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WEAKLY RELAXED α-SEMI-PSEUDOMONOTONE SET-VALUED VARIATIONAL-LIKE INEQUALITIES

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ABSTRACT. *In this paper, we introduce weakly relaxed α-pseudomonotonicity and* weakly relaxed α-semi-pseudomonotonicity of set-valued maps. Using the KKM technique, we obtain existence of solutions to the variational-like inequalities with weakly relaxed α-pseudomonotone set-valued maps in reflexive Banach spaces. We also present the solvability of the variational-like inequalities with weakly relaxed α-semi-pseudomonotone set-valued maps in arbitrary Banach spaces using Kakutani-Fan-Glicksberg fixed point theorem.

1. INTRODUCTION

Monotonicity of a defining map is very important together with continuity and convexity in variational inequality problems, complementarity problems, optimization problems, programming problems, equilibrium problems, game theory and so on.

In recent years, many authors[1)] obtained some important generalizations of monotonicity such as quasi-monotonicity, pseudo-monotonicity, relaxed monotonicity, p-monotonicity, semi-monotonicity, etc. Recently, Verma [12] studied a class of nonlinear variational inequalities with p-monotone and p-Lipschitz maps in reflexive Banach spaces and gave some existence theorems of solutions. In 1999, Chen [2] introduced a class of variational inequalities with semi-monotone single-valued maps

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[1)] *e. g.*, Chang, Lee & Chen [1], Chen [2], Cottle & Yao [3], Giannessi [6], Goeleven & Motreanu [7], Hadjisavvas & Schaible [8], Kang, Huang & Lee [9], Verma [12, 13], Yang & Chen [14] and the references therein.

in nonreflexive Banach spaces which are monotone in the second variable and continuous in the first, and obtained several existence results of solutions by the Kakutani-Fan-Glicksberg fixed point theorem. In 2003, Fang & Huang [5] considered the existence of solutions to variational-like inequalities with generalized monotone single-valued maps and generalized semi-monotone single-valued maps.

For set-valued case, Kassay & Kolumban [10] considered existence of solutions to variational inequalities given with semi-pseudomonotone set-valued maps.

Inspired and motivated by Chang, Lee & Chen [1], Chen [2], Fang & Huang [5], Kassay & Kolumban [10], we introduce weakly relaxed α-pseudomonotonicity and weakly relaxed α-semi-pseudomonotonicity of set-valued maps. Using the KKM technique, we obtain existence of solutions to the variational-like inequalities with weakly relaxed α-pseudomonotone set-valued maps in reflexive Banach spaces. We also present the solvability of the variational-like inequalities with weakly relaxed α-semi-pseudomonotone set-valued maps in arbitrary Banach spaces using Kakutani-Fan-Glicksberg fixed point theorem. The results presented in this paper extend and improve the corresponding results of Chang, Lee & Chen [1], Chen [2], Fang & Huang [5], Goeleven & Motreanu [7], Kang, Huang & Lee [9], Siddiqi, Ansari & Kazmi [11] and Verma [12, 13]. Our proofs are similar to the proofs in Kang, Huang & Lee [9].

2. Weakly relaxed α-pseudomonotone set-valued variational-like inequalities

In this section, we consider weakly relaxed α-pseudomonotone set-valued variational-like inequalities in a real reflexive Banach space E with its topological dual E^*.

Definition 2.1. Let K be a nonempty subset of E. A set-valued map $T : K \to 2^{E^*}$ is said to be weakly relaxed α-pseudomonotone if there exists a function $\alpha : E \to \mathbb{R}$ with $\alpha(tz) = k(t) \cdot \alpha(z)$ for $z \in E$ and $t \in (0,1)$, where k is a function from (0,1) to (0,1) with $\lim_{t\to 0} \frac{k(t)}{t} = 0$, such that for every pair of points x, $y \in K$ and for every $u \in T(x)$, we have

$$\langle u, \eta(y,x)\rangle + f(y,x) \geq 0$$

implies

$$\langle v, \eta(y,x)\rangle + f(y,x) \geq \alpha(y-x) \quad \text{for some } v \in T(y),$$

where $\eta : K \times K \to E$ is a map and $f : K \times K \to \mathbb{R}$ is a function.

Remark 2.1. Let $E = C_0$ and $K = \{x \in \ell_\infty : \|x\| \leq 1\}$. Define $\eta : K \times K \to \ell_\infty$ by $\eta(x,y) = (x_i^2 + y_i^2)_{i=1}^\infty$ for $x = (x_i)$, $y = (y_i) \in K\backslash\{0\}$ and $\eta(0,0) = (1)_{i=1}^\infty$. Then $\eta(x,y)$ is not the difference $x - y$.

Definition 2.2. Let K be a nonempty convex subset of E, and $T : K \to 2^{E^*}$ and $\eta : K \times K \to E$ be maps. T is said to be η-hemicontinuous if for any fixed x, $y \in K$, the map

$$t \longmapsto \langle T(x + t(y - x)), \eta(y,x)\rangle \quad \text{for} \quad 0 < t < 1,$$

is upper semicontinuous at 0^+.

Definition 2.3. Let K be a nonempty subset of E, $T : K \to 2^{E^*}$ and $\eta : K \times K \to E$ be two maps, and $f : K \times K \to \mathbb{R} \cup \{+\infty\}$ be a proper function. T is said to be η-coercive with respect to f if there exists an $x_0 \in K$ such that

$$\frac{\langle u - u_0, \eta(x, x_0)\rangle + f(x, x_0)}{\|\eta(x, x_0)\|} \to \infty$$

whenever $\|x\| \to \infty$ for all $u \in T(x)$ and $u_0 \in T(x_0)$.

Theorem 2.1. *Let K be a convex subset of E, $T : K \to 2^{E^*}$ be an η-hemicontinuous and weakly relaxed α-pseudomonotone set-valued map, and $f : K \times K \to \mathbb{R} \cup \{+\infty\}$ a proper function. Assume that*

(i) *for fixed $v \in E^*$, $x \mapsto \langle v, \eta(x,\cdot)\rangle + f(x,\cdot)$ is convex, and*
(ii) *$\eta(x,x) = \bar{0}$ and $f(x,x) = 0$ for $x \in K$.*

Then the following variational inequalities (2.1) *and* (2.2) *are equivalent.*
Find $x \in K$ such that for each $y \in K$, there exists $u \in T(x)$ satisfying

$$\langle u, \eta(y,x)\rangle + f(y,x) \geq 0. \tag{2.1}$$

Find $x \in K$ such that for each $y \in K$, there exists $v \in T(y)$ satisfying

$$\langle v, \eta(y,x)\rangle + f(y,x) \geq \alpha(y - x). \tag{2.2}$$

Proof. By the weakly relaxed α-pseudomonotonicity of T, (2.1) implies (2.2).

Conversely, let x be a point of K such that for $y \in K$ with $f(y,x) < \infty$ there exists $v \in T(y)$ satisfying $\langle v, \eta(y,x)\rangle + f(y,x) \geq \alpha(y - x)$.

Put $y_t = (1-t)x + ty$, $t \in (0,1)$, then $y_t \in K$. It follows that

$$\langle v_t, \eta(y_t, x)\rangle + f(y_t, x) \geq \alpha(y_t - x), \text{ for some } \quad v_t \in T(y_t).$$

Since $\alpha(y_t - x) = \alpha(t(y-x)) = k(t)\alpha(y-x)$, by the conditions (i) and (ii) we have

$$t\left(\langle v_t, \eta(y,x)\rangle + (y,x)\right) \geq k(t)\alpha(y - x).$$

Hence

$$\langle v_t, \eta(y,x)\rangle + f(y,x) \geq \frac{k(t)}{t}\alpha(y-x).$$

Since T is η-hemicontinuous and weakly relaxed α-pseudomonotone, by letting $t \to 0$ we have

$$\langle u, \eta(y,x)\rangle + f(y,x) \geq 0$$

for some $u \in T(x)$ and for all $y \in K$ with $f(y,x) < \infty$. When $f(y,x) = +\infty$, the inequality $\langle u, \eta(y,x)\rangle + f(y,x) \geq 0$ is trivial. Therefore $x \in K$ is a solution of (2.1). □

Remark 2.2. Theorem 2.1 generalizes Fang & Huang [5, Theorem 2.1], Kang, Huang & Lee [9, Theorem 2.1], Verma [12, Theorem 2.1] and Verma [13, Theorem 2.1].

Definition 2.4. Let K be a nonempty subset of a vector space E. A set-valued map $T : K \to 2^E$ is said to be a KKM-map if for any finite subset N of K, we have

$$\mathrm{co}(N) \subset \bigcup_{x\in N} T(x).$$

Theorem 2.2 (Fan-KKM Theorem: Fan [4]**).** *Let E be a topological vector space, $K \subset E$ an arbitrary subset, and $T : K \to 2^E$ a KKM-map. If all the sets $T(x)$ are closed in E and at least one of them is compact, then $\bigcap\{T(x) : x \in K\}$ is nonempty.*

Theorem 2.3. *Let K be a nonempty bounded closed convex subset of a real reflexive Banach space E. Let $T : K \to 2^{E^*}$ be an η-hemicontinuous and weakly relaxed α-pseudomonotone map with nonempty compact values, and $f : K \times K \to \mathbb{R} \cup \{+\infty\}$ a proper function. Assume that*

(i) *for fixed $v \in E^*$, $x \mapsto \langle v, \eta(x,\cdot)\rangle + f(x,\cdot)$ is convex, weakly lower semicontinuous,*
(ii) *$\eta(x,y) + \eta(y,x) = \bar{0}$ and $f(x,y) + f(y,x) = 0$ for x, $y \in K$, and*
(iii) *$\alpha : E \to \mathbb{R}$ is weakly lower semicontinuous.*

Then problem (2.1) *is solvable.*

Proof. Define a set-valued map $F : K \to 2^E$ as

$$F(y) = \{x \in K | \langle u, \eta(y,x)\rangle + f(y,x) \geq 0 \text{ for some } u \in T(x)\} \text{ for } y \in K.$$

Then F is a KKM-map. In fact, suppose that there exist $\{y_1, y_2, \dots, y_n\}$ in K and $t_i > 0$ with $\sum_{i=1}^{n} t_i = 1$ such that

$$y = \sum_{i=1}^{n} t_i y_i \notin \bigcup_{i=1}^{n} F(y_i).$$

Then for all $v \in T(y)$

$$\langle v, \eta(y_i, y)\rangle + f(y_i, y) < 0, \quad i = 1, 2, \dots, n.$$

It follows that

$$\begin{aligned} 0 &= \langle v, \eta(y, y)\rangle + f(y, y) \\ &= \left\langle v, \eta\left(\sum_{i=1}^{n} t_i y_i, y\right)\right\rangle + f\left(\sum_{i=1}^{n} t_i y_i, y\right) \\ &\le \sum_{i=1}^{n} t_i \left(\langle v, \eta(y_i, y)\rangle + f(y_i, y)\right) \\ &< 0, \end{aligned}$$

which is a contradiction.

Define another set-valued map $G : K \to 2^E$ by

$$G(y) = \{x \in K : \text{ for some } v \in T(y), \langle v, \eta(y, x)\rangle + f(y, x) \ge \alpha(y - x)\}$$

for $y \in K$. Then by the relaxed α-pseudomonotonicity of T it follows that G is also a KKM-map. On the other hand, let $\{x_\beta\}$ be a net in $G(y)$ converging weakly to x. Then for some $v_\beta \in T(y)$,

$$\langle v_\beta, \eta(y, x_\beta)\rangle + f(y, x_\beta) \ge \alpha(y - x_\beta) \text{ for } \beta \in I.$$

Hence $T(y)$ is compact, so that we may assume that $\{v_\beta\}$ converges to some $v \in T(y)$. Since $x \mapsto \langle v, \eta(\cdot, x)\rangle + f(\cdot, x)$ is weakly upper semicontinuous, and α is weakly lower semicontinuous, we have

$$\begin{aligned} \langle v, \eta(y, x)\rangle + f(y, x) &\ge \overline{\lim_{\beta}} \left(\langle v_\beta, \eta(y, x_\beta)\rangle + f(y, x_\beta)\right) \\ &\ge \underline{\lim_{\beta}} \left(\langle v_\beta, \eta(y, x_\beta)\rangle + f(y, x_\beta)\right) \\ &\ge \underline{\lim_{\beta}}\, \alpha(y - x_\beta) \\ &\ge \alpha(y - x). \end{aligned}$$

It follows that $x \in G(y)$ and $G(y)$ is weakly closed for all $y \in K$. Since K is bounded closed and convex, K is weakly compact, so $G(y)$ is weakly compact in K for each $y \in K$. It follows from Theorem 2.1 and Fan-KKM theorem that

$$\bigcap_{y\in K} F(y) = \bigcap_{y\in K} G(y) \neq \emptyset.$$

Hence there exists an $x \in K$ such that for each $y \in K$, there exists $u \in T(x)$ satisfying

$$\langle u, \eta(y,x)\rangle + f(y,x) \geq 0.$$

□

Theorem 2.4. *Let K be a nonempty unbounded closed convex subset of a real reflexive Banach space E. Let $T : K \to 2^{E^*}$ be an η-hemicontinuous and relaxed α-pseudomonotone set-valued map, and $f : K \times K \to \mathbb{R} \cup \{+\infty\}$ a proper function. Assume that*

(i) *for fixed $v \in E^*$, $x \mapsto \langle v, \eta(x,\cdot)\rangle + f(x,\cdot)$ is convex, weakly lower semicontinuous,*

(ii) *$\eta(x,y) + \eta(y,x) = \bar{0}$ and $f(x,y) + f(y,x) = 0$ for x, $y \in K$, and*

(iii) *$\alpha : E \to \mathbb{R}$ is weakly lower semicontinuous.*

If T is η-coercive with respect to f, then problem (2.1) is also solvable.

Proof. Let $B_r = \{y \in E : \|y\| \leq r\}$ and consider the following problem;

Find $x_r \in K \cap B_r$ such that for each $y \in K \cap B_r$, there exists $u_r \in T(x_r)$ satisfying

$$\langle u_r, \eta(y,x_r)\rangle + f(y,x_r) \geq 0. \tag{2.3}$$

By Theorem 2.3, problem (2.3) has a solution $x_r \in K \cap B_r$. Choose $r > \|x_0\|$ with x_0 in the coercivity condition of Definition 2.3. Since $x_0 \in K \cap B_r$, we have $u_r \in T(x_r)$ satisfying

$$\langle u_r, \eta(x_0,x_r)\rangle + f(x_0,x_r) \geq 0.$$

Moreover

$$\begin{aligned}
&\langle u_r, \eta(x_0,x_r)\rangle + f(x_0,x_r) \\
&\quad = -\langle u_r - u_0, \eta(x_r,x_0)\rangle + f(x_0,x_r) + \langle u_0, \eta(x_0,x_r)\rangle \\
&\quad \leq -\langle u_r - u_0, \eta(x_r,x_0)\rangle + f(x_0,x_r) + \|u_0\|\|\eta(x_r,x_0)\| \\
&\quad \leq \|\eta(x_r,x_0)\| \left(-\frac{\langle u_r - u_0, \eta(x_r,x_0)\rangle + f(x_r,x_0)}{\|\eta(x_r,x_0)\|} + \|u_0\| \right)
\end{aligned}$$

for $u_0 \in T(x_0)$.

Now if $\|x_r\| = r$ for all r, we may choose r large enough so that the above inequality and the η-coercivity of T with respect to f imply that

$$\langle u_r, \eta(x_0, x_r)\rangle + f(x_0, x_r) < 0,$$

which contradicts

$$\langle u_r, \eta(x_0, x_r)\rangle + f(x_0, x_r) \geq 0.$$

Hence there exists r such that $\|x_r\| < r$. For any $y \in K$, we can choose $\varepsilon \in (0,1)$ small enough such that $x_r + \varepsilon(y - x_r) \in K \cap B_r$. It follows from (2.4) that

$$\langle u_r, \eta(x_r + \varepsilon(y - x_r), x_r)\rangle + f(x_r + \varepsilon(y - x_r), x_r) \geq 0$$

for some $u_r \in T(x_r)$. By the conditions (i) and (ii), we have

$$\langle u_r, \eta(y, x_r)\rangle + f(y, x_r) \geq 0.$$

Thus $x_r \in K$ is a solution of (2.1). □

Remark 2.3. Theorem 2.3 and Theorem 2.4 generalize the known results in Fang & Huang [5], Hadjisavvas & Schaible [8] and corresponding results in Goeleven & Motreanu [7], Kang, Huang & Lee [9], Siddiqi, Ansari & Kazmi [11] and Verma [13].

3. Weakly relaxed α-semi-pseudomonotone set-valued variational-like inequalities

Chen [2] considered the following variational inequality **(P-1)** for a semi-monotone single-valued map $A : K \times K \to E^*$, where K is a bounded closed convex subset of E^{**}, the second dual of a real Banach space E.

(P-1) Find an $x \in K$ such that

$$\langle A(x, x), y - x\rangle \geq 0 \quad \text{for all} \quad y \in K.$$

And then, very recently Fang & Huang [5] considered the following generalized variational-like inequality **(P-2)** for a semi-monotone single-valued map $A : K \times K \to E^*$.

(P-2) Find an $x \in K$ such that

$$\langle A(x, x), \eta(y, x)\rangle + f(y) - f(x) \geq 0 \quad \text{for all} \quad y \in K,$$

where $\eta : K \times K \to E^{**}$ is a map and $f : K \to \mathbb{R} \cup \{\infty\}$ is a function.

In 2000, Kassay & Kolumban [10] considered the existence of solutions to the following variational inequalities **(P-3)** and **(P-4)** for semi-pseudomonotone set-valued maps $A : K \times K \to 2^E$, where K is a nonempty convex subset of E^*.
(P-3) Find an $x \in K$ such that

$$\sup_{u \in A(x,x)} \langle u, y - x \rangle \geq 0 \quad \text{for all} \quad y \in K.$$

(P-4) (Minty-type problem)
Find an element $x \in K$ such that

$$\sup_{u \in A(y,x)} \langle u, y - x \rangle \geq 0 \quad \text{for all} \quad y \in K.$$

They showed that **(P-3)** and **(P-4)** are equivalent, and they obtained the existence results which generalizes the corresponding results for semi-monotone single-valued maps.

In this section, we consider the existence of solutions to the following variational-like inequality for a weakly relaxed α-semi-pseudomonotone set-valued map $A : K \times K \to 2^{E^*}$, where K is a nonempty closed convex subset of E^{**};
Find $x \in K$ such that for each $y \in K$ there exists $u \in A(x,x)$ satisfying

$$\langle u, \eta(y,x) \rangle + f(y,x) \geq 0. \tag{3.1}$$

Definition 3.1. Let $\eta : K \times K \to E^{**}$ be a map and $f : K \times K \to \mathbb{R}$ be a function. Let K be a nonempty subset of E^{**}. A set-valued map $A : K \times K \to 2^{E^*}$ is said to be weakly relaxed α-semi-pseudomonotone if the following conditions hold;

(a) for each fixed $w \in K$, $A(w, \cdot) : K \to 2^{E^*}$ is weakly relaxed α-pseudomonotone, *i. e.*, there exists a function $\alpha : E^{**} \to \mathbb{R}$ with $\alpha(tz) = k(t)\alpha(z)$ for $z \in E^{**}$, where $k : (0,1) \to (0,1)$ is a function with $\lim_{t \to 0} \frac{k(t)}{t} = 0$, such that for every pair of points x, $y \in K$ and for every $u \in A(w,x)$, we have

$$\langle u, \eta(y,x) \rangle + f(y,x) \geq 0$$

implies

$$\langle v, \eta(y,x) \rangle + f(y,x) \geq \alpha(y - x) \quad \text{for some } v \in T(y).$$

(b) for each fixed $y \in K$, $A(\cdot, y)$ is completely continuous, *i. e.*, for any net $\{x_\beta\}$ in E^{**} such that $x_\beta \overset{*}{\rightharpoonup} x_0$, every net $\{v_\beta\}$ in E^* with $v_\beta \in A(x_\beta, y)$ has a convergent subnet, of which limit belongs to $A(x_0, y)$ in the norm topology of E^*, where $\rightharpoonup$ denotes the weak* convergence in E^{**}.

Theorem 3.1. *Let E be a real Banach space and K a nonempty bounded closed convex subset of E^{**}. Let $A : K \times K \to 2^{E^*}$ be a weakly relaxed α-semi-pseudomonotone set-valued map, and $f : K \times K \to \mathbb{R} \cup \{+\infty\}$ a proper function such that*

(i) *for fixed $v \in E^*$, $x \mapsto \langle v, \eta(x, \cdot)\rangle + f(x, \cdot)$ is linear, weakly lower semicontinuous,*
(ii) *$\eta(x,y) + \eta(y,x) = \bar{0}$ and $f(x,y) + f(y,x) = 0$ for x, $y \in K$,*
(iii) *$\alpha : E^{**} \to \mathbb{R}$ is convex, weakly lower semicontinuous, and*
(iv) *for each $x \in K$, $A(x, \cdot) : K \to 2^{E^*}$ is finite dimensional continuous.*

Then problem (3.1) is solvable.

Proof. Let $F \subset E^{**}$ be a finite dimensional subspace with $K_F := K \cap F \neq \emptyset$. For each $w \in K$, we consider the following problem;

Find $x_0 \in K_F$ such that for each $y \in K_F$, there exists $u_0 \in A(w, x_0)$ satisfying

$$\langle u_0, \eta(y, x_0)\rangle + f(y, x_0) \geq 0. \tag{3.2}$$

For each $w \in K_F$, since $A(w, \cdot)$ is weakly relaxed α-pseudomonotone and continuous on a bounded closed convex subset K_F of F, by Theorem 2.3, the problem (3.2) has a solution x_w in K_F. If we define a set-valued map $T : K_F \to 2^{K_F}$ as follows;

$$T(w) = \big\{x \in K_F : \text{ for } y \in K_F \text{ there exists } u \in A(w,x) \text{ such that } \langle u, \eta(y,x)\rangle + f(y,x) \geq 0\big\},$$

then $T(w)$ is nonempty, since $x_w \in T(w)$. By Theorem 2.1, $T(w)$ is equal to the set

$$\big\{x \in K_F : \text{ for } y \in K_F \text{ there exists } v \in A(w,y) \text{ such that } \langle v, \eta(y,x)\rangle + f(y,x) \geq \alpha(y-x)\big\}.$$

By conditions (i), (ii) and (iii), $T : K_F \to 2^{K_F}$ has a nonempty bounded closed and convex set-values, and T is upper semicontinuous by the complete continuity of $A(\cdot, y)$. By the Kakutani-Fan-Glicksberg fixed point theorem, T has a fixed point w_0 in K_F, *i. e.*, for each $y \in K_F$, there exists $u \in A(w_0, w_0)$ satisfying

$$\langle u, \eta(y, w_0)\rangle + f(y, w_0) \geq 0. \tag{3.3}$$

Let $\mathcal{F} = \{F : F \text{ is a finite-dimensional subspace of } E^{**} \text{ with } K_F \neq \emptyset\}$ and, for $F \in \mathcal{F}$,

$$W_F := \big\{x \in K : \langle v, \eta(y,x)\rangle + f(y,x) \geq \alpha(y-x) \text{ for } y \in K_F \text{ and for some } v \in A(x,y)\big\}.$$

By Theorem 2.1 and (3.3), we know that W_F is nonempty and bounded. Since the weak* closure $\overline{W_F}$ of W_F in E^{**} is weak* compact in E^{**}, for any $F_i \in \mathcal{F}$ $(i = 1, 2, \ldots, n)$, we know that $W_{\bigcup_i F_i} \subset \bigcap_i W_{F_i}$, so $\{\overline{W_F} : F \in \mathcal{F}\}$ has the finite intersection property. Therefore it follows that

$$\bigcap_{F \in \mathcal{F}} \overline{W_F} \neq \emptyset.$$

Let $x \in \bigcap_{F \in \mathcal{F}} \overline{W_F}$. We claim that for each $y \in K$, there exists $u \in A(x, x)$ satisfying $\langle u, \eta(y, x)\rangle + f(y, x) \geq 0$. Indeed, for each $w \in K$, let $F \in \mathcal{F}$ be such that $w \in K_F$ and $x \in K_F$. There exists a net $\{x_\beta\}$ in W_F such that $x_\beta \rightharpoonup x$. It follows that

$$\langle v_\beta, \eta(y, x_\beta)\rangle + f(y, x_\beta) \geq \alpha(y - x_\beta)$$

for all $y \in K_F$ and for some $v_\beta \in A(x_\beta, y)$. Hence we have

$$\langle v, \eta(y, x)\rangle + f(y, x) \geq \alpha(y - x)$$

for all $y \in K$ and for some $v \in A(x, y)$ by using the complete continuity of $A(\cdot, y)$ and the assumptions on f, η and α. From Theorem 2.1, for each $y \in K$, there exists $u \in A(x, x)$ satisfying

$$\langle u, \eta(y, x)\rangle + f(y, x) \geq 0.$$

□

Theorem 3.2. *Let E be a real Banach space and K a nonempty unbounded closed convex subset of E^{**}. Let $A : K \times K \to 2^{E^*}$ be a weakly relaxed α-semi-pseudomonotone set-valued map, and $f : K \times K \to \mathbb{R} \cup \{+\infty\}$ a proper function such that*

(i) *for fixed $v \in E^*$, $x \mapsto \langle v, \eta(x, \cdot)\rangle + f(x, \cdot)$ is linear, lower semicontinuous,*
(ii) *$\eta(x, y) + \eta(y, x) = \bar{0}$ and $f(x, y) + f(y, x) = 0$ for x, $y \in K$,*
(iii) *$\alpha : E^{**} \to \mathbb{R}$ is convex, weakly lower semicontinuous,*
(iv) *for each $x \in K$, $A(x, \cdot) : K \to 2^{E^*}$ is finite dimensional continuous, and*
(v) *there exists an $x_0 \in K$ such that*

$$\underline{\lim}_{\|x\| \to \infty} \langle u, \eta(x, x_0)\rangle + f(x, x_0) > 0$$

for all $x \in K$ and for all $u \in A(x, x)$. Then problem (3.1) is solvable.

Proof. Denote the closed ball with radius r and center at 0 in E^{**} by B_r. By Theorem 3.1, there exists a solution $x_r \in B_r \cap K$ such that for each $y \in B_r \cap K$

there exists $u_r \in A(x_r, x_r)$ satisfying

$$\langle u_r, \eta(y, x_r)\rangle + f(y, x_r) \geq 0.$$

Let r be large enough so that $x_0 \in B_r$, therefore there exists $u_r \in A(x_r, x_r)$ such that

$$\langle u_r, \eta(x_0, x_r)\rangle + f(x_0, x_r) \geq 0.$$

By the condition (v), we know that $\{x_r\}$ is bounded. So we may suppose that $x_r \rightharpoonup x$ as $r \to \infty$. It follows from Theorem 2.1 that for each $v_r \in A(x_r, y)$

$$\langle v_r, \eta(y, x_r)\rangle + f(y, x_r) \geq \alpha(y - x_r)$$

for all $y \in K$.

Letting $r \to \infty$, we have for all $v \in A(x, y)$

$$\langle v, \eta(y, x)\rangle + f(y, x) \geq \alpha(y - x)$$

for all $y \in K$.

Again by Theorem 2.1 we have for $y \in K$, there exists $u \in A(x, x)$ satisfying

$$\langle u, \eta(y, x)\rangle + f(y, x) \geq 0.$$

□

Remark 3.1. Theorems 3.1 and 3.2 improve and generalize Chen [2, Theorems 2.1–2.6], Fang & Huang [5, Theorems 3.1 and 3.2] and Kang, Huang & Lee [9, Theorems 3.1 and 3.2], respectively.

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