EVALUATIONS OF THE IMPROPER INTEGRALS
\[ \int_0^\infty \frac{\sin^{2m}(\alpha x)}{(x^{2n})} \, dx \text{ AND } \int_0^\infty \frac{\sin^{2m+1}(\alpha x)}{(x^{2n+1})} \, dx \]

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ABSTRACT. In this article, using the L'Hospital rule, mathematical induction, the
trigonometric power formulae and integration by parts, some integral formulae for
the improper integrals \( \int_0^\infty \frac{\sin^{2m}(\alpha x)}{(x^{2n})} \, dx \) and \( \int_0^\infty \frac{\sin^{2m+1}(\alpha x)}{(x^{2n+1})} \, dx \) are
established, where \( m \geq n \) are all positive integers and \( \alpha \neq 0 \).

1. INTRODUCTION

The following improper integral is well-known and is synonymous with names of
Laplace and Dirichlet
\[ \int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2}. \]  
(1)

In fact, in 1781, it was first obtained using the residue method by Euler. It can be
found in standard textbooks for undergraduate students, for examples Rudin [12, pp. 226–227] and Staff Room of Higher Mathematics at Xi'an Jiaotong University
[16, pp. 168–170].

Depending on the partial fraction decomposition
\[ \frac{1}{\sin t} = \frac{1}{t} + \sum_{i=1}^{\infty} (-1)^i \left( \frac{1}{t-n\pi} + \frac{1}{t+n\pi} \right), \]  
(2)
an elegant calculation of formula (1) is provided in Klambauer [6, pp. 436–437] and
Klambauer [7, pp. 382–384], due to the noted geometer N. I. Lobatschevski.
Another polished proof of identity (1) is given in Klambauer [7, pp. 381–382].
As exercises in She & Liu [13, p. 53, p. 147, and p. 335] and Spiegel [15, p. 495], using the Laplace transform, the Parseval identities of sine and cosine Fourier transforms and the residue theorem, the following formulae are requested to compute:

\[
\int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}, \quad \int_0^\infty \frac{\sin^4 t}{t^2} dt = \frac{\pi}{2}, \quad \int_0^\infty \frac{\sin^3 t}{t^3} dt = \frac{3\pi}{8}. \tag{3}
\]

In Tan [17, pp. 74–75 and p. 84], using the Mellin transform and by approaches in theory of Fourier analysis or theory of residues, the following formulae are obtained:

\[
\int_0^\infty \cos(tx)x^z \frac{dx}{x} = \Gamma(z)t^{-z} \cos \frac{\pi z}{2}, \quad \text{Re}(z) > 0, \quad t > 0; \tag{4}
\]

\[
\int_0^\infty \sin(tx)x^z \frac{dx}{x} = \Gamma(z)t^{-z} \sin \frac{\pi z}{2}, \quad \text{Re}(z) > -1, \quad t > 0; \tag{5}
\]

\[
\int_0^\infty \frac{\sin x}{x^z} dx = \frac{\pi}{2\Gamma(z)} \sin \frac{\pi z}{2}. \tag{6}
\]

Especially, taking \( t = 1 \) and \( z \to 0 \) in (5) or taking \( z = 1 \) in (6) produces (1).

The following generalisation of formula (1) can be found in Erdélyi, Magnus, Oberhettinger & Tricomi [2, 3] and Gradshteyn & Ryzhik [4, p. 458, No. 3.836.5]:

\[
\frac{2}{\pi} \int_0^\infty \left( \frac{\sin x}{x} \right)^n \cos(bx)dx = n(2^{n-1}n!) \sum_{k=0}^{[r]} (-1)^k \binom{n}{k} (n - b - 2k)^{n-1}, \tag{7}
\]

where \( 0 \leq b < n, \ n \geq 1, \ r = \frac{n-b}{2} \), and \([r]\) is the largest integer contained in \( r \).

In Sofo [14], some general results related to formulae (1) and (7) were obtained.

In Berger [1, p. 663] and Kuang [8, p. 606], the following inequality is given:

\[
\int_{-\infty}^{\infty} \left| \frac{\sin t}{t} \right|^p dt \leq \pi \frac{2}{p}, \quad p \geq 2; \tag{8}
\]

Equality is valid only if \( p = 2 \).

The integral (1) and other integral formulae stated above are useful and arising in research of damping vibration and other science or engineering. This was mentioned in Staff Room of Higher Mathematics at Xi'an Jiaotong University [16, p. 170].

Recently, Luo & Guo [9] and Luo & Qi [10, 11] obtained the following

**Theorem A** (Luo & Guo [9]). For \( k \in \mathbb{N} \) and \( \alpha \neq 0 \), we have

\[
\int_0^\infty \left( \frac{\sin(\alpha x)}{x} \right)^{2k+1} dx = \frac{\text{sgn} \alpha \sum_{i=0}^{k} (-1)^i (2k - 2i + 1)^{2k} C_{2k+1}^i}{4^k (2k)!} \cdot \alpha^{2k} \cdot \frac{\pi}{2}, \tag{9}
\]
\[ \int_0^\infty \left( \frac{\sin(\alpha x)}{x} \right)^{2k} dx = \text{sgn} \alpha \sum_{i=0}^{k-1} (-1)^i (k - i)^{2k-1} C_{2k}^i . \alpha^{2k-1} \cdot \frac{\pi}{2}. \] (10)

If taking \( k = 1 \) in (9), the formula (1) follows.

In this article, using the L'Hospital rule, mathematical induction, trigonometric power formulae and integration by parts, we will establish integral formulae of the improper integrals

\[ \int_0^\infty \frac{\sin^2m(\alpha x)}{x^{2n}} dx \quad \text{and} \quad \int_0^\infty \frac{\sin^{2m+1}(\alpha x)}{x^{2n+1}} dx, \]

where \( m \geq n \) are all positive integers and \( \alpha \neq 0 \). The following theorem holds.

**Theorem 1.** Let \( m, n \in \mathbb{N}, m \geq n, \) and \( \alpha \neq 0. \) Then

\[ \int_0^\infty \frac{\sin^{2m+1}(\alpha x)}{x^{2n+1}} dx = (-1)^{m+n} \text{sgn} \alpha \sum_{i=0}^{m} (-1)^i (2m - 2i + 1)^{2n} C_{2m+1}^i . \alpha^{2n} \cdot \frac{\pi}{2}, \] (11)

\[ \int_0^\infty \frac{\sin^{2m}(\alpha x)}{x^{2n}} dx = (-1)^{m+n} \text{sgn} \alpha \sum_{i=0}^{m-1} (-1)^i (m - i)^{2n-1} C_{2m}^i . \alpha^{2n-1} \cdot \frac{\pi}{2}. \] (12)

As a direct consequence of Theorem 1, the following integral formula holds.

**Corollary 1.** Let \( m \) be a nonnegative integer, then we have

\[ \int_0^\infty \frac{\sin^{2m+1}(\alpha x)}{x} dx = \text{sgn} \alpha \cdot \frac{(2m)!}{4^m (m!)^2} \cdot \frac{\pi}{2}. \] (13)

It is obvious that Theorem 1 generalizes the formula (1), Theorem A and other results above.

2. **Lemmas**

The following trigonometric power formulae are the basis and key of our proof for Theorem 1.

**Lemma 1.** (Group of compilation [5, p. 41 and p. 280] and Weisstein [18]) For \( \alpha > 0 \) and \( k \in \mathbb{N} \), we have

\[ \int_0^\infty \frac{\sin(\alpha x)}{x} dx = \frac{\pi}{2}, \] (14)
\[
\sin^{2k+1} x = \frac{1}{2^{2k}} \sum_{i=0}^{k} (-1)^{k+i} C^{i}_{2k+1} \sin[(2k - 2i + 1)x], \quad (15)
\]

\[
\sin^{2k} x = \frac{1}{2^{2k-1}} \left[ \sum_{i=0}^{k-1} (-1)^{k+i} C^{i}_{2k} \cos[2(k - i)x] + \frac{1}{2} C^{k}_{2k} \right], \quad (16)
\]

where \( C^{k}_{n} = \frac{n!}{(n-k)!k!} \).

The following three combinatorial identities can be regarded as by-products, enabling us to employ the L’Hospital rule in the proof of Theorem 1. They can also be found in Luo & Guo [9].

**Lemma 2.** For \( 1 \leq m \leq k \) and \( k \in \mathbb{N} \), we have

\[
\sum_{i=0}^{k} (-1)^{i}(2k - 2i + 1)^{2m-1} C^{i}_{2k+1} = 0, \quad (17)
\]

\[
\sum_{i=0}^{k-1} (-1)^{k+i} C^{i}_{2k} + \frac{1}{2} C^{k}_{2k} = 0. \quad (18)
\]

For \( 1 \leq \ell \leq k - 1 \) and \( 2 \leq k \in \mathbb{N} \), we have

\[
\sum_{i=0}^{k-1} (-1)^{i}(k - i)^{2\ell} C^{i}_{2k} = 0. \quad (19)
\]

**Proof.** By the trigonometric power formula (15), it is easy to see that

\[
\lim_{x \to 0} \frac{\sum_{i=0}^{k} (-1)^{k+i} C^{i}_{2k+1} \sin[(2k - 2i + 1)x]}{x^{2k}} = 2^{2k} \lim_{x \to 0} \frac{\sin^{2k+1} x}{x^{2k}} = 0, \quad (20)
\]

thus

\[
\sum_{i=0}^{k} (-1)^{i} C^{i}_{2k+1} \sin[(2k - 2i + 1)x] = o(x^{2k}) \quad \text{as} \quad x \to 0, \quad (21)
\]

then, for \( 0 \leq j \leq 2k \),

\[
\left( \sum_{i=0}^{k} (-1)^{i} C^{i}_{2k+1} \sin[(2k - 2i + 1)x] \right)^{(j)} = o(x^{2k-j}) \quad \text{as} \quad x \to 0, \quad (22)
\]

therefore, for any natural number \( 1 \leq m \leq k \), we have

\[
0 = \lim_{x \to 0} \left( \sum_{i=0}^{k} (-1)^{i} C^{i}_{2k+1} \sin[(2k - 2i + 1)x] \right)^{(2m-1)}
\]

\[
= \lim_{x \to 0} \left( (-1)^{m-1} \sum_{i=0}^{k} (-1)^{i}(2k - 2i + 1)^{2m-1} C^{i}_{2k+1} \cos[(2k - 2i + 1)x] \right), \quad (23)
\]
\[
= (-1)^{m-1} \sum_{i=0}^{k} (-1)^i (2k - 2i + 1)^{2m-1} C_{2k+1}^i.
\]

Identity (17) follows.

By the trigonometric power formula (16), it is not difficult to obtain

\[
\lim_{x \to 0} \sum_{i=0}^{k-1} (-1)^{k+i} C_{2k}^i \cos[2(k - i)x] + \frac{1}{2} C_{2k}^k = 2^{2k-1} \lim_{x \to 0} \frac{\sin^{2k} x}{x^{2k-1}} = 0,
\]

hence

\[
\sum_{i=0}^{k-1} (-1)^{k+i} C_{2k}^i \cos[2(k - i)x] + \frac{1}{2} C_{2k}^k = o(x^{2k-1}) \quad \text{as } x \to 0,
\]

consequently

\[
0 = \lim_{x \to 0} \sum_{i=0}^{k-1} (-1)^{k+i} C_{2k}^i \cos[2(k - i)x] + \frac{1}{2} C_{2k}^k
\]

\[
= \sum_{i=0}^{k-1} (-1)^{k+i} C_{2k}^i + \frac{1}{2} C_{2k}^k
\]

and, for \(1 \leq j \leq 2k - 1\),

\[
\left( \sum_{i=0}^{k-1} (-1)^{k+i} C_{2k}^i \cos[2(k - i)x] + \frac{1}{2} C_{2k}^k \right)^{(j)} = o(x^{2k-j-1}) \quad \text{as } x \to 0,
\]

then, for any positive integer \(\ell\) such that \(1 \leq \ell \leq k - 1\), we have

\[
0 = \lim_{x \to 0} \left( \sum_{i=0}^{k-1} (-1)^{k+i} C_{2k}^i \cos[2(k - i)x] + \frac{1}{2} C_{2k}^k \right)^{(2\ell)}
\]

\[
= \lim_{x \to 0} \left( (-1)^{\ell} \sum_{i=0}^{k} (-1)^i (2k - 2i + 1)^{2\ell} C_{2k+1}^i \cos[(2k - 2i + 1)x] \right)
\]

\[
= (-1)^{\ell} \sum_{i=0}^{k} (-1)^i (2k - 2i + 1)^{2m-1} C_{2k+1}^i.
\]

Identities (18) and (19) follow. The proof is complete. \(\square\)

3. Proof of Theorem 1

Let \(t = \alpha x\), then, by straightforward computation, we have

\[
\int_0^\infty \frac{\sin^\ell(\alpha x)}{x^s} \, dx = \alpha^{s-1} \text{sgn } \alpha \int_0^\infty \frac{\sin^\ell t}{t^s} \, dt.
\]
From Lemma 1 and Lemma 2, using the L’Hospital rule and integration by parts yields

\[
\int_0^\infty \frac{\sin^{2m+1}x}{x^{2n+1}} dx = \frac{1}{2^{2m}} \int_0^\infty \sum_{i=0}^m (-1)^{m+i} C_{2m+1}^i \frac{\sin[(2m - 2i + 1)x]}{x^{2n+1}} dx
\]

\[
= \frac{(-1)^{2j-1}!(2n-j)!}{2^{2m}(2n)!} \times \left\{ \sum_{i=0}^m (-1)^{m+i}(2m - 2i + 1)^j \frac{1}{x^{2n-j+1}} \sin[(2m - 2i + 1)x + \frac{(j-1)\pi}{2}] \right\}_{0}^{\infty}
\]

\[
= \frac{(-1)^n}{2^{2m}(2n)!} \sum_{i=0}^m (-1)^{m+i}(2m - 2i + 1)^{2n} \frac{C_{2m+1}^i \sin[(2m - 2i + 1)x]}{x^{2n-j+1}} dx
\]

\[
= \frac{(-1)^{m+n}}{2^{2m}(2n)!} \sum_{i=0}^m (-1)^i(2m - 2i + 1)^{2n} C_{2m+1}^i \int_0^\infty \frac{\sin[(2m - 2i + 1)x]}{x} dx
\]

\[
= \frac{(-1)^{m+n}}{2^{2m}(2n)!} \sum_{i=0}^m (-1)^i(2m - 2i + 1)^{2n} C_{2m+1}^i \frac{\pi}{2}
\]

where \(1 \leq j \leq 2n\). Combining with (29), the formula (11) follows.

By formula (16), using the L’Hospital rule, from Lemma 2, integration by parts gives us

\[
\int_0^\infty \frac{\sin^{2m}x}{x^{2n}} dx = \frac{1}{2^{2m-1}} \int_0^\infty \sum_{i=0}^{m-1} (-1)^{k+i} C_{2m}^i \cos[2(m - i)x] + \frac{1}{2} C_{2m}^m dx
\]

\[
= \frac{1}{2^{2m-1}} \cdot \frac{1}{2n-1} \int_0^\infty \left[ \sum_{i=0}^{m-1} (-1)^{m+i} C_{2m}^i \cos[2(m - i)x] + \frac{1}{2} C_{2m}^m \right] d\left( \frac{1}{x^{2n-1}} \right)
\]

\[
= \frac{1}{(2m-1) \cdot 2^{2n-1}} \left[ \sum_{i=0}^{m-1} (-1)^{m+i} C_{2m}^i \cos[2(m - i)x] + \frac{1}{2} C_{2m}^m \right]_{0}^{\infty}
\]

\[
+ \int_0^\infty \sum_{i=0}^{m-1} (-1)^{m+i}[2(m - i)]C_{2m}^i \sin[2(m - i)x] dx
\]

\[
= \frac{1}{(2m-1) \cdot 2^{2n-1}} \int_0^\infty \sum_{i=0}^{m-1} (-1)^{m+i}[2(m - i)]C_{2m}^i \sin[2(m - i)x] dx
\]
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\[ = \frac{(2n - j - 2)!}{2^{2m-1}(2n-1)!} \left\{ \sum_{i=0}^{m-1} (-1)^{m+i}[2(m-i)]^j C_{2m}^i \frac{\sin[2(m-i)x]}{x^{2n-j-1}} \right. \]

\[ = \left. \int_0^\infty \sum_{i=0}^{m-1} (-1)^{m+i}[2(m-i)]^{j+1} C_{2m}^i \frac{\sin[2(m-i)x]}{x^{2n-j-1}} \right. \]

\[ = \frac{(-1)^n}{2^{2m-1}(2n-1)!} \int_0^\infty \sum_{i=0}^{m-1} (-1)^{m+i}[2(m-i)]^{2n-1} C_{2m}^i \frac{\sin[2(m-i)x]}{x} \]

\[ = \frac{(-1)^n}{2^{2m-1}(2n-1)!} \sum_{i=0}^{m-1} (-1)^{m+i}[2(m-i)]^{2n-1} C_{2m}^i \int_0^\infty \frac{\sin[2(m-i)x]}{x} \]

\[ = \frac{(-1)^{m+n}}{2^{2m-2n}} \sum_{i=0}^{m-1} (-1)^i (m-i)^{2n-1} C_{2m}^i \frac{\pi}{2}, \]

where \(1 \leq j \leq 2(n - 1)\). On combining with (29), the second formula (12) follows.

The proof of Theorem 1 is thus complete.

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