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PACKING DIMENSION OF MEASURES ON A RANDOM CANTOR SET

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ABSTRACT. Packing dimension of a set is an upper bound for the packing dimensions of measures on the set. Recently the packing dimension of statistically self-similar Cantor set, which has uniform distributions for contraction ratios, was shown to be its Hausdorff dimension. We study the method to find an upper bound of packing dimensions and the upper Rényi dimensions of measures on a statistically quasi-self-similar Cantor set (its packing dimension is still unknown) which has non-uniform distributions of contraction ratios. As results, in some statistically quasi-self-similar Cantor set we show that every probability measure on it has its subset of full measure whose packing dimension is also its Hausdorff dimension almost surely and it has its subset of full measure whose packing dimension is also its Hausdorff dimension almost surely for almost all probability measure on it.

1. Introduction

Multifractal theory about random fractal made it possible to compare the Hausdorff dimension ([6], [8]) with packing dimension ([8], [13]) of statistically self-similar Cantor set ([6]). In fact, in 1994, Falconer and Olsen ([7], [11]) published their papers concerning the Hausdorff and packing dimensions of statistically self-similar set to show that the Hausdorff and packing dimensions of the support of measures are equal. It means that the statistically self-similar Cantor set is Taylor-regular with probability 1. In 1997, it ([1]) was shown that a perturbed type random Cantor set which is a statistically quasi-self-similar Cantor set in some sense has a similar result for Hausdorff dimension as the statistically self-similar Cantor set. A perturbed type random Cantor set as a variation of statistically self-similar set permits some trembling in each

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stage of distributions of contraction ratio random variables. The key idea to consider the perturbed type random Cantor set is to control the errors that can be allowed in the distribution to construct statistically self-similar Cantor set. Unlike the multifractal study to use a measure on the Cantor set, we use the local measure ([3]), which is not a measure, to investigate the packing dimension. We note that Falconer and Olsen ([7], [11]) used a covering function which is a pre-type of a measure to be applied to energy theory to find out its Hausdorff dimension, which can be called a global approach for the investigation of Cantor set in some sense, while using the local measure can be called a local approach for the study of Cantor sets in some sense. Anyway as an attempt to find the packing dimension of a statistically quasi-self-similar set or a perturbed type random Cantor set whose distributions of contraction ratios are not uniform, we study an upper bound for the packing dimensions of Borel probability measures defined on the set using the local approaches. We recall *a deranged Cantor set* ([2]). Let I_ϕ =[0,1]. We can obtain the left subinterval $I_{\tau,1}$ and the right subinterval $I_{\tau,2}$ of I_τ deleting middle open subinterval of I_τ inductively for each $\tau \in \{1,2\}^n$, where $n = 0, 1, 2, \ldots$. Consider $E_n = \cup_{\tau\in\{1,2\}^n} I_\tau$. Then $\{E_n\}$ is a decreasing sequence of closed sets. For each n, we put $\mid I_{\tau,1} \mid / \mid I_\tau \mid = c_{\tau,1}$ and $\mid I_{\tau,2} \mid / \mid I_\tau \mid = c_{\tau,2}$ for all $\tau \in \{1,2\}^n$, where $\mid I \mid$ denotes the diameter of I. We call $F = \bigcap_{n=0}^{\infty} E_n$ *a deranged Cantor set.* We note that if $x \in F$, then there is $\sigma \in \{1,2\}^{\mathbb{N}}$ such that $\bigcap_{k=0}^{\infty} I_{\sigma|k} = \{x\}$ (Here $\sigma|k = i_1, i_2, \ldots, i_k$ where $\sigma = i_1, i_2, \ldots, i_k, i_{k+1}, \ldots$). Hereafter, we use $\sigma \in \{1,2\}^{\mathbb{N}}$ and $x \in F$ as the same identity freely. Now we also recall the definition of *the local packing measure* ([3]) $q^s(\sigma)$ of σ in F, $q^s(\sigma) = \limsup_{k\to\infty}(c_1^s + c_2^s)(c_{\sigma|1,1}^s + c_{\sigma|1,2}^s)(c_{\sigma|2,1}^s + c_{\sigma|2,2}^s)\cdots(c_{\sigma|k,1}^s + c_{\sigma|k,2}^s)$ for $\sigma \in F$. Considering the above definition, we note that a deranged Cantor set is a perturbed Cantor set ([4]) in a local sense. In this paper we define a statistically quasi-self-similar Cantor set using the local approaches. We give some extra condition to the perturbed type random Cantor set to make a statistically quasi-self-similar Cantor set for a convenient local approach. Then we find that every probability measure μ defined on $\{1,2\}^{\mathbb{N}}$ (or the statistically quasi-self-similar Cantor set) whose Hausdorff dimension is s almost surely which is a real number from the expectation equation related to random contraction ratios has its own subset of full measure whose packing dimension is s also. Similarly we also find that almost every ω (or the statistically quasi-self-similar Cantor set) has its own subset of full measure whose packing dimension is s also for almost all probability measure μ.

2. Main results

Consider a probability space $(\Omega, \mathfrak{F}, P)$ (cf. [6]) such that the sample space

$$\Omega = \{(c_1, c_2, c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}, c_{1,1,1}, \dots, c_\tau, \dots) : 0 < a \le c_\tau \le b < \frac{1}{2}\},$$

where $\tau \in \{1,2\}^n, n \in \mathbb{N}$, and $\mathfrak{F} = \sigma(\mathfrak{F}_1, \mathfrak{F}_2, \dots, \mathfrak{F}_n, \dots)$, where

$$\mathfrak{F}_n = \sigma(\{(\alpha_1, \beta_1] \times \cdots \times (\alpha_{2+\cdots+2^n}, \beta_{2+\cdots+2^n}] \times [a,b] \times [a,b] \times \cdots : \\ a \le \alpha_i < \beta_i \le b\}).$$

For $\omega = (c_1, c_2, c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}, c_{1,1,1}, \dots, c_\tau, \dots)$, if we define $C_\tau(\omega) = c_\tau$, we easily see that C_τ is a random variable.

We note that if $\omega \in \Omega$ is given, there corresponds a deranged Cantor set $F(\omega)$, where $F(\omega)$ is determined by the sequence of contraction ratios $\omega = (c_1, c_2, c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}, c_{1,1,1}, \dots, c_\tau, \dots)$.

If P is a probability measure on $(\Omega, \mathfrak{F})$ such that $C_{j,1}$ and $C_{j,2}$ have the same distribution as C_1 and C_2 respectively where $j \in \{1,2\}^n$ and $n = 0, 1, 2, \dots$, and assume that C_j are independent random variables, except $C_{j,1}$ and $C_{j,2}$ for each j, we call $F(\omega)$ *a statistically self-similar Cantor set or self-similar random Cantor set.* In [7] and [11], it was shown that the Hausdorff dimension $\dim_H F(\omega)$ and the packing dimension $\dim_p F(\omega)$ of $F(\omega)$ are s where $E(C_1^s + C_2^s) = 1$ for P-almost all $\omega \in \Omega$.

From now on, we assume that C_j and $C_{j'}$ are independent random variables if $j \in \{1,2\}^n$ and $j' \in \{1,2\}^m$ with $n \ne m$ without any restriction on the distributions of contraction ratio random variables, which we call $F(\omega)$ *a random Cantor set.*

We get some result about the packing dimension $\dim_p F(\omega)$ for the random Cantor set $F(\omega)$ using the local packing measures q^s. From now on, we mean $X_k(\sigma, \omega)$ by

$$X_k(\sigma,\omega) = (C_1^s + C_2^s)(C_{\sigma|1,1}^s + C_{\sigma|1,2}^s)(C_{\sigma|2,1}^s + C_{\sigma|2,2}^s) \cdots (C_{\sigma|k,1}^s + C_{\sigma|k,2}^s)(\omega)$$

for some fixed s related to the expectation without confusion.

Lemma 1. *Let μ be a probability measure on $(\{1,2\}^{\mathbb{N}}, \mathfrak{G})$ where*

$$\mathfrak{G} = \sigma(\mathfrak{G}_1, \mathfrak{G}_2, \dots, \mathfrak{G}_n, \dots)$$

with $\mathfrak{G}_n = \sigma(\{\tau \times \{1,2\}^{\mathbb{N}} : \tau \in \{1,2\}^n\})$. For each $\sigma \in \{1,2\}^{\mathbb{N}}$ and each $s \in (0,1)$ in $X_k(\sigma,\omega)$, $X_k(\sigma,\omega)$ is $\mathfrak{G}_k \times \mathfrak{F}_{k+1}$-measurable.

Hence $\limsup_{k\to\infty} X_k(\sigma,\omega)$ is $\mathfrak{G} \times \mathfrak{F}$-measurable.

Proof. For $j \le k$ and l=1 or 2 , since

$$\begin{aligned}&\{(\sigma,\omega)\in\{1,2\}^{\mathbb{N}}\times\Omega : C^s_{\sigma|j,l}(\omega)<\beta\}\\ =&\bigcup_{\tau\in\{1,2\}^j}\tau\times\{1,2\}^{\mathbb{N}}\times\{\omega\in\Omega : C^s_{\tau,l}(\omega)<\beta\},\end{aligned}$$

$X_k(\sigma,\omega)$ is $\mathfrak{G}_k\times\mathfrak{F}_{k+1}$-measurable. □

PROPOSITION A ([3]). *Fix $\omega\in\Omega$ and $s\in(0,1)$ in $X_k(\sigma,\omega)$. Let E_ω be a Borel subset of the deranged Cantor set $F(\omega)$.*

If $E_\omega\subset\{\sigma\in F(\omega) : \limsup_{k\to\infty}X_k(\sigma,\omega)<\infty\}$, then $\dim_p E_\omega\le s$.

Proof. We define

$$\mu_s(I_\tau)=\frac{|I_\tau|^s}{(c_1^s+c_2^s)(c^s_{i_1,1}+c^s_{i_1,2})\cdots(c^s_{i_1,i_2,\ldots,i_{k-1},1}+c^s_{i_1,i_2,\ldots,i_{k-1},2})}$$

for each $\tau=i_1,i_2,\ldots,i_{k-1},i_k$, where $i_j\in\{1,2\}$. Then μ_s is extended to a Borel measure on $F(\omega)$ for each $s\in(0,1)$ and $\mu_s(F(\omega))=1$ (see [6], [8]).

Let $t>s$. Then clearly $q^t(\sigma)=0$. We note that given a small positive number r, there exists k such that $|I_{\sigma|k+1}|\le r<|I_{\sigma|k}|$ and $|I_{\sigma|k+1}|/|I_{\sigma|k}|=c_{\sigma|k+1}>a$ for all k.

Then for all $\sigma\in E_\omega$,

$$\begin{aligned}&\liminf_{r\to 0}\frac{\mu_t(B_r(\sigma))}{r^t}\\ &\ge\liminf_{k\to\infty}\frac{a^t}{(c_1^t+c_2^t)(c^t_{\sigma|1,1}+c^t_{\sigma|1,2})(c^t_{\sigma|2,1}+c^t_{\sigma|2,2})\cdots(c^t_{\sigma|k,1}+c^t_{\sigma|k,2})}\\ &=\infty.\end{aligned}$$

Thus the t-dimensional packing measure of E_ω is 0 by the packing density theorem [8]. Hence $\dim_p(E_\omega)\le s$. □

We say that

$$\sum_{n=1}^{\infty}\log E(C^s_{\sigma|n-1,1}+C^s_{\sigma|n-1,2})$$

converges so fast for each $\sigma\in\{1,2\}^{\mathbb{N}}$, if

$$\prod_{k=n}^{\infty}[E(C^s_{\sigma|k,1}+C^s_{\sigma|k,2})-1](2b^s)^n\to 0$$

as $n \to \infty$ for each $\sigma \in \{1,2\}^{\mathbb{N}}$. We call a random Cantor set $F(\omega)$ satisfying the above condition *a statistically quasi-self-similar Cantor set.*

We recall a martingale in the limit([10]) which is an adapted sequence $\{X_n\}$ satisfying

$$\lim_{m\to\infty} \sup_{n\geq m} |E(X_n|\mathfrak{F}_m) - X_m| = 0.$$

LEMMA 2. *Assume that for each $\sigma \in \{1,2\}^{\mathbb{N}}$, $\sum_{n=1}^{\infty} \log E(C^s_{\sigma|n-1,1} + C^s_{\sigma|n-1,2})$ converges so fast. Then for each $\sigma \in \{1,2\}^{\mathbb{N}}$, $\{X_n(\sigma,\omega)\}_{n=1}^{\infty}$ is a martingale in the limit.*

Proof. Fix $\sigma \in \{1,2\}^{\mathbb{N}}$. Then $\prod_{k=n}^{\infty}[E(C^s_{\sigma|k,1} + C^s_{\sigma|k,2}) - 1](2b^s)^n \to 0$ as $n \to \infty$. Given $0 < \epsilon < \frac{1}{4}$, there is N such that $\frac{-\epsilon}{(2b^s)^n} \leq \prod_{k=n}^{\infty}[E(C^s_{\sigma|k,1} + C^s_{\sigma|k,2}) - 1] \leq \frac{\epsilon}{(2b^s)^n}$ for all $n \geq N$. Since $2b^s \geq 1$, $\max\{\frac{2}{1+c_n}, \frac{2}{1-c_n}\} \leq 3$ for all n, where $0 < c_n = \frac{\epsilon}{(2b^s)^{n+1}} < \frac{1}{4}$. Then for all $n \geq m \geq N$,

$$\frac{1-c_{m+1}}{1+c_{m+1}} \leq \frac{1-c_{m+1}}{1+c_{n+1}} \leq \prod_{k=m+1}^{n} E(C^s_{\sigma|k,1} + C^s_{\sigma|k,2})$$
$$\leq \frac{1+c_{m+1}}{1-c_{n+1}} \leq \frac{1+c_{m+1}}{1-c_{m+1}}.$$

Since $\frac{1+c_{m+1}}{1-c_{m+1}} \leq 1 + 3c_{m+1}$ and $\frac{1-c_{m+1}}{1+c_{m+1}} \geq 1 - 3c_{m+1}$,

$$|\prod_{k=m+1}^{n} E(C^s_{\sigma|n,1} + C^s_{\sigma|n,2}) - 1| \leq \frac{3\epsilon}{(2b^s)^{m+1}}.$$

Since $X_m(\sigma,\omega) \leq (2b^s)^{m+1}$, for all $n \geq m \geq N$,

$$|E(X_n(\sigma,\omega)|\mathfrak{F}_m) - X_m(\sigma,\omega)| \leq 3\epsilon.$$

□

THEOREM 3. *Let μ be any probability measure on $\{1,2\}^{\mathbb{N}}$.*

If $\sum_{n=1}^{\infty} \log E(C^s_{\sigma|n-1,1} + C^s_{\sigma|n-1,2})$ converges so fast for each $\sigma \in \{1,2\}^{\mathbb{N}}$, then for P-almost every $\omega \in \Omega$, there is $F'(\omega) \subset F(\omega)$ such that $\dim_p(F'(\omega)) \leq s$ where $\mu(F'(\omega)) = 1$.

Proof. Consider a set

$$E = \{(\sigma,\omega) \in \{1,2\}^{\mathbb{N}} \times \Omega : \limsup_{k\to\infty} X_k(\sigma,\omega) < \infty\}.$$

Then E is $\mathfrak{G} \times \mathfrak{F}$-measurable by Lemma 1. For each $\sigma \in \{1,2\}^{\mathbb{N}}$, we have $\limsup_{k\to\infty} X_k(\sigma,\omega) < \infty$ for P-almost all $\omega \in \Omega$. For, if for each $\sigma \in \{1,2\}^{\mathbb{N}}$ we let $X_k(\sigma)(\omega) = X_k(\sigma,\omega)$, then $X_k(\sigma)$ is $\mathfrak{F}_{k+1}$-measurable. Now by Lemma 2, for each $\sigma \in \{1,2\}^{\mathbb{N}}$ the sequence $\{X_k(\sigma)\}$ forms an L^1-bounded martingale in the limit with respect to the increasing sequence $(\mathfrak{F}_n)$ of sub-σ-fields of $\mathfrak{F}$, hence converges ([10]) to some random variable $X(\sigma)$ for P-almost all ω. Since $E(X(\sigma)) \le \liminf E(X_k(\sigma)) < \infty$ by Fatou lemma, $X(\sigma) < \infty$ P-almost surely.

Now we apply the Fubini theorem. That is,

$$\int\int 1_E dP d\mu = \int\int 1_E d\mu dP$$

gives $1 = \int\int 1_E d\mu dP$, where 1_E is the characteristic function of E. Hence $\mu(E_\omega) = 1$ for P-almost every ω where $E_\omega = \{\sigma \in \{1,2\}^{\mathbb{N}} : (\sigma,\omega) \in E\} \in \mathfrak{G}$. For, suppose not: there is $A \in \mathfrak{F}$ with $P(A) > 0$ such that $\mu(E_\omega) < 1$ for $\omega \in A$. Then $A = \bigcup_{n=1}^{\infty} A_n$ where $A_n = \{\omega \in \Omega : \mu(E_\omega) < 1 - \frac{1}{n}\} \in \mathfrak{F}$, so there is some n such that $P(A_n) > 0$ since A_n is increasing to A if n is increasing. This gives $\int_A \mu(E_\omega)dP = \int_{A_n} \mu(E_\omega)dP + \int_{A\setminus A_n} \mu(E_\omega)dP \le (1-\frac{1}{n})P(A_n) + P(A\setminus A_n) < P(A)$. Hence $\int\int 1_E d\mu dP = \int_{A^c} \mu(E_\omega)dP + \int_A \mu(E_\omega)dP < P(A^c) + P(A) = 1$. A contradiction arises. Therefore if we put $F'(\omega) = E_\omega$ then the conclusion follows from Proposition A. □

REMARK. If we assume that for each $\sigma \in \{1,2\}^{\mathbb{N}}$,

$$\sum_{n=1}^{\infty} \log E(C^s_{\sigma|n-1,1} + C^s_{\sigma|n-1,2})$$

converges with an extra condition that

$$E(C^s_{\sigma|n-1,1} + C^s_{\sigma|n-1,2}) \ge 1$$

for all n, instead of the condition that for each $\sigma \in \{1,2\}^{\mathbb{N}}$,

$$\sum_{n=1}^{\infty} \log E(C^s_{\sigma|n-1,1} + C^s_{\sigma|n-1,2})$$

converges so fast, we also get the same results in this paper. For, if these are assumed, then $\{X_n(\sigma)\}$ forms an L^1-bounded submartingale for each $\sigma \in \{1,2\}^{\mathbb{N}}$ with respect to the increasing sequence $(\mathfrak{F}_n)$ of sub-σ-fields of $\mathfrak{F}$, hence converges to some random variable $X(\sigma)$ for P-almost all ω.

COROLLARY 4. *Let μ be any probability measure on $\{1,2\}^{\mathbb{N}}$.*

If $\sum_{n=1}^{\infty} \log E(C^s_{\sigma|n-1,1} + C^s_{\sigma|n-1,2})$ *converges so fast for each* $\sigma \in \{1,2\}^{\mathbb{N}}$ *and* $\dim_H F(\omega) \geq s$ *for* P*-almost all* $\omega \in \Omega$*, then for* P*-almost every* $\omega \in \Omega$ *there is* $F^*(\omega) \subset F(\omega)$ *such that* $\dim_p(F^*(\omega)) = s$ *where* $\mu(F^*(\omega)) = 1$.

Proof. Since for P-almost all $\omega \in \Omega$ $\dim_H(F(\omega)) \geq s$, $\dim_p(F(\omega)) \geq s$ for P-almost all $\omega \in \Omega$. We note that every Borel set in complete separable metric space has its subset whose packing dimension is less than or equal to its own packing dimension ([9], [12]). If we put $F^*(\omega) = F'(\omega) \cup E(\omega)$ where $F'(\omega)$ is in the proof of Theorem 3 and $E(\omega) \subset F(\omega)$ is the above subset satisfying $\dim_p(E(\omega)) = s$, for P-almost every $\omega \in \Omega$ there is $F^*(\omega) \subset F(\omega)$ such that $\dim_p(F^*(\omega)) = s$ with $\mu(F^*(\omega)) = 1$ where $F'(\omega) \subset F^*(\omega) \subset F(\omega)$. □

REMARK. If we loosen some conditions on self-similar random Cantor set, we get a generalized definition. If P is a probability measure on $(\Omega, \mathfrak{F})$ such that $C_{j,1}$ and $C_{j,2}$ have the same distribution as L_n and R_n respectively where $j \in \{1,2\}^{n-1}$ and $L_n = C_{i,1}$, $R_n = C_{i,2}$ and $i \in \{1\}^{n-1}$ for each $n = 1, 2, \ldots$, and assume that C_j are independent random variables, except $C_{j,1}$ and $C_{j,2}$ for each j, we call $F(\omega)$ a perturbed type random Cantor set ([1]).

In [1], it was shown that the Hausdorff dimension $\dim_H F(\omega)$ of $F(\omega)$ is s for P-almost all $\omega \in \Omega$ if $\sum_{n=1}^{\infty} \log E(L^s_n + R^s_n)$ converges.

We need an additional condition to the condition that $\sum_{n=1}^{\infty} \log E(L^s_n + R^s_n)$ converges in the perturbed type random Cantor set to guarantee the informations about packing dimension. We require a stronger condition that $\prod_{k=n}^{\infty}[E(L^s_k + R^s_k) - 1](2b^s)^n \to 0$ as $n \to \infty$, which also implies $\sum_{n=1}^{\infty} \log E(L^s_n + R^s_n)$ converges.

We say that $\sum_{n=1}^{\infty} \log E(L^s_n + R^s_n)$ *converges so fast* if $\prod_{k=n}^{\infty}[E(L^s_k + R^s_k) - 1](2b^s)^n \to 0$ as $n \to \infty$. If $\sum_{n=1}^{\infty} \log E(L^s_n + R^s_n)$ converges so fast, we easily see (cf. Lemma 2) that $\{X_k(\sigma,\omega)\}_{k=1}^{\infty}$ is a martingale in the limit ([10]) for each σ. That is, if a perturbed type random Cantor set satisfies the condition that $\sum_{n=1}^{\infty} \log E(L^s_n + R^s_n)$ converges so fast then the perturbed type random Cantor set is a statistically quasi-self-similar Cantor set.

COROLLARY 5. *Let* μ *be any probability measure on* $\{1,2\}^{\mathbb{N}}$.

If $\sum_{n=1}^{\infty} \log E(L^s_n + R^s_n)$ *converges so fast, then for* P*-almost every* $\omega \in \Omega$ *there is* $F^*(\omega) \subset F(\omega)$ *such that* $\dim_p(F^*(\omega)) = s$ *where* $\mu(F^*(\omega)) = 1$.

Proof. For P-almost all $\omega \in \Omega$ $\dim_H F(\omega) = s$([1] and above Remark), so by the above Corollary it follows. □

REMARK. We note that there is one to one correspondence between the probability measure μ supported on $\{1,2\}^{\mathbb{N}}$ and the sequence of the portion ratios $(p_\phi, p_1, p_2, p_{1,1}, p_{1,2}, p_{2,1}, p_{2,2}, p_{1,1,1}, \ldots, p_\tau, \ldots)$ where $p_\tau \in (0,1)$ and $\tau \in \{1,2\}^n$ where $n = 0, 1, 2, \ldots$ (here we define $\mu(\tau \times \{1\} \times \{1,2\}^{\mathbb{N}}) = p_\tau \mu(\tau \times \{1,2\}^{\mathbb{N}})$ and $\mu(\tau \times \{2\} \times \{1,2\}^{\mathbb{N}}) = (1 - p_\tau)\mu(\tau \times \{1,2\}^{\mathbb{N}}$)). Clearly there is one to one correspondence between the probability measure μ supported on $\{1,2\}^{\mathbb{N}}$ and the Borel probability measure $\mu(\omega)$ supported on $F(\omega)$ where $\mu(\omega)$ is a natural measure from μ. Therefore we can identify three of them without confusion henceforth. In particular, for each $\omega \in \Omega$ we will call the Borel probability measure $\mu(\omega)$ supported on $F(\omega)$ corresponding to the probability measure μ supported on $\{1,2\}^{\mathbb{N}}$ a Borel probability measure on $F(\omega)$ induced by μ. We also note that a probability measure μ on $(\{1,2\}^{\mathbb{N}}, \mathfrak{G})$ where

$$\mathfrak{G} = \sigma(\mathfrak{G}_1, \mathfrak{G}_2, \ldots, \mathfrak{G}_n, \ldots)$$

with $\mathfrak{G}_n = \sigma(\{\tau \times \{1,2\}^{\mathbb{N}} : \tau \in \{1,2\}^n\})$ is a Borel probability measure on the coding space $\{1,2\}^{\mathbb{N}}$ with an ultra metric.

LEMMA 6. *Let $(M, \mathfrak{M}, m)$ be any Borel probability measure space of probability measures supported on $\{1,2\}^{\mathbb{N}}$, where*

$$\mathfrak{M} = \sigma(\mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_n, \ldots)$$

with $\mathfrak{M}_n = \sigma(\{B \times (0,1)^N : B \in \mathfrak{B}((0,1)^{\frac{n(n+1)}{2}}\})$, where $\mathfrak{B}(S)$ is the Borel sets of S.

Then $E = \{(\mu, \omega) \in M \times \Omega : \mu\{\sigma : \limsup_{k \to \infty} X_k(\sigma, \omega) < \infty\} = 1\}$ is $\mathfrak{M} \times \mathfrak{F}$-measurable.

Proof. It is not difficult to show that

$$\begin{aligned} &\{(\mu, \omega) : \mu\{\sigma : \limsup_{k \to \infty} X_k(\sigma, \omega) < \infty\} = 1\} \\ &= \bigcap_{l=1}^{\infty} \bigcup_{\beta=1}^{\infty} \{(\mu, \omega) : \mu\{\sigma : \limsup_{k \to \infty} X_k(\sigma, \omega) < \beta\} \geq 1 - \frac{1}{l}\}. \end{aligned}$$

And we easily see that

$$\begin{aligned}
&\{(\mu,\omega) : \mu\{\sigma : \limsup_{k\to\infty} X_k(\sigma,\omega) < \beta\} \ge \gamma\} \\
&= \{(\mu,\omega) : \lim_{n\to\infty} \mu[\bigcap_{k\ge n}\{\sigma : X_k(\sigma,\omega) < \beta\}] \ge \gamma\} \\
&= \bigcap_{h=1}^{\infty}\bigcup_{n=1}^{\infty}\{(\mu,\omega) : \mu[\bigcap_{k\ge n}\{\sigma : X_k(\sigma,\omega) < \beta\}] \ge \gamma - \frac{1}{h}\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\{(\mu,\omega) : \mu\{\sigma : \limsup_{k\to\infty} X_k(\sigma,\omega) < \infty\} = 1\} \\
&= \bigcap_{l=1}^{\infty}\bigcup_{\beta=1}^{\infty}\bigcap_{h=1}^{\infty}\bigcup_{n=1}^{\infty}\{(\mu,\omega) : \mu[\bigcap_{k\ge n}\{\sigma : X_k(\sigma,\omega) < \beta\}] \ge 1 - \frac{1}{l} - \frac{1}{h}\}.
\end{aligned}$$

Noting that

$$\begin{aligned}
&\{(\mu,\omega) : \mu[\bigcap_{k\ge n}\{\sigma : X_k(\sigma,\omega) < \beta\}] \ge \alpha\} \\
&= \{(\mu,\omega) : \lim_{p\to\infty} \mu[\bigcap_{n\le k\le p}\{\sigma : X_k(\sigma,\omega) < \beta\}] \ge \alpha\} \\
&= \bigcap_{p=n}^{\infty}\{(\mu,\omega) : \mu[\bigcap_{n\le k\le p}\{\sigma : X_k(\sigma,\omega) < \beta\}] \ge \alpha\},
\end{aligned}$$

essentially we only need to show that

$$\{(\mu,\omega) : \mu\{\sigma : X_n(\sigma,\omega) < \beta, X_{n+1}(\sigma,\omega) < \beta\} \ge \alpha\}$$

is $\mathfrak{M} \times \mathfrak{F}$-measurable.

Let $T_n = \{1,2\}^n$ and note that $X_n(\sigma,\omega) = X_n(\sigma',\omega)$ where $\sigma|n = \sigma'|n$. Then

$$\begin{aligned}
&\{(\mu,\omega) : \mu\{\sigma : X_n(\sigma,\omega) < \beta, X_{n+1}(\sigma,\omega) < \beta\} \ge \alpha\} \\
&= \bigcup_{S\subset T_{n+1}} [\{\mu : \mu(S) \ge \alpha\} \times \{\omega : X_n(\sigma,\omega) < \beta, \\
&X_{n+1}(\sigma,\omega) < \beta, \sigma \in S \times \{1,2\}^{\mathbb{N}}, \\
&X_n(\sigma',\omega) \ge \beta, X_{n+1}(\sigma',\omega) \ge \beta, \sigma' \in [T_{n+1} \setminus S] \times \{1,2\}^{\mathbb{N}}\}] \\
&\in \mathfrak{M}_{n+1} \times \mathfrak{F}_{n+2}.
\end{aligned}$$

□

COROLLARY 7. *Let* $(M, \mathfrak{M}, m)$ *be any Borel probability measure space of Borel probability measures on* $\{1,2\}^{\mathbb{N}}$. *If* $\sum_{n=1}^{\infty} \log E(C^s_{\sigma|n-1,1} + C^s_{\sigma|n-1,2})$ *converges so fast for each* $\sigma \in \{1,2\}^{\mathbb{N}}$, *then for* P*-almost every* $\omega \in \Omega$ *there is* $F'_\mu(\omega) \subset F(\omega)$ *such that* $\dim_p(F'_\mu(\omega)) \leq s$ *where* $\mu(F'_\mu(\omega)) = 1$ *for* m*-almost all* $\mu \in M$. *Further if we assume an additional condition that for* P*-almost all* $\omega \in \Omega$ $\dim_H F(\omega) \geq s$, *then for* P*-almost every* $\omega \in \Omega$ *there is* $F^*_\mu(\omega) \subset F(\omega)$ *such that* $\dim_p(F^*_\mu(\omega)) = s$ *where* $\mu(F^*_\mu(\omega)) = 1$ *for* m*-almost all* $\mu \in M$.

Proof. For every $\mu \in M$, we see that $\mu\{\sigma : \limsup_{k\to\infty} X_k(\sigma, \omega) < \infty\} = 1$ for P-almost all $\omega \in \Omega$ in the proof of Theorem 3.

Let $E = \{(\mu, \omega) \in M \times \Omega : \mu\{\sigma : \limsup_{k\to\infty} X_k(\sigma, \omega) < \infty\} = 1\}$. Then E is $\mathfrak{M} \times \mathfrak{F}$-measurable by Lemma 6.

Applying the Fubini theorem, we see that

$$\int\int 1_E dP dm = \int\int 1_E dm dP$$

and

$$1 = \int\int 1_E dm dP.$$

Hence we easily see that $m(E_\omega) = 1$ for P-almost every ω where $E_\omega = \{\mu \in M : (\mu, \omega) \in E\} \in \mathfrak{M}$ using the same arguments in the proof of Theorem 3. Thus for P-almost every $\omega \in \Omega$, $\mu\{\sigma : \limsup_{k\to\infty} X_k(\sigma, \omega) < \infty\} = 1$ m-almost all $\mu \in M$. Say $\{\sigma : \limsup_{k\to\infty} X_k(\sigma, \omega) < \infty\} = F'_\mu(\omega)$. Further for P-almost every ω, if we consider $\mu \in E_\omega$, using the same arguments in the proof of Corollary 4 we see that there is $F^*_\mu(\omega) \subset F(\omega)$ such that $\dim_p(F^*_\mu(\omega)) = s$ where $\mu(F^*_\mu(\omega)) = 1$ and $\sum_{n=1}^{\infty} \log E(C^s_{\sigma|n-1,1} + C^s_{\sigma|n-1,2})$ converges so fast. □

LEMMA 8. *Let* μ *be a Borel probability measure on a bounded Borel set* E *in* $\mathbf{R}$. *Then* $\overline{R}(\mu) \leq \inf\{\dim_p F : \mu(F) = 1$ *for Borel* $F \subset E\}$, *where* $\overline{R}(\mu)$ *is the upper Rényi dimension* ([5]) *of* μ.

Proof. Assume that $t > \dim_p F$ where Borel $F \subset E$ with $\mu(F) = 1$. Then $\limsup_{r\to 0} \frac{\log \mu(B_r(x))}{\log r} \leq \dim_p F$ $\mu - a.s.$ and we can choose a Borel subset G of F with $\mu(G) = 1$ such that $\limsup_{r\to 0} \frac{\log \mu(B_r(x))}{\log r} \leq t$ for all $x \in G$. Put

$$G_m = \{x \in G : \frac{\log \mu(B_r(x))}{\log r} \leq t \text{ for } 0 < r < \frac{1}{m}\}, m \in N.$$

Using Lemma 4 in [5], we see that $\overline{R}(\mu) \leq t + \mu(F \setminus G_m)$. Since $G_m \uparrow G$, $\overline{R}(\mu) \leq t$. □

COROLLARY 9. *Let μ be a Borel probability measure on $\{1,2\}^{\mathbb{N}}$ and $\sum_{n=1}^{\infty} \log E(C^s_{\sigma|n-1,1} + C^s_{\sigma|n-1,2})$ converges so fast for each $\sigma \in \{1,2\}^{\mathbb{N}}$.*

Then $\overline{R}(\mu(\omega)) \leq s$ where $\mu(\omega)$ is a Borel probability measure on $F(\omega)$ induced by μ for P-almost all ω. Further, P-almost every $\omega \in \Omega$, $\overline{R}(\mu(\omega)) \leq s$ for m-almost all Borel probability measure $\mu \in M$ where $\mu(\omega)$ is a Borel probability measure supported on $F(\omega)$ induced by μ.

Proof. Obvious from Corollary 4, Lemma 8 and Corollary 7. □

REMARK. Let μ be a Borel probability measure on $\{1,2\}^{\mathbb{N}}$.

We note that the upper packing dimension $\dim_p^* \mu$([8]) of measure μ on E can be shown ([8], Proposition 10.3) as

$$\dim_p^* \mu = \inf\{\dim_p F : \mu(F) = 1 \text{ for Borel } F \subset E\}.$$

We also note that $\dim_p \mu \leq \dim_p^* \mu$, where $\dim_p \mu$ is the packing dimension of measure μ on E ([8]). Hence in the random Cantor set, where for each $\sigma \in \{1,2\}^{\mathbb{N}}$, $\sum_{n=1}^{\infty} \log E(C^s_{\sigma|n-1,1} + C^s_{\sigma|n-1,2})$ converges so fast, P-almost every $\omega \in \Omega$, $\dim_p(\mu(\omega)) \leq \dim_p^*(\mu(\omega)) \leq s$ (See Corollary 7) for m-almost all Borel probability measure $\mu \in M$ where $\mu(\omega)$ is a Borel probability measure supported on $F(\omega)$ induced by μ.

COROLLARY 10. *Let $(M, \mathfrak{M}, m)$ be any Borel probability measure space of Borel probability measures on $\{1,2\}^{\mathbb{N}}$. If $\sum_{n=1}^{\infty} \log E(L_n^s + R_n^s)$ converges so fast, then for P-almost every $\omega \in \Omega$ there is $F_\mu^*(\omega) \subset F(\omega)$ such that $\dim_p(F_\mu^*(\omega)) = s$ where $\mu(F_\mu^*(\omega)) = 1$ for m-almost all $\mu \in M$.*

Proof. Obvious from Corollary 7. □

REMARK. The results in Corollary 5 and 10 are different in the sense that we fix μ in Corollary 5 and we do ω in Corollary 10. Further in Corollary 5 the μ has its own subset of full measure for almost all ω and in Corollary 10 ω has its own subset of full measure for almost all μ.

REMARK. In fact, in the study of multifractal theory, it is important to know the upper bounds of the packing dimensions of measures. The packing dimension of a given fractal is considered as its upper bound. But using above facts, we can find its upper bound even though the packing dimension of the perturbed type random Cantor set is still unknown.

That is, P-almost every $\omega \in \Omega$, $\dim_p(\mu(\omega)) \leq \dim_p^*(\mu(\omega)) \leq s$ for m-almost all Borel probability measure $\mu \in M$ where $\mu(\omega)$ is a Borel probability measure supported on $F(\omega)$ induced by μ.

Here we note that statistically self-similar Cantor set has packing dimension s ([7], [11]), hence $\dim_p(\mu(\omega)) \leq \dim_p^*(\mu(\omega)) \leq s$ for all Borel probability measure $\mu \in M$ for P-almost every $\omega \in \Omega$, which is a stronger result than ours but still restrictive, not to control the statistically quasi-self-similar Cantor set.

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