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A CONSTRUCTION OF HERGLOTZ SPACES

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Abstract. Let $W(z)$ be a power series with operator coefficients such that multiplication by $W(z)$ is contractive in $\operatorname{ext} \mathcal{C}(z)$. The overlapping space $\mathcal{E}(W)$ of $\mathcal{D}(W)$ in $\mathcal{C}(z)$ is a Herglotz space with Herglotz function $\varphi(z)$ which satisfies

$$\varphi(z) + \varphi^*(z^{-1}) = 2[1 - W^*(z^{-1})W(z)].$$

1. Introduction

The anti-space of a scalar product space $(\mathcal{K}, \langle\cdot,\cdot\rangle_k)$ is $(\mathcal{K}, -\langle\cdot,\cdot\rangle_k)$. A scalar product space $\mathcal{K}$ is called a *Krein space* if it admits a fundamental decomposition of the form $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ where $\mathcal{K}_+$ is a Hilbert space and $\mathcal{K}_-$ is the anti-space of a Hilbert space. Given a Krein space $\mathcal{K}$ with fundamental decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$, define an operator J on $\mathcal{K}$ by $Jf = f_+ - f_-$ for $f = f_+ + f_-$ where $f_+ \in \mathcal{K}_+$ and $f_- \in \mathcal{K}_-$. The operator J is called the *fundamental symmetry* for the given fundamental decomposition. The space $\mathcal{K}$ can be considered as a Hilbert space $(\mathcal{K}, [,]_k)$ where

$$[f,g]_{\mathcal{K}} = \langle Jf, g\rangle_{\mathcal{K}} \quad (\langle f,g\rangle_{\mathcal{K}} = [Jf,g]_{\mathcal{K}}).$$

All J-norms defined by different decompositions are equivalent.

Let $\mathcal{H}$ and $\mathcal{C}$ be Krein spaces. A continuous linear transformation

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{C} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{H} \\ \mathcal{C} \end{pmatrix}$$

is called a *linear system.* $\mathcal{H}$ is called a *state space* and $\mathcal{C}$ is called a *coefficient space* of the system. The transfer function $W(z)$ of the linear system is defined by

$$W(z) = D + zC(I - zA)^{-1}B.$$

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A linear system is said to be observable if $\bigcap_{n=0}^{\infty} \operatorname{Ker}(CA^n f) = \{0\}$. In this case, there is an one-to-one correspondence between f in $\mathcal{H}$ and a power series $f(z) = \sum_{n=0}^{\infty} CA^n f z^n$. An observable linear system is said to be in a canonical form if the elements of the state space are power series which converge in a neighborhood of the origin. For canonical linear system,

$$\begin{aligned} A(f(z)) &= \frac{[f(z) - f(0)]}{z}, \\ Bc &= \frac{[W(z) - W(0)]c}{z}, \\ C(f(z)) &= f(0), \quad \text{and} \\ Dc &= W(0)c. \end{aligned}$$

Write a Krein space $\mathcal{C}$ as the orthogonal sum of a Hilbert space $\mathcal{C}_+$ and the anti-space $\mathcal{C}_-$ of Hilbert space. Let J be the fundamental symmetry for the given decompositon. The space $\mathcal{C}(z)$ of square summable power series is the set of power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with coefficients in $\mathcal{C}$ such that $\sum_{n=0}^{\infty} \langle Ja_n, a_n \rangle$ is finite. The space $\mathcal{C}(z)$ is considered as a Krein space with the unique scalar product such that

$$\langle f(z), f(z) \rangle_{\mathcal{C}(z)} = \sum_{n=0}^{\infty} \langle a_n, a_n \rangle.$$

The space $\mathcal{C}(z)$ is the state space of a canonical linear system whose transfer function is identically zero.

Alpay [1] has shown that a canonical linear system is not uniquely determined by transfer function if the state space is a Krein space. A construction of the state space of a canonical linear system is given by de Branges [4, 6] if multiplication by its transfer function is contractive. Complementation theoy can be used to construct the state space of a canonical linear system with given transfer function (*cf.* Branges [5]).

Let $W(z)$ be a power series with operator coefficients. A linear system with transfer function $W(z)$ is said to be in *extended canonical form* if the elements of the state space $\mathcal{D}$ are pairs $(f(z), g(z))$ of power series with coefficients in $\mathcal{C}$ and the linear system

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{on} \quad \begin{pmatrix} \mathcal{D} \\ \mathcal{C} \end{pmatrix}$$

is defined by

$$A(f(z), g(z)) = \Big(\frac{[f(z) - f(0)]}{z}, zg(z) - W^*(z) f(0) \Big),$$

$$\begin{aligned} Bc &= \left(\frac{[W(z)-W(0)]c}{z}, [1-W^*(z)W(0)]c\right), \\ C\big(f(z),g(z)\big) &= f(0), \quad \text{and} \\ Dc &= W(0)c. \end{aligned}$$

The $\mathcal{C}(z) \times \mathcal{C}(z)$ is the state space of an extended canonical linear system whose transfer function is identically zero.

The purpose of this paper is to characterize the overlapping space of an extended canonical space. In section 3, we show that the overlapping space of an extended canonical space is a *Hergrotz space.* We investigate that the overlapping space of an extended canonical space can be considered as a complementary space.

2. Existence of a canonical linear system

The constrution of a canonical linear systems made by Ando [2] can be applied to the generalization of the complementation theory.

Theorem 2.1 (de Branges [5]). *Let P is a contractive self-adjoint transformation of a Krein space $\mathcal{H}$ into itself. Then there are unique Krein spaces $\mathcal{P}$ and $\mathcal{Q}$ exist which are contained contractively and continuously in $\mathcal{H}$ such that P is the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$ and $1-P$ is the adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$. The inequality*

$$\langle c,c\rangle_{\mathcal{H}} \leq \langle a,a\rangle_{\mathcal{P}} + \langle b,b\rangle_{\mathcal{Q}}$$

holds whenever $c = a+b$ with $a \in \mathcal{P}$ and $b \in \mathcal{Q}$ and every element $c \in \mathcal{H}$ admits some such decomposition for which equality holds. The space $\mathcal{Q}$ is called the complementary space of $\mathcal{P}$ in $\mathcal{H}$.

Let $W(z)$ be a power series with operator coefficient. Assume multiplication by $W(z)$ in $\mathcal{C}(z)$ is contractive. From the Theorem 2.1 we can prove that there is unique Krein space $\mathcal{H}(W)$ which is the state space of a canonical linear system whose transfer function is $W(z)$.

The *Krein completion* can be used to construct an extended canonical linear system (*cf.* Dijksma, Langer & Snoo [8]).

Theorem 2.2 (Yang [10]). *Assume that $W(z)$ is a power series with operator coefficient such that multiplication by $W(z)$ is an everywhere defined transformation* *in $\mathcal{C}(z)$. Then there is a Krein space $\mathcal{D}(W)$ which is the state space of an extended linear system whose transfer function is $W(z)$.*

There is a relationship between $\mathcal{D}(W)$ and $\mathcal{H}(W)$.

Theorem 2.3 (de Branges & Rovnyak [3]). *Let*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be a canonical linear system with transfer function $W(z)$ whose state space is $\mathcal{H}(W)$. Assume multiplication by $W(z)$ is contractive in $\mathcal{C}(z)$. Then a partial isometric transfer function exists of $\mathcal{H}(W)$ onto $\mathcal{D}(W)$ which maps f into the pair

$$\left(\sum_{n=0}^{\infty} CA^n f z^n, \sum_{n=0}^{\infty} B^* A^{*n} f z^n\right).$$

3. Herglotz space $\mathcal{E}$

A krein space whose elements are pairs $(f(z), g(z))$ of power series with vector coefficients is called a Herglotz space if it satisfies

(1) $T : (f(z), g(z)) \to ([f(z)-f(0)]/z, zg(z)+f(0))$ has an isometric adjoint trasformation.

(2) $S : (f(z), g(z)) \to f(0)$ is a continuous transformation.

Since S is a continuous transformation, we can characterize the reproducing kernel function of a Herglotz space.

Theorem 3.1. *Let $\mathcal{E}$ be a Herglotz space. Then there is a power series $\varphi(z)$ with coefficients in $\mathcal{C}$ such that $(\varphi(z), [\varphi^*(z)-\bar{\varphi}(0)]/z)$ and $([\varphi(z)-\varphi(0)]/z, \varphi(0)+\varphi^*(z)-\bar{\varphi}(0))$ belong to $\mathcal{E}$,*

$$\langle f(0), c\rangle_{\mathcal{C}} = \langle (f(z), g(z)), (\varphi(z)c, [\varphi^*(z) - \bar{\varphi}(0)]c/z)\rangle_{\mathcal{E}} \quad and$$
$$\langle g(0), c\rangle_{\mathcal{C}} = \langle (f(z), g(z)), ([\varphi(z) - \varphi(0)]c/z, [\varphi(0) + \varphi^*(z) - \bar{\varphi}(0)]c)\rangle_{\mathcal{E}}$$

holds for every element $f(z)$ in $\mathcal{E}$ and c in $\mathcal{C}$.

Proof. Since $S(f(z), g(z)) = f(0)$ is continuous, there is a reproducing Kernel function $(\varphi(z), \phi(z))$ in $\mathcal{E}$ such that

$$\langle f(0), c\rangle_{\mathcal{C}} = \langle (f(z), g(z)), (\varphi(z)c, \phi(z)c)\rangle_{\mathcal{E}}.$$

Write $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\phi(z) = \sum_{n=0}^{\infty} b_n z^n$.

$$\langle T^n(\varphi(z), \phi(z)), (\varphi(z)c, \phi(z)c)\rangle_{\mathcal{E}} = \langle(\varphi(z), \phi(z)), T^{*n}(\varphi(z)c, \phi(z)c)\rangle_{\mathcal{E}}$$

implies $\bar{b}_{n-1} = a_n$ for each n. Hence we have $\phi(z) = \frac{[\varphi^*(z) - \bar{\varphi}(0)]}{z}$.

Since T^*T is an identity transformation,

$$T^*\big(f(z), g(z)\big) = \big(zf(z) + g(0), \frac{[g(z) - g(0]}{z}\big).$$

The identity

$$\begin{aligned}
&\Big\langle (f(z), g(z)),\ T(\varphi(z)c, \frac{[\varphi^*(z) - \bar{\varphi}(0)]c}{z})) \Big\rangle_{\mathcal{E}} \\
&= \Big\langle (f(z), g(z)),\ (\frac{[\varphi(z) - \varphi(0)]c}{z}, [\varphi^*(z) - \bar{\varphi}(0) + \varphi(0)]c)) \Big\rangle_{\mathcal{E}} \\
&= \Big\langle T^*(f(z), g(z)), (\varphi(z)c, \frac{[\varphi^*(z) - \bar{\varphi}(0)]c}{z})\Big\rangle_{\mathcal{E}} \\
&= \Big\langle (zf(z) + g(0), [g(z) - g(0)]/z), (\varphi(z)c, \frac{[\varphi^*(z) - \bar{\varphi}(0)]c}{z})\Big\rangle_{\mathcal{E}} \\
&= \Big\langle g(0), c\Big\rangle_{\mathcal{C}}.
\end{aligned}$$

holds for every $\big(f(z), g(z)\big) \in \mathcal{E}$.

This completes the proof of the theorem.

The power series $\varphi(z)$ is called the *Herglotz function* of the Herglotz space $\mathcal{E}$. It is convinent to write

$$\langle f(0), c\rangle_{\mathcal{C}} = \Big\langle (f(z), g(z)), \Big(\frac{[\varphi(z) + \bar{\varphi}(0)]c}{2}, \frac{[\varphi^*(z) - \bar{\varphi}(0)]c}{2z}\Big)\Big\rangle_{\mathcal{E}} \quad \text{and}$$

$$\langle g(0), c\rangle_{\mathcal{C}} = \Big\langle (f(z), g(z)), \Big(\frac{[\varphi(z) - \bar{\varphi}(0)]c}{2z}, \frac{[\varphi^*(z) + \bar{\varphi}(0)]c}{2}\Big)\Big\rangle_{\mathcal{E}}.$$

Assume multiplication by $W(z)$ is contractive in $\mathcal{C}(z)$. There is a Krein space $\mathcal{D}(W)$ which is the state space of an extended canonical linear system whose transfer function is $W(z)$. Let $\mathcal{E}(W)$ be the set of pairs $\big(f(z), g(z)\big)$ of elements in $\mathcal{C}(z)$ such that $\big(W(z)f(z), -g(z)\big)$ is in $\mathcal{D}(W)$. The space $\mathcal{E}(W)$ becomes a Krein space with the scalar product

$$\begin{aligned}
&\langle (f(z), g(z)), (u(z), v(z))\ \rangle_{\mathcal{E}(W)} \\
&\qquad = \langle (f(z), u(z))\rangle_{\mathcal{C}(z)} + \langle (W(z)f(z), -g(z)), (W(z)u(z), -v(z))\rangle_{\mathcal{D}(W)}.
\end{aligned}$$

The space $\mathcal{E}(W)$ is called *an overlapping space* of $\mathcal{D}(W)$ in $\mathcal{C}(z)$.

Theorem 3.2. *Assume multiplication by $W(z)$ is contractive in $\mathcal{C}(z)$. Then the overlapping space $\mathcal{E}(W)$ of $\mathcal{D}(W)$ in $\mathcal{C}(z)$ is a Herglotz space.*

Proof. Let

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{D}(W) \\ \mathcal{C} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{D}(W) \\ \mathcal{C} \end{pmatrix}$$

be an extended canonical linear system whose transfer function is $W(z)$.

For $(f(z), g(z)) \in \mathcal{E}(W)$, $(W(z)f(z), -g(z)) \in \mathcal{D}(W)$. The identity

$$\left(W(z)\frac{[f(z)-f(0)]}{z}, -[f(0)+zg(z)]\right) = A\big(W(z)f(z), -g(z)\big) - B\big(f(0)\big)$$

implies that

$$\left(W(z)\frac{[f(z)-f(0)]}{z}, -[f(0)+zg(z)]\right) \in \mathcal{D}(W).$$

We have

$$\left(\frac{[f(z)-f(0)]}{z}, [f(0)+zg(z)]\right) \in \mathcal{E}(W).$$

Define $T : \mathcal{E}(W) \longrightarrow \mathcal{E}(W)$ by

$$T\Big(f(z), g(z)\Big) = \Big(\frac{[f(z)-f(0)]}{z}, [f(0)+zg(z)]\Big).$$

Then the identity

$$\begin{aligned}
&\Big\langle T(f(z), g(z)),\ (u(z), v(z)) \Big\rangle_{\mathcal{E}(W)} \\
&= \left\langle \frac{[f(z)-f(0)]}{z}, u(z) \right\rangle_{\mathcal{C}(z)} \\
&\quad + \left\langle \frac{(W(z)[f(z)-f(0)]}{z}, -[f(0)+zg(z)]), (W(z)u(z), -v(z)) \right\rangle_{\mathcal{D}(W)} \\
&= \langle f(z), zu(z)\rangle_{\mathcal{C}(z)} + \langle (W(z)f(z), -g(z)), A^*(W(z)u(z), -v(z))\rangle_{\mathcal{D}(W)} \\
&\quad - \langle f(0), B^*(W(z)u(z), -v(z))\rangle_{\mathcal{C}} \\
&= \langle f(z), zu(z)+v(0)\rangle_{\mathcal{C}(z)} \\
&\quad + \left\langle (W(z)f(z), -g(z)), \left(W(z)[zu(z)+v(0)], -\frac{[v(z)-v(0)]}{z}\right) \right\rangle_{\mathcal{D}(W)} \\
&= \left\langle (f(z), g(z)), \left(zu(z)+v(0), \frac{[v(z)-v(0)]}{z}\right) \right\rangle_{\mathcal{E}(W)}.
\end{aligned}$$

holds for every $(f(z), g(z))$ and $(u(z), v(z))$ in $\mathcal{E}(W)$. T has an isometric adjoint since T^*T is the idnetity transformation.

This completes the proof of the theorem. □

Let $\operatorname{ext}\mathcal{C}(z)$ be the set of Laurent series $f(z) = \sum_{-\infty}^{\infty} a_n z^n$ with coefficients in $\mathcal{C}$ such that $\sum_{-\infty}^{\infty} \langle Ja_n, a_n\rangle_{\mathcal{C}}$ is finite. The space $\operatorname{ext}\mathcal{C}(z)$ is considered as a Krein space with unique scalar product such that

$$\langle f(z), f(z)\rangle_{\operatorname{ext}\mathcal{C}(z)} = \sum_{-\infty}^{\infty} \langle a_n, a_n\rangle_{\mathcal{C}}.$$

Assume multiplication by $W(z)$ is everywhere defined transformation in $\operatorname{ext}\mathcal{C}(z)$. Then the adjoint of multiplication by $W(z)$ in $\operatorname{ext}\mathcal{C}(z)$ is everywhere defined transformation in $\operatorname{ext}\mathcal{C}(z)$ and it is then multiplication by $W^*(z^{-1})$ in $\operatorname{ext}\mathcal{C}(z)$.

Define $\mathcal{P}$ be the set of Laurant series of the form

$$f(z) + z^{-1}g(z^{-1})$$

where $(f(z), g(z)) \in \mathcal{E}(W)$. The space $\mathcal{P}$ becomes a Krein space with scalar product

$$\langle f(z) + z^{-1}g(z^{-1}), f(z) + z^{-1}g(z^{-1})\rangle_{\mathcal{P}} = \langle (f(z), g(z)), (f(z), g(z))\rangle_{\mathcal{E}(W)}.$$

The complementation theory can be used to characterize the overlapping space $\mathcal{E}(W)$ of $\mathcal{D}(W)$ in $\mathcal{C}(z)$.

Theorem 3.3. *Assume that multiplicaion by $W(z)$ is contractive transformation in $\operatorname{ext}\mathcal{C}(z)$. Then $\mathcal{P}$ is contained contractively and continuously in $\operatorname{ext}\mathcal{C}(z)$ and the adjoint of the inclusion of the space in $\operatorname{ext}\mathcal{C}(z)$ is multiplication by $1 - W^*(z^{-1})W(z)$ in $\operatorname{ext}\mathcal{C}(z)$.*

Proof. Let $f(z)$ be in $\operatorname{ext}\mathcal{C}(z)$. Write $f(z)$ by $f(z) = f_1(z) + z^{-1}f_2(z^{-1})$ where $f_1(z)$, $f_2(z) \in \mathcal{C}(z)$. Choose $g_1(z)$ in $\mathcal{C}(z)$ so that the identity

$$\langle g_1(z), h(z)\rangle_{\operatorname{ext}\mathcal{C}(z)} = \langle W(z)f(z), W(z)h(z)\rangle_{\operatorname{ext}\mathcal{C}(z)}$$

holds for every $h(z) \in \mathcal{C}(z)$. Let $g(z) = g_1(z) + z^{-1}f_2(z^{-1})$. Then $f(z) - g(z)$ and $z^{-1}g(z^{-1}) - W^*(z)W(z^{-1})z^{-1}f(z^{-1})$ belong to $\mathcal{C}(z)$. Furthermore

$$(W(z)[f(z) - g(z)], -z^{-1}g(z^{-1}) + W^*(z)W(z^{-1})z^{-1}f(z^{-1}))$$

belongs to $\mathcal{D}(W)$ and the identity

$$\begin{aligned}\langle (W(z)[f(z)-g(z)], -z^{-1}g(z^{-1})+W^*(z)W(z^{-1})z^{-1}f(z^{-1})), (W(z)u(z), -v(z))\rangle_{\mathcal{D}} \\ = \langle g(z), u(z) + z^{-1}v(z^{-1})\rangle_{\mathcal{C}(z)}\end{aligned}$$

holds for every $(u(z), v(z)) \in \mathcal{E}(W)$. It implies that

$$([f(z) - g(z)], z^{-1}g(z^{-1}) - W^*(z)W(z^{-1})z^{-1}f(z^{-1}))$$

belongs to $\mathcal{E}(W)$. Hence

$$[f(z) - g(z)] + z^{-1}[zg(z) - W^*(z^{-1})W(z)zf(z)] = [1 - W^*(z^{-1})W(z)]f(z) \in \mathcal{P}.$$

The identity

$$\begin{aligned}
&\langle([1 - W^*(z^{-1})W(z)]f(z), u(z) + z^{-1}v(z^{-1})\rangle_{\mathcal{P}} \\
&= \langle([f(z) - g(z)], z^{-1}g(z^{-1}) - W^*(z)W(z^{-1})z^{-1}f(z^{-1})), (u(z), v(z))\rangle_{\mathcal{E}(W)} \\
&= \langle f(z), u(z) + z^{-1}v(z^{-1})\rangle_{\operatorname{ext}\mathcal{C}(z)}
\end{aligned}$$

holds for every $(u(z), v(z)) \in \mathcal{E}(W)$. It implies multiplication by $1 - W^*(z^{-1})W(z)$ in $\operatorname{ext}\mathcal{C}(z)$ is the adjoint of the inclusion of $\mathcal{P}$ in $\operatorname{ext}\mathcal{C}(z)$.

This completes the proof of the theorem. □

For every $f(z) \in \operatorname{ext}\mathcal{C}(z)$, $[1 - W^*(z^{-1})W(z)]f(z) \in \mathcal{P}$. Now the complementary theory can be used. Assume that multiplicaion by $W(z)$ is contractive transformation in $\operatorname{ext}\mathcal{C}(z)$. There is unique Krein space $\mathcal{M}$ such that $\mathcal{M}$ is contained continuously and contractively in $\operatorname{ext}\mathcal{C}(z)$ and such that the adjoint of inclusion of the space in $\operatorname{ext}\mathcal{C}(z)$ coincides with multiplication by $[1 - W^*(z^{-1})W(z)]$ in $\operatorname{ext}\mathcal{C}(z)$. By the uniqueness, the space $\mathcal{P}$ is the space $\mathcal{M}$. The identity

$$\begin{aligned}
&\langle f(z) - W^*(z^{-1})W(z)f(z), f(z) - W^*(z^{-1})W(z)f(z)\rangle_{\mathcal{P}} \\
&\qquad = \langle f(z), f(z)\rangle_{\operatorname{ext}\mathcal{C}(z)} - \langle W(z)f(z), W(z)f(z)\rangle_{\operatorname{ext}\mathcal{C}(z)}
\end{aligned}$$

holds for every $f(z)$ in $\operatorname{ext}\mathcal{C}(z)$.

From the argument in the proof of the Theorem 3.3, we have the identity

$$\begin{aligned}
&\langle[f(0) - g(0)], c\rangle_{\mathcal{C}} \\
&= \Big\langle (f(z) - g(z), z^{-1}g(z^{-1}) - W^*(z)W(z^{-1})z^{-1}f(z^{-1})), \\
&\qquad \Big(\frac{[\varphi(z) + \bar{\varphi}(0)]c}{2}, \frac{[\varphi^*(z) - \bar{\varphi}(0)]c}{2z}\Big)\Big\rangle_{\mathcal{E}} \\
&= \langle f(z), \frac{1}{2}[\varphi(z) + \varphi^*(z^{-1})]c\rangle_{\operatorname{ext}\mathcal{C}(z)}.
\end{aligned}$$

The choice of $g(z)$ gives us the identity

$$\langle f(0) - g(0), c\rangle_{\mathcal{C}} = \langle f(z), [1 - W^*(z^{-1})W(z)]c\rangle_{\operatorname{ext}\mathcal{C}(z)}.$$

The overlapping space $\mathcal{E}(W)$ of $\mathcal{D}(W)$ with respect to $\mathcal{C}(z)$ is a Herglotz space with Herglotz function determined by the identity

$$\varphi(z) + \varphi^*(z^{-1}) = 2[1 - W^*(z^{-1})W(z)].$$

The factorization of transfer function W(z) is derived from a Herglotz space (*cf.* de Branges [7], Dritschel & Rovnyak [9]).

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