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ON THE RESIDUAL FINITENESS OF FUNDAMENTAL GROUPS OF GRAPHS OF CERTAIN GROUPS

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Abstract. We give a characterization for fundamental groups of graphs of groups amalgamating cyclic edge subgroups to be cyclic subgroup separable if each pair of edge subgroups has a non-trivial intersection. We show that fundamental groups of graphs of abelian groups amalgamating cyclic edge subgroups are cyclic subgroup separable, hence residually finite, if each edge subgroup is isolated in its containing vertex group.

1. Introduction

Let Γ be a finite graph and let Y be a maximal tree of Γ. The fundamental group G of Γ of groups A_v, amalgamating cyclic edge subgroups, is obtained by first taking a tree product A of the A_v according to Y and then taking HNN extensions $G = \langle A, t_1, \ldots, t_n : t_i^{-1} h_i t_i = k_i, i = 1, \ldots, n\rangle$, where all the h_i and k_i are in the vertex groups A_v. Thus fundamental groups of graphs of groups are generalizations of amalgamated free products and HNN extensions of groups. Since the Baumslag-Solitar group $\langle a, t : t^{-1}a^2t = a^3\rangle$ is not residually finite [2], residual properties of HNN extensions or fundamental groups of graphs of groups are difficult to obtain. Recently, Raptis and Varsos [6, 9] considered residual nilpotence and subgroup separability of fundamental groups of graphs of groups when the edge subgroups are of finite index in the containing vertex groups.

A group G is *cyclic subgroup separable* (briefly, π_c) if, for each pair $g, x \in G$ such that $g \notin \langle x\rangle$, there exists a normal subgroup N of finite index in G (briefly, $N \lhd_f G$) such that $g \notin N\langle x\rangle$.

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In this paper, using a characterization (Theorem 2.3 below) for HNN extensions to be π_c, we consider the cyclic subgroup separability and residual finiteness of fundamental groups of graphs of groups when the edge subgroups are infinite cyclic.

We construct a fundamental group of a bouquet of two cycles which is not residually finite (Example 2.6) if each pair of edge subgroups has the trivial intersection. However, we give a characterization for fundamental groups of graphs of groups amalgamating cyclic edge subgroups to be π_c if each pair of edge subgroups has a non-trivial intersection (Corollary 3.2). On the other hand, we also show that fundamental groups of any graph of abelian groups amalgamating cyclic edge subgroups are cyclic subgroup separable, hence residually finite, if each edge subgroup is isolated in its containing vertex group (Theorem 3.7).

2. Preliminaries

Let $\Gamma = (V, E)$ be a graph, where V is a set of vertices and E is a set of edges. To each vertex v in V, we assign a group G_v. To each edge e in E, we assign a group G_e and monomorphisms α_e and β_e embedding G_e to the vertex groups at the end of e. Then, for a maximal tree T of Γ, the *fundamental group of the graph* (Γ) *of groups* G_v amalgamating the edge subgroups is defined to be the group generated by the generators and relations of the vertex groups and additional generators t_e for each $e \in E$ with the additional relations $t_e^{-1}(g_e\alpha_e)t_e = g_e\beta_e$ for each $g_e \in G_e$ where $t_e = 1$ if e is an edge of T. Each of $G_e\alpha_e$ and $G_e\beta_e$ is called an *edge subgroup* in its containing vertex group. It is well-known that the fundamental group of a graph of groups is independent from the choice of the maximal tree [7]. In particular, if the graph Γ is a tree then the fundamental group of Γ of groups A_v is called a *tree product* (see [4]) of the A_v.

Throughout this paper we consider fundamental groups of finite graphs of groups amalgamating infinite cyclic edge subgroups. If the graph Γ is not connected, then the fundamental group of Γ is a free product of fundamental groups of the connected subgraphs of Γ. Since free products of residually finite or π_c groups are again residually finite or π_c [1], we only consider the case that the graphs are connected.

We begin with tree products of groups.

THEOREM 2.1. *Tree products of finitely generated free-by-finite or polycyclic-by-finite groups amalgamating cyclic subgroups are π_c, hence residually finite.*

Proof. It was known by [8, 3] that finitely generated free-by-finite or polycyclic-by-finite groups are weak potent, equivalently regular quotients, and π_c. Thus the result follows from Theorem 3.4 in [3]. □

As we mentioned in the beginning of the introduction, a fundamental group of a graph of groups amalgamating cyclic edge subgroups is presented by

$$G = \langle A, t_1, \dots, t_n : t_i^{-1}h_it_i = k_i, i = 1, \dots, n\rangle$$

where A is a tree product for a maximal tree of the graph and where h_i, k_i are in the vertex groups A_v of A. Thus, we need to study the residual property of HNN extensions of groups. The following results in [5] are useful for this purpose.

DEFINITION 2.2. Let A be a group and let $h, k \in A$ be of infinite order. Then A is said to be *quasi-regular at* $\{h, k\}$ if, for each given integer $\epsilon > 0$, there exist an integer $\lambda_\epsilon > 0$ and $N_\epsilon \lhd_f A$, depending on ϵ, such that $N_\epsilon \cap \langle h\rangle = \langle h^{\epsilon\lambda_\epsilon}\rangle$ and $N_\epsilon \cap \langle k\rangle = \langle k^{\epsilon\lambda_\epsilon}\rangle$.

THEOREM 2.3. [5] *Let A be π_c and let $h, k \in A$ be of infinite order. Then the HNN extension $\langle A, t : t^{-1}ht = k\rangle$ is π_c if and only if A is quasi-regular at $\{h, k\}$.*

For a finitely generated abelian group A, it is not difficult to see that A is quasi-regular at $\{h, k\}$ if and only if $\langle h\rangle \cap \langle k\rangle = 1$ or $h^s = k^{\pm s}$ for some s. Using this, we have the followings:

THEOREM 2.4. [5] *Let A be a finitely generated abelian group. Let $h, k \in A$ be of infinite order. The HNN extension $G = \langle A, t : t^{-1}ht = k\rangle$ is π_c if and only if $\langle h\rangle \cap \langle k\rangle = 1$ or $h^s = k^{\pm s}$ for some $s > 0$.*

THEOREM 2.5. [5] *Let A be a finitely generated abelian group. Let $h, k \in A$ be of infinite order. The HNN extension $G = \langle A, t : t^{-1}ht = k\rangle$ is residually finite if and only if one of the followings holds:*
(i) *If $A = \langle b\rangle$, $h = b^\alpha$ and $k = b^\beta$, then $|\alpha| = 1$ or $|\beta| = 1$ or $\alpha \pm \beta = 0$.*
(ii) *If A is not cyclic then $\langle h\rangle \cap \langle k\rangle = 1$ or $h^s = k^{\pm s}$ for some $s > 0$.*

The above two results characterized the cyclic subgroup separability and residual finiteness of fundamental groups of a graph of an abelian group if the graph has a vertex and an edge. But if a graph has a vertex and two edges, a bouquet with two cycles, then as in the next example, the fundamental group may not be even residually finite.

EXAMPLE 2.6. Let $G = \langle A, t_1, t_2 : t_1^{-1}at_1 = b, t_2^{-1}a^2b^3t_2 = a^2b^6\rangle$, where $A = \langle a, b : [a, b] = 1\rangle$. Suppose G is residually finite. Let $g =$

$[t_2ab^3t_2^{-1}, b]$. Then $g \neq 1$. Since G is residually finite, there exists $N \lhd_f G$ such that $g \notin N$ and $a \notin N$. Let $|\overline{a}| = \alpha$ and $|\overline{a}\overline{b}^3| = \beta$ in $\overline{G} = G/N$. Since $a \sim_G b$, $|\overline{b}| = \alpha$. Hence $\beta \mid \alpha$. Since $a^2b^3 \sim_G a^2b^6$ and $|\overline{a}\overline{b}^3| = \beta$, we have $\overline{a}^{2\beta}\overline{b}^{6\beta} = 1 = \overline{a}^{2\beta}\overline{b}^{3\beta}$. Thus $\overline{b}^{3\beta} = 1$ and $\overline{a}^{2\beta} = 1$. Since $a \sim_G b$, $\overline{b}^{2\beta} = 1$. Hence $\overline{b}^{\beta} = 1$. Thus $\alpha \mid \beta$. Therefore $\alpha = \beta$. Now, since $\overline{g} = [\overline{t_2ab^3t_2^{-1}}, \overline{b}] \neq 1$, we have $\overline{a}\overline{b}^3 \notin \langle \overline{a}^2\overline{b}^6 \rangle = \langle (\overline{a}\overline{b}^3)^2 \rangle$. This implies $2 \mid \beta = |\overline{a}\overline{b}^3|$. Let $\beta = \alpha = 2\alpha_1$. Then $(\overline{a}^2\overline{b}^6)^{\alpha_1} = \overline{a}^{\alpha}\overline{b}^{3\alpha} = 1$. Hence $(\overline{a}^2\overline{b}^3)^{\alpha_1} = 1$, since $a^2b^3 \sim_G a^2b^6$. Then $\overline{a}^{2\alpha_1}\overline{b}^{3\alpha_1} = 1 = \overline{a}^{2\alpha_1}\overline{b}^{6\alpha_1}$. Thus $\overline{b}^{3\alpha_1} = 1 = \overline{a}^{2\alpha_1}$ and hence $\overline{b}^{2\alpha_1} = 1$. Therefore $\overline{b}^{\alpha_1} = 1$, hence $\alpha \mid \alpha_1$, a contradiction. Thus G is not residually finite.

We note that, in the above example, G is a fundamental group of a bouquet of two cycles and all pairs of edge subgroups have the trivial intersection and the edge subgroup $\langle a^2b^6 \rangle$ is not isolated in the vertex group A. Thus, in the next section, we consider fundamental groups of graph of groups if the graph has one closed cycle, or if each pair of two edge subgroups has nontrivial intersections, or if each edge subgroup is isolated in its containing vertex group.

3. Main results

THEOREM 3.1. *Let A be π_c and let $h_i, k_i \in A$ be of infinite order such that $\langle h_i \rangle \cap \langle k_i \rangle \neq 1$ for each $i = 1, \dots, n$. Then $G_n = \langle A, t_1, \dots, t_n : t_i^{-1}h_it_i = k_i, i = 1, \dots, n \rangle$ is π_c if and only if, for each $i = 1, \dots, n$, $h_i^{s_i} = k_i^{\pm s_i}$ for some $s_i > 0$.*

Proof. We first show that $G_1 = \langle A, t_1 : t_1^{-1}h_1t_1 = k_1 \rangle$ is π_c if and only if $h_1^{s_1} = k_1^{\pm s_1}$ for some $s_1 > 0$, whenever $\langle h_1 \rangle \cap \langle k_1 \rangle \neq 1$. If G_1 is π_c then, by Theorem 2.3, A is quasi-regular at $\{h_1, k_1\}$. Hence $h_1^{s_1} = k_1^{\pm s_1}$ for some $s_1 > 0$, since $\langle h_1 \rangle \cap \langle k_1 \rangle \neq 1$. Conversely, suppose $h_1^{s_1} = k_1^{\pm s_1}$ for some $s_1 > 0$. Let ϵ be a given integer. Since $|h_1|, |k_1|$ are infinite, $h_1^i, k_1^i \notin \langle h_1^{s_1\epsilon} \rangle = \langle k_1^{s_1\epsilon} \rangle$ for all $1 \leq i < s_1\epsilon$. Now A is π_c. This implies that there exists $N_1 \lhd_f A$ such that $h_1^i, k_1^i \notin N_1\langle h_1^{s_1\epsilon} \rangle$ for all $1 \leq i < s_1\epsilon$. Hence $N_1 \cap \langle h_1 \rangle = \langle h_1^{s_1\epsilon\delta_1} \rangle$ for some $\delta_1 > 0$ and $N_1 \cap \langle k_1 \rangle = \langle k_1^{s_1\epsilon\delta_2} \rangle$ for some $\delta_2 > 0$. Then $\langle h_1^{s_1\epsilon\delta_1} \rangle = N_1 \cap \langle h_1^{s_1} \rangle = N_1 \cap \langle k_1^{s_1} \rangle = \langle k_1^{s_1\epsilon\delta_2} \rangle$. Hence $\delta_1 = \delta_2$. Therefore, $N_1 \cap \langle h_1 \rangle = \langle h_1^{s_1\epsilon\delta_1} \rangle$ and $N_1 \cap \langle k_1 \rangle = \langle k_1^{s_1\epsilon\delta_1} \rangle$ for some $\delta_1 > 0$. Thus A is quasi-regular at $\{h_1, k_1\}$. Hence G_1 is π_c by Theorem 2.3.

Suppose G_n is π_c. Since $\langle A, t_i : t_i^{-1}h_it_i = k_i\rangle$ is a subgroup of G_n, it must be π_c. Hence, by above, $h_i^{s_i} = k_i^{\pm s_i}$ for some $s_i > 0$.

Conversely, suppose, for each $i = 1, \ldots, n$, $h_i^{s_i} = k_i^{\pm s_i}$ for some $s_i > 0$. Then, by above, G_1 is π_c. Inductively, we assume $G_{n-1} = \langle A, t_1, \ldots, t_{n-1} : t_i^{-1}h_it_i = k_i, i = 1, \ldots, n-1\rangle$ is π_c. Then, as above, G_{n-1} is quasi-regular at $\{h_n, k_n\}$. Hence G_n is π_c by Theorem 2.3. □

In Lemma 2.5 of [5], it was shown that G_1 above is π_c if and only if $h_1^{s_1} = k_1^{\pm s_1}$ for some $s_1 > 0$ with the additional condition that A is weak $\langle h_1\rangle$-potent.

COROLLARY 3.2. *Let A_v be finitely generated free-by-finite or polycyclic-by-finite groups. Let G be a fundamental group of a graph of the A_v, amalgamating cyclic edge subgroups, presented by $G = \langle A, t_1, \ldots, t_n : t_i^{-1}h_it_i = k_i, i = 1, \ldots, n\rangle$, where A is a tree product of the A_v according to a maximal tree of the graph and where $\langle h_i\rangle \cap \langle k_i\rangle \neq 1$ for each $i = 1, \ldots, n$. Then G is π_c if and only if, for each $i = 1, \ldots, n$, $h_i^{s_i} = k_i^{\pm s_i}$ for some $s_i > 0$.*

Proof. The result follows from Theorem 3.1 and Theorem 2.1. □

Note that, in the above corollary, h_i and k_i may not be in the same vertex group.

COROLLARY 3.3. *Let A be finitely generated free-by-finite or polycyclic-by-finite groups. Let $h_i, k_i \in A$ be of infinite order such that $h_i^{s_i} = k_i^{\pm s_i}$ for some $s_i > 0$. Then $G = \langle A, t_1, \ldots, t_n : t_i^{-1}h_it_i = k_i, i = 1, \ldots, n\rangle$ is π_c, hence, residually finite.*

If a graph Γ has one cycle, a loop or a closed path, then the fundamental group of Γ of groups amalgamating cyclic edge subgroups is presented by $G = \langle A, t : t^{-1}ht = k\rangle$, where A is a tree product of a maximal tree of Γ.

THEOREM 3.4. *Let A_i be polycyclic-by-finite groups. Let Γ be a graph with one cycle. Let G be a fundamental group of Γ of the A_i amalgamating cyclic edge subgroups, where each edge subgroup is contained in the center of its containing vertex group. If each pair $\langle h\rangle, \langle k\rangle$ of edge subgroups has the property $\langle h\rangle \cap \langle k\rangle = 1$ or $h^s = k^{\pm s}$ for some s, then G is π_c, hence residually finite.*

Proof. Let $G = \langle A, t : t^{-1}ht = k\rangle$, where A is a tree product of a maximal tree Y of Γ and where h, k are in the centers of their containing vertex groups of the tree product A. First of all, A is π_c by Theorem 2.1.

Hence, if $h^s = k^{\pm s}$ for some s, then G is π_c by Theorem 3.1. So, we suppose that $\langle h\rangle \cap \langle k\rangle = 1$. To apply Theorem 2.3, we shall show that A is quasi-regular at $\{h, k\}$. In fact, we can show that, for any α, β, there exists $N \lhd_f A$, depending on α, β, such that $N \cap \langle h\rangle = \langle h^\alpha\rangle$ and $N \cap \langle k\rangle = \langle k^\beta\rangle$. For this, we use an induction on the number of vertex groups of the tree product A.

If A has one vertex, then $h, k \in Z(A)$. Since $\overline{A} = A/\langle h^\alpha\rangle\langle k^\beta\rangle$ is again a polycyclic-by-finite group, $\overline{A}$ is residually finite. Hence there exists $\overline{N} \lhd_f \overline{A}$ such that $\overline{N} \cap \langle \overline{h}\rangle = 1$ and $\overline{N} \cap \langle \overline{k}\rangle = 1$. Let N be the preimage of $\overline{N}$ in A. Then $N \cap \langle h\rangle = \langle h^\alpha\rangle$ and $N \cap \langle k\rangle = \langle k^\beta\rangle$, as required.

Let $A = B *_{\langle c\rangle} A_v$, where v is an extreme vertex of the maximal tree Y and B is the tree product of the subtree $Y - \{v\}$. For an induction, we suppose if, for each pair $\langle a\rangle, \langle b\rangle$ of edge subgroups of B, $\langle a\rangle \cap \langle b\rangle = 1$ then there exists $M \lhd_f B$ such that $M \cap \langle a\rangle = \langle a^\alpha\rangle$ and $M \cap \langle b\rangle = \langle b^\beta\rangle$ for any α, β.

(i) Suppose $h, k \in B$ (or similarly $h, k \in A_v$). By induction hypothesis, there exists $M \lhd_f B$ such that $M \cap \langle h\rangle = \langle h^\alpha\rangle$ and $M \cap \langle k\rangle = \langle k^\beta\rangle$ for any α, β. Let $M \cap \langle c\rangle = \langle c^s\rangle$. Since $c \in Z(A_v)$, we have a homomorphic image $\overline{A} = B/M *_{\langle \overline{c}\rangle} A_v/\langle c^s\rangle$ of A. Since $\overline{A}$ is residually finite, as before, there exists $\overline{N} \lhd_f \overline{A}$ such that $\overline{N} \cap \langle \overline{h}\rangle = 1$ and $\overline{N} \cap \langle \overline{k}\rangle = 1$. Let N be the preimage of $\overline{N}$ in A. Then $N \cap \langle h\rangle = \langle h^\alpha\rangle$ and $N \cap \langle k\rangle = \langle k^\beta\rangle$, as required.

(ii) Suppose $h \in B$ and $k \in A_v$. Since $\langle h\rangle \cap \langle k\rangle = 1$, $\langle h\rangle \cap \langle c\rangle = 1$ or $\langle k\rangle \cap \langle c\rangle = 1$. Suppose $\langle h\rangle \cap \langle c\rangle = 1$. If $\langle k\rangle \cap \langle c\rangle \neq 1$, then let $\langle k^\beta\rangle \cap \langle c\rangle = \langle c^s\rangle$. Since $\langle h\rangle \cap \langle c\rangle = 1$, by induction hypothesis there exists $M \lhd_f B$ such that $M \cap \langle h\rangle = \langle h^\alpha\rangle$ and $M \cap \langle c\rangle = \langle c^s\rangle$. Thus we have a homomorphic image $\overline{A} = B/M *_{\langle \overline{c}\rangle} A_v/\langle k^\beta\rangle$ of A. On the other hand, if $\langle k\rangle \cap \langle c\rangle = 1$, then we have a homomorphic image $\overline{A} = B/M *_{\langle \overline{c}\rangle} A_v/\langle k^\beta\rangle\langle c^s\rangle$ of A. In both cases, since $\overline{A}$ is residually finite, there exists $\overline{N} \lhd_f \overline{A}$ such that $\overline{N} \cap \langle \overline{h}\rangle = 1$ and $\overline{N} \cap \langle \overline{k}\rangle = 1$. Let N be the preimage of $\overline{N}$ in A. Then $N \cap \langle h\rangle = \langle h^\alpha\rangle$ and $N \cap \langle k\rangle = \langle k^\beta\rangle$, as required.

Therefore A is quasi-regular at $\{h, k\}$, hence G is π_c by Theorem 2.3. □

COROLLARY 3.5. *Let Γ be a graph with one cycle. Let G be a fundamental group of Γ of finitely generated abelian groups A_i amalgamating cyclic edge subgroups. If each pair $\langle h\rangle, \langle k\rangle$ of edge subgroups has the* *property* $\langle h\rangle \cap \langle k\rangle = 1$ *or* $h^s = k^{\pm s}$ *for some* s, *then* G *is* π_c, *hence residually finite.*

We note that the conditions, $\langle h\rangle \cap \langle k\rangle = 1$ or $h^s = k^{\pm s}$ for some s, are best possible, since the Baumslag-Solitar group $\langle a, t : t^{-1}a^2t = a^3\rangle$ is not residually finite and the group $\langle a, t : t^{-1}at = a^2\rangle$ is residually finite by Theorem 2.5.

LEMMA 3.6. *Let* A *be* π_c *and let* $h_i, k_i \in A$ *be of infinite order. Suppose, for each* $\epsilon > 0$, *there exist* λ *and* $N_\epsilon \triangleleft_f A$ *such that* $N_\epsilon \cap \langle h_i\rangle = \langle h_i^{\lambda\epsilon}\rangle$ *and* $N_\epsilon \cap \langle k_i\rangle = \langle k_i^{\lambda\epsilon}\rangle$ *for each* i. *Then* $G_n = \langle A, t_1, \ldots, t_n : t_i^{-1}h_it_i = k_i, i = 1, \ldots, n\rangle$ *is* π_c.

Proof. Since A is quasi-regular at $\{h_1, k_1\}$, $G_1 = \langle A, t_1 : t_1^{-1}h_1t_1 = k_1\rangle$ is π_c by Theorem 2.3. Inductively, suppose

$$G_{n-1} = \langle A, t_1, \ldots, t_{n-1} : t_i^{-1}h_it_i = k_i, i = 1, \ldots, n-1\rangle$$

is π_c. Then, by assumption, for each $\epsilon > 0$ there exist λ and $N_\epsilon \triangleleft_f A$ such that, in $\overline{A} = A/N_\epsilon$, $|\overline{h}_i| = \epsilon\lambda = |\overline{k}_i|$ for each $1 \le i \le n$. Hence, there exists a homomorphism $\pi : G_{n-1} \to \overline{G}_{n-1}$, where $\overline{G}_{n-1} = \langle \overline{A}, t_1, \ldots, t_{n-1} : t_i^{-1}\overline{h}_it_i = \overline{k}_i, i = 1, \ldots, n-1\rangle$ and $\overline{A} = A/N_\epsilon$. Since $\overline{A}$ and $|\overline{h}_i| = |\overline{k}_i|$ are finite, $\overline{G}_{n-1}$ is π_c by [1]. Hence $\overline{G}_{n-1}$ is residually finite. Thus there exists $\overline{M} \triangleleft_f \overline{G}_{n-1}$ such that $\overline{M} \cap \langle \overline{h}_n\rangle = 1 = \overline{M} \cap \langle \overline{k}_n\rangle$. Let $M = \pi^{-1}(\overline{M})$. Then $M \triangleleft_f G_{n-1}$, $M \cap \langle h_n\rangle = \langle h_n^{\epsilon\lambda}\rangle$ and $M \cap \langle k_n\rangle = \langle k_n^{\epsilon\lambda}\rangle$. Thus G_{n-1} is quasi-regular at $\{h_n, k_n\}$. Hence G_n is π_c by Theorem 2.3. □

Recall that a subgroup H of G is said to be *isolated* in G if, for each $a \in G$ and $n \ne 0$, $a^n \in H$ implies $a \in H$.

THEOREM 3.7. *Let* G *be a fundamental group of any graph of finitely generated abelian groups* A_i *amalgamating isolated cyclic edge subgroups. Then* G *is* π_c, *hence, residually finite.*

Proof. Let $G = \langle A, t_1, \ldots, t_n : t_i^{-1}h_it_i = k_i, i = 1, \ldots, n\rangle$, where A is a tree product of the A_i according to a maximal tree of the graph and $\langle h_i\rangle, \langle k_i\rangle$ are isolated in their respective vertex groups. We note that if $\langle h\rangle$ is isolated in a finitely generated abelian group A_s then $A_s^\epsilon \cap \langle h\rangle = \langle h^\epsilon\rangle$ for each ϵ. Thus we have a homomorphism $\pi : A \to \overline{A}$, where $\overline{A}$ is a tree product of the $\overline{A}_s = A_s/A_s^\epsilon$, since all edge subgroups are isolated in their respective vertex groups. Then $|\overline{h}_i| = \epsilon = |\overline{k}_i|$ in $\overline{A}$. Since $\overline{A}$ is π_c by Theorem 2.1, hence residually finite, there exists $\overline{N} \triangleleft_f \overline{A}$ such that $\overline{N} \cap \langle \overline{h}_i\rangle = 1 = \overline{N} \cap \langle \overline{k}_i\rangle$ for each $i = 1, \ldots, n$. Let $N = \pi^{-1}(\overline{N})$. Then $N \triangleleft_f A$, $N \cap \langle h_i\rangle = \langle h_i^\epsilon\rangle$ and $N \cap \langle k_i\rangle = \langle k_i^\epsilon\rangle$ for each $i = 1, \ldots, n$. Since A is π_c, it follows from Lemma 3.6 that G is π_c. □

We note that, in Example 2.6, the edge subgroup $\langle a^2b^6\rangle$ is not isolated in the free abelian group $A = \langle a, b\rangle$, and the resulting group G is not residually finite.

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