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NORMAL INTERPOLATION PROBLEMS IN ALG$\mathcal{L}$

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ABSTRACT. Let X and Y be operators acting on a Hilbert space and let $\mathcal{L}$ be a subspace lattice of orthogonal projections on the space containing 0 and I. We investigate normal interpolation problems in Alg$\mathcal{L}$: Given operators X and Y acting on a Hilbert space, when does there exist a normal operator A in Alg$\mathcal{L}$ such that $AX = Y$?

1. Introduction

A number of authors have considered the equation $Ax = y$, where x and y repersent given vectors in Hilbert space and the bounded operator A is to be found subject to certain criteria.

Lance [6] initiated the discussion by considering a nest $\mathcal{N}$ and asking what conditions on x and y will guarantee the existence of an operator A in Alg$\mathcal{N}$ such that $Ax = y$. This result was used to find a new proof of Ringrose's characterization of the Jacobson radical. Hopenwasser [3] extended Lance's result to the case where the nest $\mathcal{N}$ is replaced by an arbitrary commutative subspace lattice $\mathcal{L}$; the conditions in both cases read the same. Munch [7] considered the problem of finding a Hilbert-Schmit operator A in Alg$\mathcal{N}$ that maps x to y, where upon Hopenwasser [4] again extended to Alg$\mathcal{L}$. In [1], authors studied the problem of finding A so that $Ax = y$ and A is required to lie in certain ideals contained in Alg$\mathcal{L}$ (for a nest $\mathcal{L}$).

When an operator maps one thing to another, we think of the operator as an interpolating operator and the equation representing the mapping as the interpolation equation. The equations $Ax = y$ and $AX = Y$ are

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indistinguishable if spoken aloud, but we mean the change to capital letters to indicate that we intend to look at fixed operators X and Y, and ask under what conditions there will exist an operator A satisfying the equation $AX = Y$. Let x and y be vectors in a Hilbert space. Then $\langle x, y\rangle$ means the inner product of vectors x and y. Note that the "vector interpolation" problem is a special case of the "operator interpolation" problem. Indeed, if we denote by $x \otimes u$ the rank-one operator defined by the equation $x\otimes u(w) = \langle w, u\rangle\, x$, and if we set $X = x\otimes u$, and $Y = y\otimes u$, then the equations $AX = Y$ and $Ax = y$ represent the same restriction on A.

The simplest case of the operator interpolation problem relaxes all restrictions on A, requiring it simply to be a bounded operator. In this case, the existence of A is nicely characterized by the well-known factorization theorem of Douglas :

THEOREM D[2]. *Let Y and X be bounded operators on a Hilbert space $\mathcal{H}$. The following statements are equivalent:*

(1) *$range(Y^*) \subseteq range(X^*)$;*

(2) *$Y^*Y \leq \lambda^2 X^*X$ for some $\lambda \geq 0$;*

(3) *there exists a bounded operator A on $\mathcal{H}$ so that $AX = Y$.*

Moreover, if (1), (2), and (3) are valid, then there exists a unique operator A so that

(a) *$\|A\|^2 = \inf\{\mu : Y^*Y \leq \mu X^*X\}$;*

(b) *$ker[Y^*] = ker[A^*]$; and*

(c) *$range[A^*] \subseteq range[X]^-$.*

We establish some notations and conventions. A (commutative) subspace lattice $\mathcal{L}$ is a strongly closed lattice of (commutative) projections acting on a Hilbert space $\mathcal{H}$. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant.

2. Results

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a subspace lattice of orthogonal projections acting on $\mathcal{H}$ containing 0 and I. Then Alg$\mathcal{L}$ is the algebra of all bounded linear operator on $\mathcal{H}$ that leave invariant all projections in $\mathcal{L}$. Let M be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{M}$ means the closure of M and $M^\perp$ the orthogonal complement of M. A bounded operator A is *normal* if $AA^* = A^*A$. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers. In this paper, we use the convention $\frac{0}{0} = 0$, when necessary.

THEOREM 1. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Let X and Y be operators acting on $\mathcal{H}$. Assume that the range$X$ is dense in $\mathcal{H}$. Then the following statements are equivalent.*

(1) *There exists a normal operator A in Alg$\mathcal{L}$ such that $AX = Y$ and every E in $\mathcal{L}$ reduces A.*

(2) $\sup\left\{\frac{\|\sum_{i=1}^{n} E_i Y f_i\|}{\|\sum_{i=1}^{n} E_i X f_i\|} : n \in \mathbb{N}, f_i \in \mathcal{H} \text{ and } E_i \in \mathcal{L}\right\} < \infty$ *and there is an operator T acting on $\mathcal{H}$ such that*

$$\langle Xf, Tg\rangle = \langle Yf, Xg\rangle$$

and

$$\langle Tf, Tg\rangle = \langle Yf, Yg\rangle$$

for all f and g in $\mathcal{H}$.

PROOF. (1) $\Rightarrow$ (2). If there exists a normal operator A in Alg$\mathcal{L}$ such that $AX = Y$ and every E in $\mathcal{L}$ reduces A, then

$$\sup\left\{\frac{\|\sum_{i=1}^{n} E_i Y f_i\|}{\|\sum_{i=1}^{n} E_i X f_i\|} : n \in \mathbb{N},\ f_i \in \mathcal{H} \text{ and } E_i \in \mathcal{L}\right\} < \infty$$

by Theorem 1 [5]. Let $A^*X = T$. Then

$$\begin{aligned}\langle Tf, Tg\rangle &= \langle A^*Xf, A^*Xg\rangle \\ &= \langle AA^*Xf, Xg\rangle \\ &= \langle A^*AXf, Xg\rangle \\ &= \langle AXf, AXg\rangle \\ &= \langle Yf, Yg\rangle \text{ and} \\ \langle Xf, Tg\rangle &= \langle Xf, A^*Xg\rangle \\ &= \langle AXf, Xg\rangle \\ &= \langle Yf, Xg\rangle \text{ for all } f \text{ and } g \text{ in } \mathcal{H}.\end{aligned}$$

Conversely, under the first condition of hypothesis, there is an operator A in Alg$\mathcal{L}$ such that $AX = Y$ and every E in $\mathcal{L}$ reduces A by Theorem 1 [5]. Since $\langle Yf, Xg\rangle = \langle Xf, Tg\rangle$, $\langle AXf, Xg\rangle = \langle Yf, Xg\rangle = \langle Xf, Tg\rangle$

for all f and g in $\mathcal{H}$. Since the range$X$ is dense in $\mathcal{H}$, $A^*X = T$. Since $\langle Yf, Yg\rangle = \langle Tf, Tg\rangle$,

$$\begin{aligned}\langle AXf, Yg\rangle &= \langle Yf, Yg\rangle \\ &= \langle Tf, Tg\rangle \\ &= \langle A^*Xf, Tg\rangle \\ &= \langle Xf, ATg\rangle \text{ for all } f \text{ and } g \text{ in } \mathcal{H}.\end{aligned}$$

Since the range$X$ is dense in $\mathcal{H}$, $A^*Y = AT$. Hence $A^*AX = AA^*X$. Since the range$X$ is dense in $\mathcal{H}$, $A^*A = AA^*$. □

THEOREM 2. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let X and Y be operators acting on $\mathcal{H}$.*

$$\text{Let } \mathcal{M} = \left\{\sum_{i=1}^{n} E_i X f_i : n \in \mathbb{N},\ E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H}\right\}.$$

Assume that $EYf \in \overline{\mathcal{M}}$ for all E in $\mathcal{L}$ and all f in $\mathcal{H}$. Then the following statements are equivalent.

(1) There exists a normal operator A in Alg$\mathcal{L}$ such that $AX = Y$ and every E in $\mathcal{L}$ reduces A.

(2) $\sup\left\{\frac{\|\sum_{i=1}^{n} E_i Y f_i\|}{\|\sum_{i=1}^{n} E_i X f_i\|} : n\in\mathbb{N}, E_i\in\mathcal{L} \text{ and } f_i\in\mathcal{H}\right\} < \infty$ *and there is an operator T acting on $\mathcal{H}$ such that*

$$\langle EYf, Yg\rangle = \langle ETf, Tg\rangle, \langle EYf, Xg\rangle = \langle EXf, Tg\rangle$$

and $Th \in \overline{\mathcal{M}}$ for all E in $\mathcal{L}$ and all f, g and h in $\mathcal{H}$.

PROOF. (1) ⇒ (2). By Theorem 1, we can get the first result. Let $T = A^*X$. Then

$$\begin{aligned}\langle EYf, Yg\rangle &= \langle ETf, Tg\rangle, \\ \langle EYf, Xg\rangle &= \langle EXf, Tg\rangle\end{aligned}$$

and $Th \in \overline{\mathcal{M}}$ for all $E \in \mathcal{L}$ and for all f, g, h in $\mathcal{H}$.

Conversely, under the conditions of hypothesis, there exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$ and every E in $\mathcal{L}$ reduces A by

Theorem 2 [5]. Since $\langle EYf, Xg\rangle = \langle EXf, Tg\rangle$ for all f, g in $\mathcal{H}$ and all E in $\mathcal{L}$,

$$\left\langle A(\sum_{i=1}^{n} E_i X f_i), Xg \right\rangle = \left\langle \sum_{i=1}^{n} E_i Y f_i, Xg \right\rangle = \left\langle \sum_{i=1}^{n} E_i X f_i, Tg \right\rangle,$$

$n \in \mathbb{N}$, $E_i \in \mathcal{L}$ and $f_i \in \mathcal{H}$. Let $f \in \overline{\mathcal{M}}$. Then $\langle Af, Xg\rangle = \langle f, Tg\rangle$. Since $\langle x, Th\rangle = 0$ and $\langle Ax, Xh\rangle = 0$ for all x in $\overline{\mathcal{M}}^{\perp}$ and all h in $\mathcal{H}$, $A^*X = T$. Since $\langle EYf, Yg\rangle = \langle ETf, Tg\rangle$ for all E in $\mathcal{L}$ and all f, g in $\mathcal{H}$,

$$\begin{aligned}
&\left\langle A(\sum_{i=1}^{n} E_i X f_i), Yg \right\rangle \\
&= \left\langle \sum_{i=1}^{n} E_i Y f_i, Yg \right\rangle \\
&= \left\langle \sum_{i=1}^{n} E_i T f_i, Tg \right\rangle \\
&= \left\langle \sum_{i=1}^{n} E_i A^* X f_i, Tg \right\rangle \\
&= \left\langle A^*(\sum_{i=1}^{n} E_i X f_i), Tg \right\rangle \\
&= \left\langle \sum_{i=1}^{n} E_i X f_i, ATg \right\rangle, \quad n \in \mathbb{N},\ E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H}.
\end{aligned}$$

So $\langle Af, Yg\rangle = \langle f, ATg\rangle$ for all f in $\overline{\mathcal{M}}$. Since $EYf \in \overline{\mathcal{M}}$ and $Th \in \overline{\mathcal{M}}$ for all f, h in $\mathcal{H}$ and all E in $\mathcal{L}$, $ATh \in \overline{\mathcal{M}}$. Since $\langle x, Th\rangle = 0$ for all x in $\overline{\mathcal{M}}^{\perp}$ and h in $\mathcal{H}$, $\langle Ax, Yg\rangle = \langle x, ATg\rangle = 0$. So $A^*Y = AT$ and hence $A^*AX = AA^*X$. Since $AE = EA$ and $EA^* = A^*E$ for all E in $\mathcal{L}$, $A^*Af = AA^*f$ for all f in $\overline{\mathcal{M}}$. Since $A^*X = T$ and $EYf \in \overline{\mathcal{M}}$ for all E in $\mathcal{L}$ and all f in $\mathcal{H}$, $0 = \langle EYf, x\rangle = \langle AEXf, x\rangle = \langle EXf, A^*x\rangle$ for all x in $\overline{\mathcal{M}}^{\perp}$. So $A^*x \in \overline{\mathcal{M}}^{\perp}$ for all x in $\overline{\mathcal{M}}^{\perp}$. Hence $AA^*x = 0$ for all x in $\overline{\mathcal{M}}^{\perp}$. Thus $A^*A = AA^*$. □

THEOREM 3. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Let $X_1, \cdots, X_n$, $Y_1, \cdots, Y_n$ be operators acting on $\mathcal{H}$. Assume that the range of one of the X_p's is dense in $\mathcal{H}(p = 1, 2, \cdots, n)$, let it be X_1. Then the following statements are equivalent.*

(1) There exists a normal operator A in Alg$\mathcal{L}$ such that $AX_j = Y_j (j = 1, 2, \cdots, n)$ and every E in $\mathcal{L}$ reduces A.

(2) $\sup\left\{\frac{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}Y_i f_{k,i}\|}{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_i f_{k,i}\|} : l \leq n, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} < \infty$ *and there are operators $T_r (r = 1, 2, \cdots, n)$ acting on $\mathcal{H}$ such that*

$$\langle Y_p f, Y_q g\rangle = \langle T_p f, T_q g\rangle$$

and

$$\langle Y_p f, X_q g\rangle = \langle X_p f, T_q g\rangle$$

all f, g in $\mathcal{H}$ and all $p, q = 1, 2, \cdots, n$.

PROOF. (1) $\Rightarrow$ (2). Under the conditions of hypothesis except that A is normal,

$$\sup\left\{\frac{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}Y_i f_{k,i}\|}{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_i f_{k,i}\|} : l \leq n, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} < \infty$$

by Theorem 3 [5]. Let $A^*X_r = T_r (r = 1, 2, \cdots, n)$. Then

$$\begin{aligned}\langle X_p f, T_q g\rangle &= \langle X_p f, A^*X_q g\rangle \\ &= \langle AX_p f, X_q g\rangle \\ &= \langle Y_p f, X_q g\rangle \text{ and} \\ \langle T_p f, T_q g\rangle &= \langle A^*X_p f, A^*X_q g\rangle \\ &= \langle AA^*X_p f, X_q g\rangle \\ &= \langle A^*AX_p f, X_q g\rangle \\ &= \langle AX_p f, AX_q g\rangle \\ &= \langle Y_p f, Y_q g\rangle\end{aligned}$$

for all f, g in $\mathcal{H}$ and all $p, q = 1, 2, \cdots, n$.

Conversely, under the first condition of hypothesis, there exists an operator A in Alg$\mathcal{L}$ such that $AX_j = Y_j$ and every E in $\mathcal{L}$ reduces A $(j = 1, 2, \cdots, n)$ by Theorem 4 [5]. Since $\langle Y_1 f, X_q g\rangle = \langle X_1 f, T_q g\rangle$ for all f, g

in $\mathcal{H}$ and all $q = 1, 2, \cdots, n$, $\langle AX_1 f, X_q g\rangle = \langle Y_1 f, X_q g\rangle = \langle X_1 f, T_q g\rangle$. Hence $A^* X_q = T_q$ for all $q = 1, 2, \cdots, n$. Since $\langle Y_p f, Y_q g\rangle = \langle T_p f, T_q g\rangle$ for all f, g in $\mathcal{H}$ and all $p, q = 1, 2, \cdots, n$,

$$\begin{aligned}\langle AX_1 f, Y_q g\rangle &= \langle Y_1 f, Y_q g\rangle \\ &= \langle T_1 f, T_q g\rangle \\ &= \langle A^* X_1 f, T_q g\rangle \\ &= \langle X_1 f, AT_q g\rangle .\end{aligned}$$

Since the rangeX_1 is dense in $\mathcal{H}$, $A^* Y_q = AT_q$ for all $q = 1, 2, \cdots, n$. So $A^* AX_q = AA^* X_q$ for all $q = 1, 2, \cdots, n$. Hence $A^* A = AA^*$ because the rangeX_1 is dense in $\mathcal{H}$. □

THEOREM 4. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let $X_1, \cdots, X_n, Y_1, \cdots, Y_n$ be operators acting on $\mathcal{H}$.*

$$\text{Let } \mathcal{N} = \left\{ \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_i f_{k,i} : l \leq n, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H} \right\}.$$

Assume that $EY_p f \in \overline{\mathcal{N}}$ for all E in $\mathcal{L}$, all f in $\mathcal{H}$ and all $p = 1, 2, \cdots, n$. Then the following statements are equivalent.

(1) There is a normal operator A in Alg$\mathcal{L}$ such that $AX_p = Y_p$ ($p = 1, 2, \cdots, n$) and every E in $\mathcal{L}$ reduces A.

$$(2)\ \sup \left\{ \frac{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_i f_{k,i} \|}{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \|} : l \leq n, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H} \right\} < \infty$$
and there are operators $T_r (r = 1, 2, \cdots, n)$ acting on $\mathcal{H}$ such that

$$\langle EY_p f, Y_q y\rangle = \langle ET_p f, T_q y\rangle, \langle EY_p f, X_q y\rangle = \langle EX_p f, T_q y\rangle$$

and

$$\langle g, T_p f\rangle = 0$$

for all E in $\mathcal{L}$, all f, y in $\mathcal{H}$, all g in $\overline{\mathcal{N}}^{\perp}$ and all $p, q = 1, 2, \cdots, n$.

PROOF. (1) $\Rightarrow$ (2). By Theorem 3, we can get all results except that $\langle g, T_p f\rangle = 0$ for all f in $\mathcal{H}$, all g in $\overline{\mathcal{N}}^{\perp}$ and all $p = 1, 2, \cdots, n$. Since $A^* X_p = T_p$ and $Ag = 0$ for all g in $\overline{\mathcal{N}}^{\perp}$ and all $p = 1, 2, \cdots, n$, $\langle g, T_p f\rangle = \langle g, A^* X_p f\rangle = \langle Ag, X_p f\rangle = 0$ for all f in $\mathcal{H}$.

Conversely, under the first condition of hypothesis, there is an operator A in Alg$\mathcal{L}$ such that $AX_j = Y_j (j = 1, 2, \cdots, n)$ and every E in $\mathcal{L}$ reduces A by Theorem 5 [5]. Since $\langle EY_p f, X_q y\rangle = \langle EX_p f, T_q y\rangle$ for all E in $\mathcal{L}$, all f, y in $\mathcal{H}$ and all $p, q = 1, 2, \cdots, n$,

$$\left\langle A(\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_i f_{k,i}), X_q y\right\rangle = \left\langle \sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}Y_i f_{k,i}, X_q y\right\rangle$$
$$= \left\langle \sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_i f_{k,i}, T_q y\right\rangle,$$

$m_i \in \mathbb{N}$, $l \leq n$, $E_{k,i} \in \mathcal{L}$, $f_{k,i} \in \mathcal{H}$. So $\langle Af, X_q y\rangle = \langle f, T_q y\rangle$, for $f \in \overline{\mathcal{N}}$, $y \in \mathcal{H}$ and $q = 1, 2, \cdots, n$. Since $\langle g, T_q y\rangle = 0$ and $\langle Ag, X_q y\rangle = 0$ for all g in $\overline{\mathcal{N}}^{\perp}$ and all y in $\mathcal{H}$, $A^* X_q = T_q$ for all $q = 1, 2, \cdots, n$. Since $\langle EY_p f, Y_q y\rangle = \langle ET_p f, T_q y\rangle$ for all E in $\mathcal{L}$, all f, y in $\mathcal{H}$ and all $p, q = 1, 2, \cdots, n$,

$$\left\langle A(\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}), Y_p f\right\rangle = \left\langle \sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}Y_i f_{k,i}, Y_p f\right\rangle$$
$$= \left\langle \sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}T_i f_{k,i}, T_p f\right\rangle$$
$$= \left\langle \sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}A^* X_i f_{k,i}, T_p f\right\rangle$$
$$= \left\langle \sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_i f_{k,i}, AT_p f\right\rangle.$$

So $\langle Ax, Y_p f\rangle = \langle x, AT_p f\rangle$ for all x in $\overline{\mathcal{N}}$, all f in $\mathcal{H}$ and all $p = 1, 2, \cdots, n$. Since $EY_p f \in \overline{\mathcal{N}}$ for all E in $\mathcal{L}$, all f in $\mathcal{H}$ and all $p = 1, 2, \cdots, n$, $\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i} \in \overline{\mathcal{N}}$, for $m_i \in \mathbb{N}$, $l \leq n$, $E_{k,i} \in \mathcal{L}$ and

$$(*) \qquad f_{k,i} \in \mathcal{H}.$$

Since $\langle g, T_p f\rangle = 0$ for all g in $\overline{\mathcal{N}}^{\perp}$, all f in $\mathcal{H}$ and $p = 1, 2, \cdots, n$, $T_p f \in \overline{\mathcal{N}}$. Hence $AT_p f \in \overline{\mathcal{N}}$ by $(*)$ for all f in $\mathcal{H}$ and all $p = 1, 2, \cdots, n$.

Since $\langle g, AT_pf\rangle = 0$, $\langle Ag, Y_pf\rangle = \langle g, AT_pf\rangle$ for all g in $\overline{\mathcal{N}}^{\perp}$, all f in $\mathcal{H}$ and all $p = 1, 2, \cdots, n$. So $A^*Y_p = AT_p$ for all $p = 1, 2, \cdots, n$. Thus $A^*AX_p = AA^*X_p (p = 1, 2, \cdots, n)$. Hence $A^*Af = AA^*f$ for all f in $\overline{\mathcal{N}}$. Since $AX_p = Y_p$ and $EY_pf \in \overline{\mathcal{N}}$, $0 = \langle EY_pf, g\rangle = \langle AEX_pf, g\rangle = \langle EX_pf, A^*g\rangle$, $E \in \mathcal{L}$, $g \in \overline{\mathcal{N}}^{\perp}$, $f \in \mathcal{H}$ and $p = 1, 2, \cdots, n$. So $A^*g \in \overline{\mathcal{N}}^{\perp}$ for all g in $\overline{\mathcal{N}}^{\perp}$. Thus $AA^*g = 0$ and $AA^*g = A^*Ag$ for all g in $\overline{\mathcal{N}}^{\perp}$. Hence $AA^* = A^*A$. □

If we modify the proofs of Theorems 3 and 4 a little bit, we can prove the following theorems. So we will omit their proofs.

THEOREM 5. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Let $\{X_n\}$ and $\{Y_n\}$ be two infinite sequences of operators acting on $\mathcal{H}$. Assume that the range of one of X_n's is dense in $\mathcal{H}$. Then the following statements are equivalent.*

(1) *There is a normal operator A in Alg$\mathcal{L}$ such that $AX_j = Y_j (j = 1, 2, \cdots)$ and every E in $\mathcal{L}$ reduces A.*

(2) $\sup \left\{ \frac{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}Y_if_{k,i}\|}{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_if_{k,i}\|} : m_i, l \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H} \right\} < \infty$
and there is a sequence $\{T_n\}$ of operators acting on $\mathcal{H}$ such that

$$\langle Y_pf, Y_qy\rangle = \langle T_pf, T_qy\rangle$$

and

$$\langle Y_pf, X_qy\rangle = \langle X_pf, T_qy\rangle$$

for all f, y in $\mathcal{H}$ and all $p, q = 1, 2, \cdots$.

THEOREM 6. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let $\{X_n\}$ and $\{Y_n\}$ be two infinite sequences of operators acting on $\mathcal{H}$.*

$$\text{Let } \mathcal{K} = \left\{ \sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_if_{k,i} : m_i, l \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H} \right\}.$$

Assume that $EY_pf \in \overline{\mathcal{K}}$ for all E in $\mathcal{L}$, all f in $\mathcal{H}$ and all $p = 1, 2, \cdots$. Then the following statements are equivalent.

(1) *There is a normal operator A in Alg$\mathcal{L}$ such that $AX_j = Y_j (j = 1, 2, \cdots)$ and every E in $\mathcal{L}$ reduces A.*

(2) $\sup\left\{\frac{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}Y_i f_{k,i}\|}{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_i f_{k,i}\|} : m_i, l \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} < \infty$
and there is a sequence $\{T_n\}$ *of operators acting on* $\mathcal{H}$ *such that*

$$\langle EY_p f, Y_q y\rangle = \langle ET_p f, T_q y\rangle, \langle EY_p f, X_q y\rangle = \langle EX_p f, T_q y\rangle$$

and

$$\langle g, T_p f\rangle = 0$$

for all E *in* $\mathcal{L}$, *all* f, y *in* $\mathcal{H}$, *all* g *in* $\overline{\mathcal{K}}^{\perp}$ *and all* $p, q = 1, 2, \cdots$.

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