values at $p$, we can also obtain a $J_0$-anti-invariant symmetric 2-tensor $\tilde{\eta}$ we want.

Since $f$ and $h$ are functions of $r$ and $\rho$ only and the $J_0$-anti-invariant symmetric 2-tensor $\eta$ is $S^1 \times S^1$-invariant, the metric $g_t = g_0 e^{\eta t}$ is also $S^1 \times S^1$-invariant. Therefore we proved the following

PROPOSITION 1. *Let $B^1 \subset \mathbb{R}^2$ be an open ball of radius 1 and $B = B^1 \times B^1 \subset \mathbb{R}^4$. With the standard symplectic structure on $\mathbb{R}^4$, there exists an $S^1 \times S^1$-invariant compatible almost-Kähler metric $g$ on $\mathbb{R}^4$ which has the scalar curvature negative on $B$ and are Euclidean on $B^c$, the complement of $B$.*

REMARK. It is not hard to show that the $S^1 \times S^1$-invariant metrics on $\mathbb{R}^4$ of Proposition 1 can also be chosen to have the form of (3.1). So it is interesting to find such a form of metrics instead of going through the perturbation of the section 4.

## 5. Generalization to $\mathbb{R}^{2n}, n \geq 3$

We generalize Proposition 1 to higher dimensions. The method is similar to 4-dimensional case.

Let $g_0$ be a metric on $\mathbb{R}^{2n}$, $n \geq 3$ , defined by

$$g_0 = \sum_{i=1}^{n}(f_i^2 dr_i^2 + \frac{r_i^2}{f_i^2} d\theta_i^2),$$

where $(r_i, \theta_i)$, $i = 1, \ldots, n$, are the polar coordinates on $\mathbb{R}^2$ and $f_i$'s are smooth positive functions on $\mathbb{R}^{2n}$ depending only on the variables $r_1, \ldots, r_n$. Let $e_{2i-1} = \frac{1}{f_i}\frac{\partial}{\partial r_i}$, $e_{2i} = \frac{f_i}{r_i}\frac{\partial}{\partial \theta_i}$ and $J_0$ be the almost complex structure defined by $J_0(e_{2i-1}) = e_{2i}, J_0(e_{2i}) = -e_{2i-1}$. Then with the standard symplectic form $\omega_0$ on $\mathbb{R}^{2n}$ the triple $(g_0, \omega_0, J_0)$ is an almost-Kähler structure on $\mathbb{R}^{2n}$. The scalar curvature of $g_0$ is

$$\begin{aligned} \frac{s_{g_0}}{2} &= \sum_{i=1}^{n}(\frac{f_{i,ii}}{f_i^3} + 3\frac{f_{i,i}}{r_i f_i^3} - 3\frac{f_{i,i}^2}{f_i^4}) - \sum_{i<j}\frac{f_{i,j}^2 + f_{j,i}^2}{f_i^2 f_j^2} \\ &= -\frac{1}{2}\sum_{i=1}^{n}\{(f_i^{-2})_{ii} + \frac{3}{r_i}(f_i^{-2})_i\} - \sum_{i<j}\frac{f_{i,j}^2 + f_{j,i}^2}{f_i^2 f_j^2}, \end{aligned}$$

where $f_{i,j} = \frac{\partial f_i}{\partial r_j}$, $f_{i,jk} = \frac{\partial^2 f_i}{\partial r_k \partial r_j}$. Set $F_i = f_i^{-2}$, $i = 1, \ldots, n$. We shall find the functions $F_i$ so that they satisfy

$$(5.1) \qquad \sum_{i=1}^{n} (F_{i,ii} + \frac{3}{r_i} F_{i,i}) = 0.$$

Set $F_{i,ii} + \frac{3}{r_i} F_{i,i} = \alpha_1^i(r_1) \cdots \alpha_n^i(r_n)$, $i = 1, \ldots, n-1$ and $F_{n,nn} + \frac{3}{r_n} F_{n,n} = -\sum_{i=1}^{n-1} \alpha_1^i(r_1) \cdots \alpha_n^i(r_n)$, where $\alpha_j^i(r_j)$, $i = 1, \ldots, n-1$, $j = 1, \ldots, n$ are smooth functions on $\mathbb{R}$ which satisfy at least

$$\alpha_j^i(r_j) = 0 \quad \text{for} \quad r_j \geq 1.$$

The functions $\alpha_j^i$'s need to be specified more. Let $k_j^i(t)$ be smooth functions on $\mathbb{R}$ satisfying the properties a), c), d) of (3.3) and $|\frac{dk_j^i}{dt}(t)|_{C_0} \leq \frac{1}{n - \sqrt[1]{n-1}} |t^3|_{C_0}$. Set $\alpha_j^i(t) = \frac{1}{t^3} \frac{dk_j^i}{dt}(t)$. Moreover we choose such functions $k_j^i$'s so that each derivative $\frac{d\alpha_j^i}{dt}$ of $\alpha_j^i$ is zero at exactly three points in the open interval $(0,1)$ and that the sets $A_j^i = \{t \in (0,1) | \alpha_j^i(t) = 0\}$ and $B_j^i = \{t \in (0,1) | \frac{d\alpha_j^i}{dt}(t) = 0\}$ are disjoint each other, i.e. $A_j^i \cap A_l^k = \phi$, $B_j^i \cap B_l^k = \phi$ for any pairs $(i,j) \neq (k,l)$ and $A_j^i \cap B_l^k = \phi$ for any pairs $(i,j)$, $(k,l)$.

Define the functions $F_i$, $i = 1, \ldots, n-1$, and $F_n$ by

$$F_i(r_1, \ldots, r_n) = \frac{\alpha_1^i(r_1) \cdots \alpha_n^i(r_n)}{\alpha_i^i(r_i)} \int_0^{r_i} (\frac{1}{y^3} \int_0^y x^3 \alpha_i^i(x)\, dx)\, dy + 1,$$

$$F_n(r_1, \ldots, r_n) = -\sum_{i=1}^{n-1} \alpha_1^i(r_1) \cdots \alpha_{n-1}^i(r_{n-1}) \int_0^{r_n} (\frac{1}{y^3} \int_0^y x^3 \alpha_n^i(x)\, dx)\, dy + 1.$$

Then the functions $F_i$, $i = 1, \ldots, n-1$, and $F_n$ satisfies the equation (5.1) and

$$F_i,\ F_n \equiv 1 \quad \text{if} \quad r_k \geq 1 \quad \text{for some } k,$$
$$F_i,\ H_n > 0 \quad \text{if} \quad r_k < 1 \quad \text{for all } k.$$

The scalar curvature becomes

$$s_{g_0}(r_1, \ldots, r_n) = -\frac{1}{2} \sum_{i<j} (\frac{F_j}{F_i^2} F_{i,j}^2 + \frac{F_i}{F_j^2} F_{j,i}^2),$$

which is zero if $r_k \geq 1$ for some $k$ and non-positive if $r_k \in (0,1)$ for all $k$. In order $s_{g_0}$ to be zero at $(r_1, \ldots, r_n)$ with all $r_k \in (0,1)$, $F^2_{i,j} = 0$ and $F^2_{j,i} = 0$ for all $i < j$. It is not difficult to check that this is impossible because we have chosen $\alpha^i_j$'s so that the sets $A^i_j$'s and $B^i_j$'s are disjoint each other. Therefore $s_{g_0}$ becomes zero only if one of $r_k$'s vanishes. At these points, by a similar argument in Section 4, we can perturb the metric $g_0$ so that its scalar curvature becomes negative.

Let $B = \{(r_1, \theta_1, \ldots, r_n, \theta_n) | 0 \leq r_k < 1, 0 \leq \theta_k < 2\pi, k = 1, \ldots, n\}$ be a polydisc and let $T^n = \overbrace{S^1 \times \cdots \times S^1}^{n-\text{times}} \subset \mathbb{R}^{2n}$ be the $n$-torus group. Our metric constructed above is $T^n$-invariant, Euclidean on $B^c$ and its scalar curvature is negative on $B$.

In summary, we proved

PROPOSITION 2. *Let $B^1 \subset \mathbb{R}^2$ be an open ball of radius 1 and $B \subset \mathbb{R}^{2n}$, $n \geq 3$, be the $n$-product of $B^1$. With the standard symplectic structure on $\mathbb{R}^{2n}$ there exists an $T^n$-invariant compatible almost-Kähler metric $g$ on $\mathbb{R}^{2n}$ which has the scalar curvature negative on $B$ and are Euclidean on $B^c$.*

*Proof of Theorem 1.* In Proposition 1 and 2 we proved the statement of Theorem 1 when the polydisc is standard, i.e. the product of the 2-dimensional discs of radius 1. But our construction of metrics just works for general polydiscs which are the products of discs of variable radii. This finishes the proof. □

In this article we have studied the scalar curvatures of almost Kähler metrics only on the Euclidean space with the standard symplectic structure. One might try to generalize this to an arbitrary almost Kähler manifold. An interesting question along this line is whether every closed symplectic manifold admits an almost Kähler metrics with negative scalar curvature.

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