

PUNCTURED TORUS REPRESENTATIONS USING THE GLUING METHOD

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ABSTRACT. A punctured torus $\Sigma(1, 1)$ is a building block of oriented surfaces. In this paper we formulate the matrix presentations of elements of the Teichmüller space of a punctured torus using the matrix presentations of a pair of pants $\Sigma(0, 3)$ and the gluing method.

Introduction

Let M be a compact connected smooth surface with $\chi(M) < 0$. Then the equivalence classes of hyperbolic structures on M form a deformation space $\mathfrak{T}(M)$ called the *Teichmüller space*.

Let $\pi = \pi_1(M)$ be the fundamental group of M . For a given hyperbolic structure on M , the action of π by deck transformation on the universal covering space \tilde{M} of M determines a homomorphism $\pi \rightarrow \mathbf{PSL}(2, \mathbb{R})$ called the *holonomy homomorphism* and it is well-defined up to conjugation in $\mathbf{PSL}(2, \mathbb{R})$. Thus the Teichmüller space $\mathfrak{T}(M)$ has a natural topology which identified with an open subset of the orbit space $\text{Hom}(\pi, \mathbf{PSL}(2, \mathbb{R}))/\mathbf{PSL}(2, \mathbb{R})$. Since holonomy homomorphisms $\pi \rightarrow \mathbf{PSL}(2, \mathbb{R})$ are isomorphic to their images, the generators of π can be presented by the conjugacy classes of matrices in $\mathbf{PSL}(2, \mathbb{R})$.

Let $M = \Sigma(g, n)$ be a compact connected oriented surface with g -genus and n -boundary components. Then M can be decomposed as a disjoint union of g punctured tori $\Sigma(1, 1)$ and $g - 2 + n$ pairs of pants $\Sigma(0, 3)$. Thus a punctured torus $\Sigma(1, 1)$ and a pair of pants $\Sigma(0, 3)$ are building blocks of an oriented surface M . The matrix presentations of a pair of pants $\Sigma(0, 3)$ are classified in the preceding paper [5]. The

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purpose of this paper is to formulate the matrix presentations of elements of the Teichmüller space of a punctured torus $\Sigma(1, 1)$ using the matrix presentations of $\Sigma(0, 3)$ and the gluing method.

In Section 1, we recall some preliminary definitions and describe the relation between the deformation space $\mathfrak{D}(M)$ of (G, X) -structures on a smooth manifold M and the orbit space $\text{Hom}(\pi, G)/G$. In Section 2, we define the hyperbolic elements of $\mathbf{SL}(2, \mathbb{R})$ and $\mathbf{PSL}(2, \mathbb{R})$ and classify the locations of fixed points and principal lines of hyperbolic elements. In Section 3, we introduce the gluing method and calculate the matrix presentations of elements of the Teichmüller space $\mathfrak{T}(\Sigma(1, 1))$.

1. (G, X) -structures on a smooth manifold M

Suppose a connected Lie group G acts on a smooth n -manifold X . A (G, X) -structure on a connected smooth n -manifold M is a maximal collection of coordinate charts $\{(U_\alpha, \psi_\alpha)\}$ such that

1. G acts *strongly effectively* on X ; i.e. if $g_1, g_2 \in G$ agree on a nonempty open set of X , then $g_1 = g_2$.
2. $\{U_\alpha\}$ is an open covering of M .
3. For each α , $\psi_\alpha : U_\alpha \rightarrow X$ is a diffeomorphism onto its image.
4. The change of coordinates is locally- (G, X) ; i.e. If (U_α, ψ_α) and (U_β, ψ_β) are two coordinate charts with $U_\alpha \cap U_\beta \neq \emptyset$, then the restriction of transition function $\psi_\beta \circ \psi_\alpha^{-1}$ to any connected component of $\psi_\alpha(U_\alpha \cap U_\beta)$ is a restriction of a (G, X) -transformation $g \in G$.

Let $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ be the upper half complex plane. Then $\mathbf{SL}(2, \mathbb{R})$ acts on \mathbb{H}^2 by

$$(1.1) \quad A \cdot z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

Since we have $A \cdot z = (-A) \cdot z$ for any $A \in \mathbf{SL}(2, \mathbb{R})$ and $z \in \mathbb{H}^2$, the Lie group $\mathbf{PSL}(2, \mathbb{R}) = \mathbf{SL}(2, \mathbb{R})/\pm I$ acts strongly effectively on \mathbb{H}^2 .

DEFINITION 1.1. The $(\mathbf{PSL}(2, \mathbb{R}), \mathbb{H}^2)$ -structures on a smooth surface M is called *hyperbolic structures* on M .

A manifold M with a (G, X) -structure is called a (G, X) -manifold. Let M and N be (G, X) -manifolds and $f : M \rightarrow N$ a smooth map. Then f is called a (G, X) -map if for each coordinate chart (U, ψ_U) on M and (V, ψ_V) on N , the composition $\psi_V \circ f \circ \psi_U^{-1} : \psi_U(f^{-1}(V) \cap U) \rightarrow \psi_V(f(U) \cap V)$ is locally- (G, X) .

The following *Development Theorem* is the fundamental fact about (G, X) -structures. See Thurston's book [7] for details.

THEOREM 1.2. *Let $p : \tilde{M} \rightarrow M$ denote a universal covering map of a (G, X) -manifold M , and π the corresponding group of covering transformations.*

1. *There exist a (G, X) -map $\mathbf{dev} : \tilde{M} \rightarrow X$ (called the developing map) and homomorphism $h : \pi \rightarrow G$ (called the holonomy homomorphism) such that for each $\gamma \in \pi$ the following diagram commutes :*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\mathbf{dev}} & X \\ \gamma \downarrow & & \downarrow h(\gamma) \\ \tilde{M} & \xrightarrow{\mathbf{dev}} & X \end{array}$$

2. *Suppose (\mathbf{dev}', h') is another pair satisfying above conditions. Then there exists a (G, X) -transformation $g \in G$ such that*

$$\mathbf{dev}' = g \circ \mathbf{dev} \quad \text{and} \quad h' = \iota_g \circ h$$

where $\iota_g : G \rightarrow G$ denotes the inner automorphism defined by g ; that is, $h'(\gamma) = (\iota_g \circ h)(\gamma) = g \circ h(\gamma) \circ g^{-1}$:

$$\begin{array}{ccccc} \tilde{M} & \xrightarrow{\mathbf{dev}} & X & \xrightarrow{g} & X \\ \gamma \downarrow & & \downarrow h(\gamma) & & \downarrow h'(\gamma) \\ \tilde{M} & \xrightarrow{\mathbf{dev}} & X & \xrightarrow{g} & X \end{array}$$

By Theorem 1.2, the *developing pair* (\mathbf{dev}, h) is unique up to the G -action by composition and conjugation respectively.

Let M be a smooth manifold. Consider a pair (f, N) where N is a (G, X) -manifold and $f : M \rightarrow N$ is a diffeomorphism. We say two pairs (f, N) and (f', N') are *equivalent* if there exists a (G, X) -diffeomorphism $g : N \rightarrow N'$ such that $g \circ f$ is isotopic to f' . The set of equivalence classes of all pairs (f, N) is called the *deformation space* of (G, X) -structures on M and denoted by $\mathfrak{D}(M)$.

DEFINITION 1.3. Let M be a connected smooth surface. The deformation space of hyperbolic structures on M is called the *Teichmüller space* and denoted by $\mathfrak{T}(M)$.

The deformation space $\mathfrak{D}(M)$ is closely related to $\text{Hom}(\pi, G)/G$ the orbit space of homomorphisms $\phi : \pi \rightarrow G$. Suppose $M = \Sigma(g, n)$ is a

compact oriented smooth surface with g -genus, n -boundary components. Then π admits $2g + n$ generators $A_1, B_1, \dots, A_g, B_g, C_1, \dots, C_n$ with a single relation

$$R = C_n \cdots C_1 B_g^{-1} A_g^{-1} B_g A_g \cdots B_1^{-1} A_1^{-1} B_1 A_1 = I.$$

From the correspondence of the homomorphism $\phi : \pi \rightarrow G$ to the image of generators, $\text{Hom}(\pi, G)$ may be identified with the collection of all $(2g + n)$ -tuples $(A_1, B_1, \dots, A_g, B_g, C_1, \dots, C_n) \in G^{2g+n}$ elements of G satisfying $R(A_1, B_1, \dots, A_g, B_g, C_1, \dots, C_n) = I$.

Taking the holonomy homomorphism of a (G, X) -structure defines a map

$$\mathbf{hol} : \mathfrak{D}(M) \longrightarrow \text{Hom}(\pi, G)/G$$

which is a local diffeomorphism. See Goldman [2] and Johnson [3] for details. For the hyperbolic structures on M , the map \mathbf{hol} on $\mathfrak{T}(M)$ is a diffeomorphism onto its image.

THEOREM 1.4. *Let M be a compact oriented surface with $\chi(M) = 2 - 2g - n < 0$. Then $\mathbf{hol} : \mathfrak{T}(M) \rightarrow \text{Hom}(\pi, \mathbf{PSL}(2, \mathbb{R}))/\mathbf{PSL}(2, \mathbb{R})$ is an embedding onto a real analytic manifold of dimension $6g - 6 + 3n$.*

Therefore the Teichmüller space $\mathfrak{T}(M)$ is diffeomorphic to $\mathbb{R}^{6g-6+3n}$ and an element of $\mathfrak{T}(M)$ will be identified with a conjugacy class of $\text{Hom}(\pi, \mathbf{PSL}(2, \mathbb{R}))$. In the next section, we shall explicitly formulate the algebraic presentation of elements of $\mathfrak{T}(M)$ for a punctured torus $M = \Sigma(1, 1)$.

2. Matrix presentations of a punctured torus

An element A of $\mathbf{SL}(2, \mathbb{R})$ is said to be *hyperbolic* if A has two distinct real eigenvalues. Thus A is hyperbolic if and only if $\text{tr}(A)^2 > 4$. A hyperbolic element A can be expressed by the diagonal matrix

$$(2.1) \quad \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}$$

via an $\mathbf{SL}(2, \mathbb{R})$ -conjugation with $\alpha^2 > 1$.

An element A of $\mathbf{PSL}(2, \mathbb{R})$ is said to be *hyperbolic* if A has two distinct fixed points on $\partial\mathbb{H}^2$. Since the absolute value of trace is still defined, A is hyperbolic if and only if $|\text{tr}(A)| > 2$.

The following theorem is due to Kuiper [6].

THEOREM 2.1. *Suppose that M is a compact connected oriented hyperbolic surface. Let the holonomy group $\Gamma = h(\pi) \subset \mathbf{PSL}(2, \mathbb{R})$ be the image of the holonomy homomorphism. Then every nontrivial element of Γ is hyperbolic.*

Let $M = \Sigma(g, n)$ be a compact connected oriented surface with g -genus and n -boundary components. If $\chi(M) = 2 - 2g - n < 0$, then there exist $2g - 3 + n$ nontrivial homotopically-distinct disjoint simply-closed curves on M such that they decompose M as the disjoint union of g punctured tori $\Sigma(1, 1)$ and $g - 2 + n$ pairs of pants $\Sigma(0, 3)$. Thus the punctured torus $\Sigma(1, 1)$ and a pair of pants $\Sigma(0, 3)$ are building blocks of an oriented surface M . For more detail, see Wolpert's paper [8].

The matrix presentations of the Teichmüller space of a pair of pants $\Sigma(0, 3)$ are classified in the preceding paper [5]. Using the matrix presentation of $\Sigma(0, 3)$ and the gluing method, we shall find expressions of the elements of the Teichmüller space $\mathfrak{T}(\Sigma(1, 1))$ of a punctured torus.

For a hyperbolic manifold M , let $\Omega = \mathbf{dev}(\tilde{M})$ be the developing image in \mathbb{H}^2 . For a non-trivial element A of the holonomy group $\Gamma \subset \mathbf{PSL}(2, \mathbb{R})$, the translation length $\ell(A)$ is defined by

$$\ell(A) = \inf_{z \in \Omega} d_P(z, A(z))$$

where d_P is the Poincaré metric on Ω . From Beardon's book [1], we get the relation

$$(2.2) \quad \left| \frac{\operatorname{tr}(A)}{2} \right| = \cosh\left(\frac{\ell(A)}{2}\right).$$

Since $\cosh^{-1}(t) = \log(t + \sqrt{t^2 - 1})$ and $|\operatorname{tr}(A)| = \alpha + \alpha^{-1}$ for $\alpha > 1$, Equation (2.2) becomes

$$\frac{\ell(A)}{2} = \log\left(\frac{\alpha + \alpha^{-1}}{2} + \sqrt{\frac{\alpha^2 + 2 + \alpha^{-2}}{4} - 1}\right) = \log(\alpha).$$

Therefore the *Fenchel-Nielsen's length parameter* ℓ can be defined as

$$(2.3) \quad \ell(A) = \log(\alpha^2)$$

for a hyperbolic element $A \in \mathbf{PSL}(2, \mathbb{R})$ with $\alpha^2 > 1$.

The *principal line* of a hyperbolic element $A \in \mathbf{PSL}(2, \mathbb{R})$ is the A -invariant unique geodesic in \mathbb{H}^2 and it is the line joining the *repelling* and *attracting* fixed points of A .

We now consider the location of the principal line of A and the relations of entries of A .

PROPOSITION 2.2. Suppose $A = \pm \begin{pmatrix} a & b \\ c & b \end{pmatrix} \in \mathbf{PSL}(2, \mathbb{R})$ is a hyperbolic transformation of \mathbb{H}^2 and z_a, z_r are finite (not infinite) fixed points of A . Then $z_a < z_r$ if and only if $(a + d)c < 0$.

Proof. Since z_a, z_r are the fixed points of the hyperbolic transformation $A(z) = \frac{az+b}{cz+d}$, they are the roots of the equation

$$(2.4) \quad cz^2 + (d - a)z - b = 0.$$

First we claim that $c \neq 0$. If $c = 0$, then $1 = \det(A) = ad$. Thus $d = a^{-1}$ and $A(z) = a^2z + ab$. This yields that ∞ is a fixed point of $A(z)$ since $a \neq 0$. It contradicts the assumption that both fixed points are finite. Since $c \neq 0$, the roots z_a, z_r of the Equation (2.4) can be expressed by

$$(2.5) \quad z_a, z_r = \frac{(a - d) \pm \sqrt{(a + d)^2 - 4}}{2c}.$$

Let w be the mid point of the fixed points z_a and z_r ; i.e.

$$w = (z_a + z_r)/2 = (a - d)/(2c).$$

Then the condition $z_a < z_r$ is equivalent to $A(w) < w$. For an easy understanding, see Figure 1. We can compute

$$\begin{aligned} A(w) - w &= \frac{a(\frac{a-d}{2c}) + b}{c(\frac{a-d}{2c}) + d} - \left(\frac{a - d}{2c} \right) \\ &= \frac{a(a - d) + 2bc}{(a + d)c} - \left(\frac{a - d}{2c} \right) = \frac{(a + d)^2 - 4}{2(a + d)c}. \end{aligned}$$

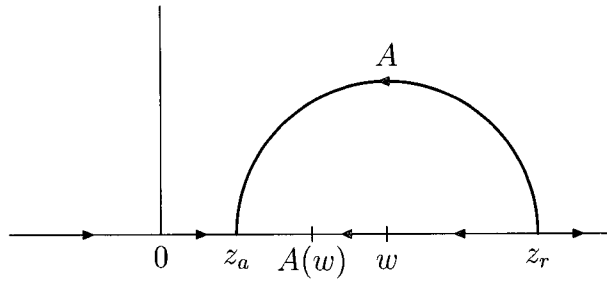
Since $(a + d)^2 > 4$, it proves $z_a < z_r$ if and only if $(a + d)c < 0$. \square

THEOREM 2.3. Suppose $A \in \mathbf{PSL}(2, \mathbb{R})$ is a hyperbolic transformation of \mathbb{H}^2 with the finite fixed points z_a, z_r . Then $0 < z_a < z_r$ if and only if $(a + d)c < 0$, $(a - d)c > 0$ and $bc < 0$.

Proof. Since the finite fixed points z_a, z_r are the roots of the Equation (2.4), we get the relations

$$(2.6) \quad z_a + z_r = \frac{a - d}{c} \quad \text{and} \quad z_a \cdot z_r = \frac{-b}{c}.$$

It proves $z_a > 0$ and $z_r > 0$ if and only if $(a - d)c > 0$ and $bc < 0$. By the Proposition 2.2, we can deduce $0 < z_a < z_r$ if and only if $(a + d)c < 0$, $(a - d)c > 0$ and $bc < 0$. \square

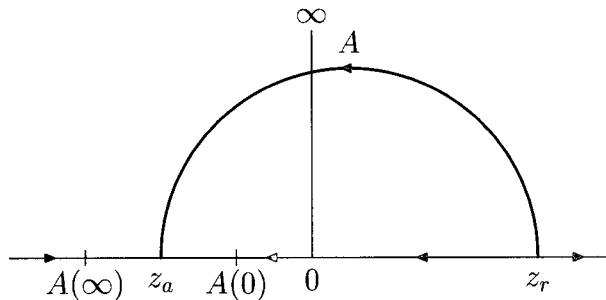
FIGURE 1. The principal line with $0 < z_a < z_r < \infty$

COROLLARY 2.4. Suppose $A \in \mathbf{PSL}(2, \mathbb{R})$ is a hyperbolic transformation of \mathbb{H}^2 with the finite fixed points z_a, z_r . If $0 < z_a < z_r$, then $a^2 < d^2$ and $bd > 0$.

Proof. From the Theorem 2.3, we have the relations $(a-d)c > 0$ and $(a+d)c < 0$. Thus $(a-d)(a+d)c^2 = (a^2 - d^2)c^2 < 0$ implies $a^2 < d^2$. Since $z_a < z_r$, the image of the origin under A should be positive as in the Figure 1. That means $A(0) = b/d > 0$. Thus we have $bd > 0$. \square

REMARK 2.5. The image of infinity under A is just less than z_a . That means the sign of $A(\infty)$ can be positive, zero, or negative.

THEOREM 2.6. Suppose $A \in \mathbf{PSL}(2, \mathbb{R})$ is a hyperbolic transformation of \mathbb{H}^2 with the finite fixed points z_a, z_r . Then $z_a < 0 < z_r$ if and only if $bc > 0$, $ac < 0$ and $bd < 0$.

FIGURE 2. The principal line with $z_a < 0 < z_r$

Proof. Suppose $z_a < 0 < z_r$. From the Equation 2.6, we can get the relation $z_a \cdot z_r < 0$ if and only if $bc > 0$. The the images of the origin

and infinity under A should be negative as in the Figure 2. That means $A(0) = b/d < 0$ and $A(\infty) = a/c < 0$. Thus we have $bd < 0$ and $ac < 0$. Conversely, the relations $bc > 0$ and $bd < 0$ derive $cd < 0$. Thus we get $(a+d)c < 0$, equivalently $z_a < z_r$. The fact $bc < 0$ implies $z_a \cdot z_r < 0$. Thus we can conclude $z_a < 0 < z_r$. \square

3. Matrix presentations of a punctured torus $\Sigma(1, 1)$

Recall that a punctured torus $\Sigma(1, 1)$ is a torus with a hole. Suppose $\Sigma(1, 1)$ is equipped with a hyperbolic structure. Since the holonomy homomorphism is isomorphic to its image, the fundamental group π of $\Sigma(1, 1)$ will be identified with

$$\pi = \langle A, B, C \in \mathbf{PSL}(2, \mathbb{R}) \mid R = CB^{-1}A^{-1}BA = I \rangle.$$

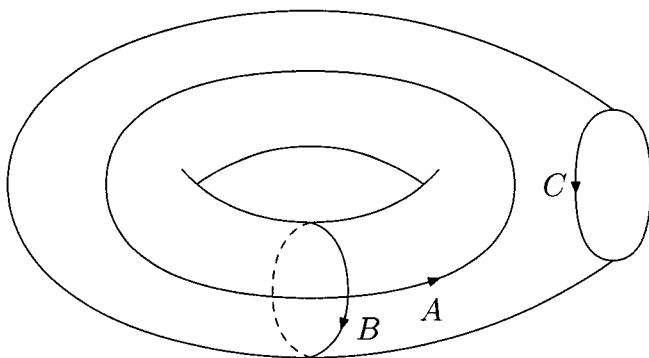


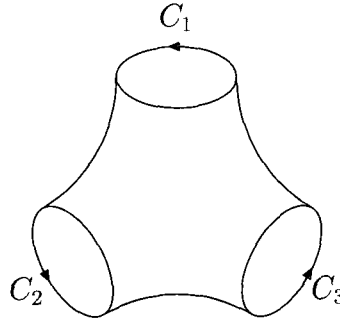
FIGURE 3. A punctured torus $M = \Sigma(1, 1)$

Let $A, B, C \in \mathbf{PSL}(2, \mathbb{R})$ represent elements of the fundamental group of M as in Figure 3. For the orientations of loops A, B , and C , see Keen's paper [4].

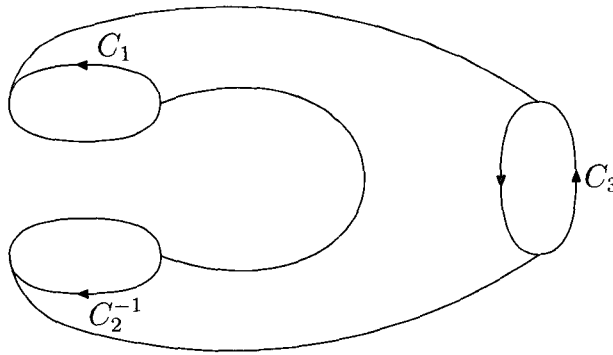
We will find the expression of the generators A, B and C of π in terms of $\mathbf{SL}(2, \mathbb{R})$ instead of $\mathbf{PSL}(2, \mathbb{R})$ because $\mathbf{SL}(2, \mathbb{R})$ is easier to compute than $\mathbf{PSL}(2, \mathbb{R})$.

Let $C_1, C_2, C_3 \in \mathbf{SL}(2, \mathbb{R})$ represent the boundary components of a pair of pants $\Sigma(0, 3)$ as in Figure 4. Then the fundamental group π of $\Sigma(0, 3)$ is identified with

$$\pi = \langle C_1, C_2, C_3 \in \mathbf{SL}(2, \mathbb{R}) \mid R = C_3C_2C_1 = I \rangle.$$

FIGURE 4. A pair of pants $M = \Sigma(0, 3)$

Suppose two boundary components C_1, C_2 of a pair of pants $\Sigma(0, 3)$ have the same translation lengths ; i.e. $\ell(C_1) = \ell(C_2)$. Then a punctured torus can be obtained by gluing two boundaries C_1, C_2 of a pair of pants $\Sigma(0, 3)$. By the orientations of boundary components C_1 and C_2 , the boundary C_1 is identified with $\pm C_2^{-1}$ up to conjugate. For an easy understanding, see the Figure 5. Thus there exists a matrix $Q \in \mathbf{SL}(2, \mathbb{R})$ such that $C_1 = Q^{-1}C_2^{-1}Q$ or $C_1 = Q^{-1}(-C_2^{-1})Q$.

FIGURE 5. Gluing boundary components C_1 with C_2^{-1}

Let λ, μ be the eigenvalues of C_1, C_2 respectively with $\lambda^2 > 1, \mu^2 > 1$. Since $\ell(C_1) = \log(\lambda^2)$ and $\ell(C_2) = \log(\mu^2)$, the condition $\ell(C_1) = \ell(C_2)$ implies $\lambda = \mu$ or $\lambda = -\mu$. Thus $C_1 = Q^{-1}C_2^{-1}Q$ if and only if $\lambda = \mu$ and $C_1 = -Q^{-1}C_2^{-1}Q$ if and only if $\lambda = -\mu$.

THEOREM 3.1. *Let $C_1, C_2, C_3 \in \mathbf{SL}(2, \mathbb{R})$ be generators of the fundamental group of a pair of pants $\Sigma(0, 3)$ with $\ell(C_1) = \ell(C_2)$.*

1. If Q is a hyperbolic matrix in $\mathbf{SL}(2, \mathbb{R})$ with $C_1 = Q^{-1}C_2^{-1}Q$, then $A := Q$, $B := C_2^{-1}$, $C := C_3$ are generators of the fundamental group of a punctured torus $\Sigma(1, 1)$.
2. If Q is a hyperbolic matrix in $\mathbf{SL}(2, \mathbb{R})$ with $C_1 = -Q^{-1}C_2^{-1}Q$, then $A := Q$, $B := -C_2^{-1}$, $C := -C_3$ are generators of the fundamental group of a punctured torus $\Sigma(1, 1)$.

Proof. Suppose $C_1 = Q^{-1}C_2^{-1}Q$. If we define $A = Q$, $B = C_2^{-1}$, $C = C_3$, then we obtain $CB^{-1}A^{-1}BA = C_3C_2Q^{-1}C_2^{-1}Q = C_3C_2C_1 = I$.

Suppose $C_1 = -Q^{-1}C_2^{-1}Q$. Similarly let $A = Q$, $B = -C_2^{-1}$, $C = -C_3$, then $CB^{-1}A^{-1}BA = (-C_3)(-C_2)Q^{-1}(-C_2^{-1})Q = C_3C_2C_1 = I$. Thus both cases, A, B, C form generators of the fundamental group of a punctured torus $\Sigma(1, 1)$. \square

Now we find the matrix presentations of the Teichmüller space of a punctured torus $\Sigma(1, 1)$ with respect to those of a pair of pants $\Sigma(0, 3)$. Since the matrices $C_1, C_2, C_3 \in \mathbf{SL}(2, \mathbb{R})$ are hyperbolic and represented up to conjugate, without loss of generality, we can assume that C_2 is a diagonal matrix.

THEOREM 3.2. *The following matrices $C_1, C_2, C_3 \in \mathbf{SL}(2, \mathbb{R})$ with*

$$(3.1) \quad a < 0, \lambda > 1, \mu^2 > 1 \quad \text{and} \quad (-a)(\mu^2 - 1) > 2|\mu| + (\lambda + \lambda^{-1})$$

form the generators of fundamental group of a pair of pants $\Sigma(0, 3)$.

$$(3.2) \quad C_1 = \begin{pmatrix} a & 1 \\ -(a - \lambda)(a - \lambda^{-1}) & -a + \lambda + \lambda^{-1} \end{pmatrix}, \quad C_2 = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix},$$

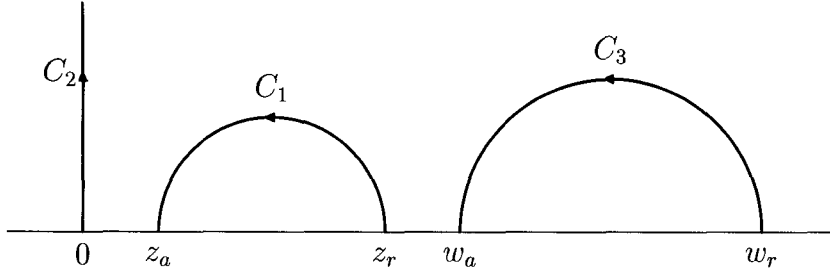
and

$$(3.3) \quad C_3 = \begin{pmatrix} \mu^{-1}(-a + \lambda + \lambda^{-1}) & -\mu \\ \mu^{-1}(a - \lambda)(a - \lambda^{-1}) & a\mu \end{pmatrix}.$$

Proof. See Kim's paper [5]. \square

The conditions in (3.1) are from the discreteness of holonomy group and the locations of principal lines of C_1, C_2 , and C_3 . See the Figure 6, for the locations of principal lines. We can easily check the relations of fixed points by the conditions in the Theorem 2.3 and Corollary 2.4.

Now we shall find a matrix $Q \in \mathbf{SL}(2, \mathbb{R})$ such that $C_1 = Q^{-1}C_2^{-1}Q$. Let $C_1, C_2 \in \mathbf{SL}(2, \mathbb{R})$ be hyperbolic matrix in (3.2). The conditions $\ell(C_1) = \ell(C_2)$ and $C_1 = Q^{-1}C_2^{-1}Q$ imply $\lambda = \mu$. Let $Q = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$.

FIGURE 6. The locations of the principal lines of C_1, C_2, C_3

After some calculations, the relation $QC_1 = C_2^{-1}Q$ induces

$$x = (a - \lambda)y \quad \text{and} \quad z = (a - \lambda^{-1})w.$$

Since $1 = \det(Q) = (a - \lambda)yw - (a - \lambda^{-1})yw = (\lambda^{-1} - \lambda)yw$, we get the relation $w = (\lambda^{-1} - \lambda)^{-1}y^{-1}$. Therefore the following $Q \in \mathbf{SL}(2, \mathbb{R})$ satisfies the condition $C_1 = Q^{-1}C_2^{-1}Q$;

$$(3.4) \quad Q = \begin{pmatrix} (a - \lambda)y & y \\ \frac{(a\lambda - 1)}{(1 - \lambda^2)} \frac{1}{y} & \frac{\lambda}{(1 - \lambda^2)} \frac{1}{y} \end{pmatrix}.$$

PROPOSITION 3.3. *Suppose we have another $\bar{Q} \in \mathbf{SL}(2, \mathbb{R})$ such that $C_1 = \bar{Q}^{-1}C_2^{-1}\bar{Q}$. Then there exists a diagonal matrix $D \in \mathbf{SL}(2, \mathbb{R})$ such that $\bar{Q} = DQ$.*

Proof. The condition $C_1 = Q^{-1}C_2^{-1}Q = \bar{Q}^{-1}C_2^{-1}\bar{Q}$ derives that $(\bar{Q}Q^{-1})C_2^{-1} = C_2^{-1}(\bar{Q}Q^{-1})$. Since C_2^{-1} is a diagonal matrix, the commutativity of $(\bar{Q}Q^{-1})$ with C_2^{-1} implies $(\bar{Q}Q^{-1})$ should be diagonal. Therefore there exists $D \in \mathbf{SL}(2, \mathbb{R})$ such that $\bar{Q} = DQ$. \square

Let $D \in \mathbf{SL}(2, \mathbb{R})$ be a diagonal matrix with entries $D_{11} = x$ and $D_{22} = x^{-1}$. Then DQ is the same shape of matrix as in (3.4), just replaced y to xy . Actually the matrix Q in (3.4) can be represented by composition of two matrices;

$$(3.5) \quad Q = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \begin{pmatrix} (a - \lambda) & 1 \\ \frac{(a\lambda - 1)}{(1 - \lambda^2)} & \frac{\lambda}{(1 - \lambda^2)} \end{pmatrix}.$$

Now we shall show that the matrix Q in (3.4) is hyperbolic.

PROPOSITION 3.4. *Let Q be the matrix in (3.4). Then*

1. $\text{tr}(Q) > 2$ if and only if $y < 0$.
2. $\text{tr}(Q) < -2$ if and only if $y > 0$.

Proof. Recall from the Theorem 3.2, we have the conditions $a < 0$ and $\lambda > 1$. Suppose $y < 0$. Then we have $(a - \lambda)y > 0$ and $\frac{\lambda}{(1-\lambda^2)}\frac{1}{y} > 0$. Thus

$$\operatorname{tr}(Q) = (a - \lambda)y + \frac{\lambda}{(1 - \lambda^2)}\frac{1}{y} \geq 2\sqrt{\frac{\lambda^2 - a\lambda}{\lambda^2 - 1}} > 2.$$

Conversely, suppose $\operatorname{tr}(Q) > 2$. Since $(a - \lambda) < 0$ and $\frac{\lambda}{(1-\lambda^2)} < 0$, the sign of y should be negative. Similarly we can show $y > 0$ if and only if $\operatorname{tr}(Q) < 2$. \square

Since Q is hyperbolic, from the Theorem 3.1, we have the following main theorem.

THEOREM 3.5. *Suppose the hyperbolic matrices in $\mathbf{SL}(2, \mathbb{R})$*

$$(3.6) \quad A = \begin{pmatrix} (a - \lambda)y & y \\ \frac{(a\lambda - 1)}{(1 - \lambda^2)}\frac{1}{y} & \frac{\lambda}{(1 - \lambda^2)}\frac{1}{y} \end{pmatrix}, \quad B = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix},$$

and

$$(3.7) \quad C = \begin{pmatrix} \lambda^{-1}(-a + \lambda + \lambda^{-1}) & -\lambda \\ \lambda^{-1}(a - \lambda)(a - \lambda^{-1}) & a\lambda \end{pmatrix}$$

satisfy $a < 0$, $\lambda > 1$, $y \neq 0$, and $(-a)(\lambda^2 - 1) > 3\lambda + \lambda^{-1}$. Then $\{A, B, C\}$ and $\{A, -B, C\}$ forms generators of the fundamental group of a punctured torus $\Sigma(1, 1)$.

Proof. First let $\mu = \pm\lambda$, then $|\mu| = \lambda > 1$. Thus the inequality in (3.1) becomes $(-a)(\lambda^2 - 1) > 3\lambda + \lambda^{-1}$. Suppose $C_1 = Q^{-1}C_2^{-1}Q$. Then $\mu = \lambda$. From Theorem 3.1, $Q = A$, $C_2^{-1} = B$, $C_3 = C$ are generators of the fundamental group of a punctured torus $\Sigma(1, 1)$. If $C_1 = -Q^{-1}C_2^{-1}Q$, then $\mu = -\lambda$ and $Q = A$, $-C_2^{-1} = -B$, $-C_3 = C$ are generators. Since the replacement of the parameter μ in C_3 to λ and that of μ in $-C_3$ to $-\lambda$ induce the same matrix C . Thus the matrix C is invariant both cases. Therefore $\{A, B, C\}$ and $\{A, -B, C\}$ forms generators of the fundamental group π of a punctured torus $\Sigma(1, 1)$. \square

See the Figure 7, for the locations of principal lines of A, B , and C . Finally we have the Main Theorem.

THEOREM 3.6. *The equivalent classes of matrices $[A]$, $[B]$ and $[C] \in \mathbf{PSL}(2, \mathbb{R})$ are representations of elements of the Teichmüller space of a punctured torus $\Sigma(1, 1)$.*

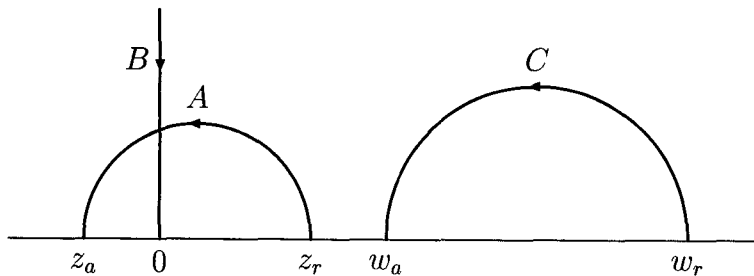


FIGURE 7. The locations of the principal lines of A, B, C

Proof. Since above hyperbolic matrices A, B and $C \in \mathbf{SL}(2, \mathbb{R})$ in Theorem 3.5 form generators of fundamental group π of $\Sigma(1, 1)$, these equivalent classes of matrices $[A], [B]$ and $[C] \in \mathbf{PSL}(2, \mathbb{R})$ represent conjugacy classes of the orbit space $\text{Hom}(\pi, \mathbf{PSL}(2, \mathbb{R}))/\mathbf{PSL}(2, \mathbb{R})$. \square

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