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A SUBFOLIATION OF A CR-FOLIATION ON A LOCALLY CONFORMAL ALMOST KÄHLER MANIFOLD

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Abstract. The present paper treats with a subfoliation of a CR-foliation $\mathcal{F}$ on an almost Hermitian manifold M. When M is locally conformal almost Kähler, it has three CR-foliations. We show that a CR-foliation $\mathcal{F}$ on such manifold M admits a canonical subfoliation $\mathcal{D}_{\mathcal{F}}^{\perp}$ defined by its totally real subbundle. Furthermore, we investigate some cohomology classes for $\mathcal{D}_{\mathcal{F}}^{\perp}$. Finally, we construct a new one from an old locally conformal almost Kähler (in particular, an almost generalized Hopf) manifold.

1. Introduction

Locally conformal Kähler geometry has been discussed by many mathematicians since Vaisman ([7]). Most of the known examples of locally conformal Kähler manifolds turn out to be generalized Hopf manifolds, that is, locally conformal Kähler manifolds with parallel Lee form ([8]). Classical Hopf manifolds are typical examples of compact generalized Hopf manifolds which are not globally conformal Kähler.

The present paper has two sources. Chen and Piccinni ([2]) studied the canonical foliations of a generalized Hopf manifold and the canonical cohomology determined by a CR-submanifold in a locally conformal Kähler manifold. On the other hand, Kashiwada ([4]) recently introduced a notion of almost generalized Hopf manifolds, which becomes a generalized Hopf manifold when the given almost complex structure is integrable.

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The main purpose of the present paper is to extend the results obtained in [2] in the context of almost Hermitian geometry from the viewpoint of foliation. In section 2, we introduce the notion of a CR-foliation on an almost Hermitian manifold. In section 3, we consider a canonical subfoliation $\mathcal{D}_{\mathcal{F}}^{\perp}$ defined by the totally real subbundle for a CR-foliation $\mathcal{F}$ on a locally conformal almost Kähler manifold. Section 4 is discussed with some cohomology classes for $\mathcal{D}_{\mathcal{F}}^{\perp}$. In section 5, we construct a new one from an old locally conformal almost Kähler (in particular, an almost generalized Hopf) manifold.

2. A CR-foliation of an almost Hermitian manifold

Let (M, J, g) be an almost Hermitian manifold of dimension $2n$, and Ω be its fundamental form given by $\Omega(X,Y) := g(JX,Y)$. Given a foliation $\mathcal{F}$ on M, there is an orthogonal decomposition

$$TM = F \oplus F^{\perp}$$

with respect to g, where F denotes the subbundle of TM tangent to $\mathcal{F}$. Correspondingly, the metric g is decomposed into $g = g_F + g_{F^{\perp}}$. Recall that $\mathcal{F}$ is said to be bundle-like if $\overset{\circ}{\nabla}_V g_{F^{\perp}} = 0$ for all $V \in \Gamma(F)$, where $\overset{\circ}{\nabla}$ denotes the Bott connection ([2], [6]).

DEFINITION. $\mathcal{F}$ is a CR-foliation on an almost Hermitian manifold (M, J, g) if its tangent bundle F admits a subbundle D_F such that

(1) D_F is the maximal complex subbundle of F, i.e., $JD_F \subset D_F$,
(2) the orthogonal complementary subbundle $D_F^{\perp}$ to D_F is totally real, i.e., $JD_F^{\perp} \subset F^{\perp}$.

From the definition, we find that each leaf $\mathcal{L}$ of a CR-foliation $\mathcal{F}$ is a CR-submanifold of M. The maximal complex subbundle and the totally real subbundle for $\mathcal{L}$ are isomorphic to D_F and $D_F^{\perp}$ respectively.

An almost Hermitian manifold (M, J, g) is said to be locally conformal almost Kähler if for each $p \in M$, there exists a neighborhood U and a function σ on U such that $\tilde{g_U} := e^{-2\sigma} g|_U$ is an almost Kähler metric with respect to J on U. It is well-known ([7]) that a characterization of a locally conformal almost Kähler manifold is the existence of a Lee form α, that is, a 1-form α such that

$$d\alpha = 0, \quad d\Omega = 2\alpha \wedge \Omega. \tag{2.1}$$

Throughout the paper, the Lee form α is assumed to be nowhere vanishing on M. Then we have natural CR-foliations as follows.

PROPOSITION 2.1. *Let M be a locally conformal almost Kähler manifold with nowhere vanishing Lee form α. Then M admits the following CR-foliations*

(1) *a flow $\mathcal{E}^\perp$ generated the Lee vector field $A := \alpha^\#$,*
(2) *a codimension 1 foliation $\mathcal{E}$ given by $\alpha = 0$,*
(3) *on the leaves of $\mathcal{E}$ a flow $\mathcal{D}^\perp$ defined by the totally real subbundle.*

Proof. Since α is nowhere vanishing on M, the subbundle $E^\perp$ generated by the Lee vector field A defines a flow (1-dimensional foliation) $\mathcal{E}^\perp$. In addition, $d\alpha = 0$ implies that $\alpha = 0$ defines an integrable subbundle E on M, and hence a codimension 1 foliation $\mathcal{E}$. The leaves of $\mathcal{E}$ are hypersurfaces of M. Its totally real subbundle $D^\perp$ satisfies $E^\perp = JD^\perp$ and so it defines a flow $\mathcal{D}^\perp$ on M. □

REMARK. Proposition 2.1 is found in [2] for the case of a locally conformal Kähler manifold.

Kashiwada ([4]) introduced a weaker notion than a generalized Hopf manifold. Denote the Levi-Civita connection of M by ∇.

DEFINITION. A locally conformal almost Kähler manifold (M, J, g) is called an almost generalized Hopf manifold if the Lee form α is ∇-parallel and $\beta := -J \circ \alpha$ is Killing.

We may assume that the Lee form α of an almost generalized Hopf manifold is of unit length. In this case we have a further CR-foliation.

PROPOSITION 2.2. *Let M be an almost generalized Hopf manifold. Then the subbundle $E^\perp \oplus D^\perp$ on M is integrable. Moreover, $\mathcal{E}, \mathcal{E}^\perp, \mathcal{D}^\perp$ and $\mathcal{E}^\perp \oplus \mathcal{D}^\perp$ are all totally geodesic and bundle-like.*

Proof. Since α is parallel, both A and $B := \beta^\#$ are unit Killing on M. Hence the corresponding foliations $\mathcal{E}^\perp$ and $\mathcal{D}^\perp$ are totally geodesic and bundle-like (see [6], [5]). This implies that $\mathcal{F}$ is also totally geodesic and bundle-like. For the foliation $\mathcal{E}^\perp \oplus \mathcal{D}^\perp$ it is proved in [4]. Indeed, it holds $\nabla_A B = \nabla_B B = 0$. □

3. A canonical subfoliation of a CR-foliation

In view of Proposition 2.1, we observe that the corresponding totally real subbundles of three CR-foliations $\mathcal{E}, \mathcal{E}^{\perp}$ and $\mathcal{D}^{\perp}$ on a locally conformal almost Kähler manifold M are $D^{\perp}, E^{\perp}$ and $D^{\perp}$ respectively. They are all integrable. This observation can be extended as follows.

PROPOSITION 3.1. *Let $\mathcal{F}$ be a CR-foliation of an almost Hermitian manifold M. Then the totally real subbundle $D_F^{\perp}$ for $\mathcal{F}$ is integrable if and only if*

$$d\Omega(V, W, X) = 0 \quad V, W \in \Gamma(D_F^{\perp}), \ X \in \Gamma(D_F).$$

Proof. A direct computation yields for each $V, W \in \Gamma(D_F^{\perp})$ and $X \in \Gamma(D_F)$

$$\begin{aligned} 3d\Omega(V, W, X) &= V\Omega(W, X) + W\Omega(X, V) + X\Omega(V, W) \\ &\quad - \Omega([V, W], X) - \Omega([W, X], V) - \Omega([X, V], W) \\ &= g([V, W], JX). \end{aligned}$$

This completes the proof. □

COROLLARY 3.2. *Under the same situation as in Proposition 3.1, if, in particular, M is locally conformal almost Kähler, then $D_F^{\perp}$ is integrable, so it defines a foliation $\mathcal{D}_{\mathcal{F}}^{\perp}$. $\mathcal{D}_{\mathcal{F}}^{\perp}$ is called the canonical subfoliation for a CR-foliation $\mathcal{F}$.*

Proof. For $V, W \in \Gamma(D_F^{\perp})$ and for $X \in \Gamma(D_F)$,

$$\begin{aligned} \frac{1}{2}d\Omega(V, W, X) &= \alpha(V)\Omega(W, X) + \alpha(W)\Omega(X, V) + \alpha(X)\Omega(V, W) \\ &= 0 \end{aligned}$$

by means of (2.1). It follows from Proposition 3.1 that $D_F^{\perp}$ is integrable. □

From Proposition 2.2 we know that if M is an almost generalized Hopf manifold then all the subfoliations of four CR-foliations are totally geodesic and bundle-like. Moreover, the converse is true. Indeed, we have the following Proposition.

PROPOSITION 3.3. *Let M be a locally conformal almost Kähler manifold. Then the followings are equivalent.*

(1) *M is an almost generalized Hopf manifold,*
(2) *non-trivial subfoliations $\mathcal{E}^{\perp}$ and $\mathcal{D}^{\perp}$ are all totally geodesic and bundle-like.*

Proof. Since α is closed, it suffices to notice that A is Killing if and only if α is parallel. It follows that both A an B are unit Killing, or equivalently $\mathcal{E}^{\perp}$ and $\mathcal{D}^{\perp}$ are all totally geodesic and bundle-like if and only if M is an almost generalized Hopf manifold. □

4. Some cohomology classes for the canonical subfoliation

In section 3, we see that a CR-foliation $\mathcal{F}$ on a locally conformal almost Kähler manifold has a canonical subfoliation $\mathcal{D}^{\perp}_{\mathcal{F}}$ defined by its totally real subbundle. In this section we investigate some cohomology classes for $\mathcal{D}^{\perp}_{\mathcal{F}}$.

PROPOSITION 4.1. *Let $\mathcal{F}$ be a CR-foliation of a locally conformal almost Kähler manifold M. Then the mean curvature vector field κ of the maximal complex subbundle D_F for $\mathcal{F}$ satisfies*

$$g(\kappa, V) = -\alpha(V) \quad V \in \Gamma(D^{\perp}_{F}). \tag{4.1}$$

Proof. For each $X \in \Gamma(D_F)$ and $V \in \Gamma(D^{\perp}_{F})$, we compute

$$\begin{aligned} g(\nabla_X X, V) &= -g(JX, J\nabla_X V) \\ &= g(JX, \Pi_{JV} X) + g(JX, (\nabla_X J)V), \end{aligned} \tag{4.2}$$

where Π denotes the Weingarten map for $\mathcal{F}$. Similarly, we find

$$g(\nabla_{JX} JX, V) = -g(X, \Pi_{JV} JX) - g(X, (\nabla_{JX} J)V). \tag{4.3}$$

We may take the trace of the previous equations with respect to a local orthonormal frame field $\{X_i, JX_i\}_{i=1}^{b}$ for D_F ($2b := \dim D_F$) such that $H(X_i, X_j) = 0$, where

$$H(Y, Z) := (\nabla_Y J)Z + \beta(Z)Y - g(Y, Z)B + \alpha(Z)JY - \Omega(Y, Z)A.$$

Then (4.2) and (4.3) imply

$$\begin{aligned}2bg(\kappa, V) &= \sum g(\nabla_{X_i} X_i, V) + g(\nabla_{JX_i} JX_i, V)\\ &= \sum g((\nabla_{X_i} J)V, JX_i) - g((\nabla_{JX_i} J)V, X_i)\\ &= -2b\alpha(V),\end{aligned}$$

which yields (4.1). □

COROLLARY 4.2. *Under the same situation as in Proposition 4.1, D_F is minimal if and only if $D_F^{\perp} \subset E = \ker \alpha$.*

Given a foliation $\mathcal{F}$ on a manifold M, we have the basic complex $(\Lambda_B(M), d_B)$ as a subcomplex of the De Rham complex $(\Lambda(M), d)$ on M. Let $H_B(\mathcal{F}) := H(\Lambda_B(M), d_B)$ be the basic cohomology for $\mathcal{F}$ (see [6]).

THEOREM 4.3. *Let M be a $2n$-dimensional locally conformal almost Kähler manifold with nowhere vanishing Lee form α. Let $\mathcal{F}$ be a compact CR-foliation of M with $D_F^{\perp} \subset E$. Then the transversal volume form ν for $\mathcal{D}_{\mathcal{F}}^{\perp} \subset \mathcal{F}$ defines a basic cohomology class*

$$c(\mathcal{D}_{\mathcal{F}}^{\perp}) := [\nu] \in H_B^{2b}(\mathcal{D}_{\mathcal{F}}^{\perp}),$$

where b denotes the complex dimension of D_F. If $\mathcal{D}_{\mathcal{F}}^{\perp}$ is minimal and D_F is integrable, then we have

(1) $c(\mathcal{D}_{\mathcal{F}}^{\perp}) \neq 0$,

(2) *if, moreover, $\mathcal{F} \subset \mathcal{E}$ then*

$$H_B^{2k}(\mathcal{D}_{\mathcal{F}}^{\perp}) \neq 0 \quad 0 < k \leq b.$$

Proof. The transversal volume form ν for $\mathcal{D}_{\mathcal{F}}^{\perp}$ is given by

$$\nu := \omega_1 \wedge \cdots \wedge \omega_{2b},$$

where $\{\omega_i\}$ is the dual frame field of an orthonormal frame field $\{X_i, JX_i\}$ of D_F. Then Corollaries 3.2 and 4.2 imply that if $D_F^{\perp} \subset F$ then $d\nu = 0$ ([2], [6]). Thus $c(\mathcal{D}_{\mathcal{F}}^{\perp}) \in H_B^{2b}(\mathcal{D}_{\mathcal{F}}^{\perp})$.

Now note that the restriction Ω_F of Ω to F is a closed 2-form satisfying

$$\Omega_F^b = (-1)^b b! \nu. \tag{4.4}$$

Since $\mathcal{F} \subset \mathcal{E}$, we have that Ω_F is harmonic. The rest of the proof follows by a similar argument as in [2]. □

For example, $\mathcal{E}$ appeared in Proposition 2.1 satisfies $D_E^\perp \subset E$. In this case, $c(\mathcal{D}_{\mathcal{E}}^\perp)$ is explicitly represented by a $2(n-1)$-form $\frac{(-1)^{n-1}}{(n-1)!}(d\beta)^{n-1}$.

From now on we consider the Godbillon-Vey class for the canonical subfoliation $\mathcal{D}_{\mathcal{F}}^\perp$. We have seen in Proposition 2.2 that in an almost generalized Hopf manifold, any $\mathcal{D}_{\mathcal{F}}^\perp$ of four CR-foliations is bundle-like. Thus it has vanishing secondary characteristic classes ([3], [6]). In general $\mathcal{D}_{\mathcal{F}}^\perp$ is not bundle-like. However, we have the following Theorem.

THEOREM 4.4. *Let M be a locally conformal almost Kähler manifold and let $\mathcal{F}$ be a CR-foliation of M. Then the Godbillon-Vey class $GV(\mathcal{D}_{\mathcal{F}}^\perp)$ is given by the formula*

$$GV(\mathcal{D}_{\mathcal{F}}^\perp) = (2b)^{2b+1}[\alpha_F \wedge (d\alpha_F)^{2b}] = 0,$$

where $2b = \dim D_F$ and α_F is the restriction of the Lee form α to $\mathcal{F}$.

Proof. Recall the definition of the Godbillon-Vey class for $\mathcal{D}_{\mathcal{F}}^\perp$ given by

$$GV(\mathcal{D}_{\mathcal{F}}^\perp) := [\psi \wedge (d\psi)^{2b}], \tag{4.5}$$

where ψ is 1-form on $F \subset TM$ satisfying $d\nu = \psi \wedge \nu$. Now from (4.4) we have

$$\begin{aligned}(-1)^b b! d\nu = b d\Omega_F \wedge \Omega_F^{b-1} &= b2\alpha_F \wedge \Omega_F^b \\ &= b(-1)^b b! 2\alpha_F \wedge \nu.\end{aligned}$$

Therefore, we may choose $\psi = 2b\alpha_{\mathcal{F}}$ and hence from (4.5) we obtain the conclusion. □

5. Examples

The simplest example of almost generalized Hopf manifolds is the Riemannian product of K-contact manifolds and the real line $\mathbf{R}$. On the other hand, Proposition 2.2 provides an example of almost generalized Hopf manifolds in the almost Hermitian submersion (onto almost Kähler manifolds) context (see [10]).

In this section, we construct a new one from an old locally conformal almost Kähler (in particular, an almost generalized Hopf) manifold. Our results extend those in [9] to the almost version. We start with recalling several definitions.

DEFINITION. An almost contact metric manifold $(N, \varphi, \xi, \eta, h)$ is said to be *almost quasi-Sasakian* if its fundamental form Φ is closed. Moreover, when Φ is closed and η is integrable (namely, $\eta \wedge d\eta = 0$), it is said to be *special almost quasi-Sasakian.*

A locally conformal special almost quasi-Sasakian manifold is characterized by ([9])

$$d\Phi = 2\omega \wedge \Phi, \;\; \eta \wedge d\eta = 0, \tag{5.1}$$

where ω is a closed 1-form satisfying

$$[\varphi, \varphi] + d\eta \otimes \xi = (\omega \wedge \eta) \otimes \xi.$$

Here $[\varphi, \varphi]$ denotes the Nijenhuis tensor of φ.

PROPOSITION 5.1. *Let (M,J,g) be a locally conformal almost Kähler manifold and τ be a closed and nowhere vanishing Pfaff form on M. Then the leaves of foliation $\mathcal{F}_\tau$ defined by $\tau = 0$ carry an induced locally conformal almost quasi-Sasakian structure, which is special if $\tau \circ J$ is integrable.*

Proof. Let φ and h be the restriction of J and g to the subbundle F_τ given by $\tau = 0$ and $\xi := -J\zeta$, where ζ denotes the unit vector field normal to F_τ. Then the dual 1-form to ξ is $\eta = \frac{\tau \circ J}{|\tau|}$. It is easy to see that (φ, ξ, η, h) yields an almost contact metric structure on the leaves of the foliation $\mathcal{F}_\tau$.

The fundamental form Φ of φ satisfies

$$\Phi(X,Y) = \Omega(X,Y), \;\; d\Phi(X,Y,Z) = d\Omega(X,Y,Z)$$

for any $X, Y, Z \in \Gamma(F_\tau)$. It follows from (2.1) that

$$d\Phi = \bar{\alpha} \wedge \Phi,$$

where $\bar{\alpha}$ is the restriction of the Lee form α on M to F_τ. □

PROPOSITION 5.2. *Let $(N, \varphi, \xi, \eta, h)$ be a locally conformal special almost quasi-Sasakian manifold. Then $\eta = 0$ defines a foliation $\mathcal{F}_\eta$ whose leaves carry an induced locally conformal almost Kähler structure.*

Proof. It is proved in [1] that the leaves of $\mathcal{F}_\eta$ carry an induced almost Hermitian structure whose fundamental form Ω satisfies

$$\Omega(X,Y) = \Phi(X,Y), \ d\Omega(X,Y,Z) = d\Phi(X,Y,Z)$$

for any $X, Y, Z \in \Gamma(F_\eta)$. Hence (5.1) implies

$$d\Omega = 2\bar{\omega} \wedge \Omega,$$

where $\bar{\omega}$ is the restriction of ω to F_η. □

Combined Proposition 5.1 and Proposition 5.2, we conclude that

COROLLARY 5.3. *Let (M,J,g) be a locally conformal almost Kähler manifold (resp. an almost generalized Hopf manifold). Let τ be a closed and nowhere vanishing Pfaff form on M. If $\tau \circ J$ is integrable then the leaves of the foliation $\mathcal{F}_\eta$ carry an induced locally conformal almost Kähler structure (resp. an almost generalized Hopf structure).*

Now let $(N, \varphi, \xi, \eta, h)$ be a $(2n-1)$-dimensional almost Kenmotsu manifold, that is, an almost contact metric manifold such that

$$d\eta = 0, \ d\Phi = 2\eta \wedge \Phi. \tag{5.2}$$

It can be characterized as a locally conformal special almost quasi-Sasakian manifold with $\omega = \eta$ in (5.1). Consider the Riemannian product manifold $M := N \times \mathbf{R}$, which admits an almost Hermitian structure. Let T be the unit tangent vector field of $\mathbf{R}$ and ω be its dual form. Then the fundamental form Ω on M satisfies

$$\Omega = \Phi - \omega \wedge \eta.$$

It follows that

$$d\Omega = 2\eta \wedge \Omega,$$

so that M is a $2n$-dimensional locally conformal almost Kähler manifold whose Lee form is η. If we take the Pfaff form τ as η in Corollary 5.3 then $\tau \circ J = \omega$, which is parallel. In this case, the leaves of the foliation $\mathcal{F}_\eta$ carry an induced almost Kähler structure.

References

[1] D. E. Blair and G. D. Ludden, *Hypersurfaces in almost contact manifolds*, Tohoku Math. J. **21** (1969), 354–362.
[2] B. Y. Chen and P. Piccinni, *The canonical foliations of a locally conformal Kähler manifold*, Ann. Math. Pura Appl. **141** (1985), 289–305.
[3] F. W. Kamber and Ph. Tondeur, *G-foliations and their characteristic classes*, Bull. Amer. Math. Soc. **84** (1978), 1086–1124.
[4] T. Kashiwada, *On $\alpha\beta$-Einstein almost generalized Hopf manifolds*, Sci. Rep. Ochanomizu Univ. **46** (1995), 1–7.
[5] H. K. Pak, *On one-dimenaional metric foliations in Einstein spaces*, Illinois J. Math. **36** (1992), 594–599.
[6] Ph. Tondeur, *Foliations on Riemannian Manifolds*, Springer-Verlag, 1988.
[7] I. Vaisman, *On locally conformal almost Kähler manifolds*, Israel J. Math. **24** (1976), 338–351.
[8] ———, *Locally conformal Kähler manifolds with parallel Lee form*, Rend. Mat. Appl. **12** (1979), 263–284.
[9] ———, *Conformal changes of almost contact metric manifolds*, Lecture Notes in Math. 792, Berlin-Heidelberg-New York (1980), 435–443.
[10] B. Watson, *Almost Hermitian submersions*, J. Differential Geom. **11** (1976), 147–165.

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