A NUMBER SYSTEM IN $\mathbb{R}^n$

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ABSTRACT. In this paper, we establish a number system in $\mathbb{R}^n$ which arises from a Haar wavelet basis in connection with decompositions of certain Cuntz algebra representations on $L^2(\mathbb{R}^n)$. Number systems in $\mathbb{R}^n$ are also of independent interest [9]. We study radix-representations of $x \in \mathbb{R}^n$:

$$x := a_1 a_{i-1} \cdots a_1 a_0 \cdot a_{-1} a_{-2} \cdots$$

as

$$x = M^i a_i + \cdots + M a_1 + a_0 + M^{-1} a_{-1} + M^{-2} a_{-2} + \cdots$$

where each $a_k \in D$, and $D$ is some specified digit set. Our analysis uses iteration techniques of a number-theoretic flavor. The viewpoint is a dual one which we term “fractals in the large vs. fractals in the small,” illustrating the number theory of integral lattice points vs. “fractions”.

1. Introduction

D. E. Knuth [12] had raised the question of describing, for a given positive integer $M$ which will be called the radix, or base, those finite sets $D$ of real numbers with the property that every real number $r$ can be represented in the form

$$(1.1) \quad r = \pm \sum_{i=-N(r)}^{\infty} a_i M^{-i}, \quad a_i = d_i(r) \in D.$$  

If every real number has a representation then the set $D$ will be called feasible for radix $M$. The most common representation is the one we are using with $M = 10$ and the set $D = \{0, 1, 2, \ldots, 8, 9\}$. If we defined the set $T = \{ r \mid r = \sum_{i=1}^{\infty} a_i 10^{-i}, a_i \in \{0, 1, 2, \ldots, 8, 9\} \}$, then the set $T$ is the closed interval $[0,1]$. On the other hand, this kind of radix

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representation in $\mathbb{R}^n, n \geq 2$, is not much known yet, but the set $\mathbf{T}$ is well known as a fractal tile, see Figure 2 in this paper. Fractal tiles are used for wavelet base in digital theory. We study the simplest number system in $\mathbb{R}^n, n \geq 2$ which is similar to the number system in $\mathbb{R}$ which we are using. In section 2, we construct a radix-representation in $\mathbb{Z}^n$ using fractals in large. In section 3, we use a fractal tile in the small to extend a radix-representation of $\mathbb{Z}^n$ to $\mathbb{R}^n$. In section 4, we show an example of radix-representation of $\mathbb{R}^2$, and as an application of the radix-representation $\mathbb{Z}^n$, we introduce decomposition of representations of certain $C^*$-algebras, so called Cuntz algebra. We give some remarks about uniqueness of the radix-representations of $x$ in $\mathbb{R}^n$.

2. A radix-representation in $\mathbb{Z}^n$

We start with elementary facts. If $M$ is a fixed positive integer with $1 < M$, then for any given positive integer $x$, there exists a nonnegative integer $l$ and a set of $l+1$ integers $a_0, a_1, \ldots, a_l$ such that $x$ may be represented uniquely in the following form:

\[(2.1) \quad x = M^l a_l + \cdots + Ma_1 + a_0\]

with $0 \leq a_i < M$ for $i \neq l$ and $0 < a_l < M$.


The following questions arise immediately:

1. What are the conditions for uniqueness in (2.1) when we consider $x \in \mathbb{Z}$ without the restriction $x > 0$? How do we apply this in $\mathbb{Z}^n$?
2. Is there a radix-representation in $\mathbb{Z}^n$, $n = 2, 3, \ldots$, with a single base $M$ in the sense of (2.1)?
3. How do we extend a radix-representation in $\mathbb{Z}^n$ to $\mathbb{R}^n$?

**Lemma 2.1.** Let $M \geq 3$ be an integer and

\[D := \left( -\frac{M}{2}, \frac{M}{2} \right] \cap \mathbb{Z}\]

a digit set. Then there exists a unique radix-representation for each integer $x$, i.e., for any $x \in \mathbb{Z}$, there exists a unique nonnegative integer $l$ and unique integers $a_0, a_1, \ldots, a_l$ where $a_j \in D$ for $0 \leq j < l$ and $a_l \in D - \{0\}$, such that

\[x = M^l a_l + M^{l-1} a_{l-1} + \cdots + Ma_1 + a_0.\]
Proof. Let \( D_0 = D \) and \( D_n = \{ mn + d \mid d \in D_0 \} \) for \( n = \ldots, -2, -1, 0, 1, 2, \ldots \). Let \( x \) be an arbitrary integer. Since \( \mathbb{Z} = \bigcup_{n \in \mathbb{Z}} D_n \) and \( D_n \cap D_m = \emptyset \) unless \( n = m \), there exists a unique integer \( n_1 \) such that \( x \in D_{n_1} \). If \( n_1 = 0 \), then \( x = a_0 \) for some \( a_0 \in D \); otherwise, we have
\[
x = Mn_1 + a_0
\]
for some \( a_0 \in D \). If \( -\frac{M}{2} < n_1 \leq \frac{M}{2} \), then replace \( n_1 \) by \( a_1 \in D \) so that we have
\[
x = Ma_1 + a_0.
\]
If not, we repeat the above step for \( n_1 \) to get
\[
x = Mx_1 + a_0 \\
= M(Mn_2 + a_1) + a_0 \\
= M^2n_2 + Ma_1 + a_0
\]
with \( a_0, a_1 \in D \). Since \( |n_1| > |n_2| > \cdots \), there exists a smallest positive integer \( l \) such that \( -\frac{M}{2} < n_l \leq \frac{M}{2} \). Put \( a_l = n_l \in D \) so that we have
\[
x = M^l a_l + \cdots + Ma_1 + a_0.
\]
This completes the proof. \( \square \)

For an integer \( M \geq 3 \), define
\[
P_M = \{ M^l a_l + \cdots + Ma_1 + a_0 \mid a_0, \ldots, a_{l-1} \in D, \ a_l \in D - \{0\} \}
\]
where the digit set \( D := (-\frac{M}{2}, \frac{M}{2}] \cap \mathbb{Z} \). There then exists an one-to-one and onto natural correspondence between the integer set \( \mathbb{Z} \) and the set \( P_M \) of polynomials by Lemma 1. Since the set
\[
I := \left\{ \sum_{k=1}^{\infty} M^{-k} a_k \mid a_k \in D \right\}
\]
is an interval of length 1, every real number \( r \in \mathbb{R} \) can be decomposed into
\[
r = p + q
\]
where \( p \in P_M \) and \( q \in I \). In the following, we shall establish a number system in \( \mathbb{R}^n \), \( n = 2, 3, \ldots \), in the manner described for \( \mathbb{R} \).

The choice of a base \( M \) and a digit set \( D \) for construction of a radix-representation in \( \mathbb{Z}^n \). Let \( M \) be an \( n \times n \) matrix with integer entries. Then \( |\det M| = N \) is a positive integer. The set \( U \)
denotes an open-closed hypercube \( \prod_{i=1}^{n} \left(-\frac{1}{2}, \frac{1}{2}\right) \) in \( \mathbb{R}^n \). By elementary linear algebra the set

\[
MU := \{ Mx \mid x \in U \}
\]

has Lebesgue measure \( N \), and the set \( MU \cap \mathbb{Z}^n \) has exactly \( N \) elements which consists a residue set modulo \( M\mathbb{Z}^n \). In particular, for \( x, y \in MU \cap \mathbb{Z}^n \), \( x - y \notin M\mathbb{Z}^n \) unless \( x = y \).

Define \( C_0 \) to be the set

\[
C_0 := \{ x = (x_1, \ldots, x_n) \in \mathbb{Z}^n \mid x_i = 0 \text{ for all but one } i, \ 1 \leq i \leq n, \ i = i_0, \ x_{i_0} = 1 \text{ or } -1 \}
\]

and let

\[
C = C_0 \cup \{0\}.
\]

**Theorem 2.2.** Let \( M \) be an \( n \times n \) matrix with integer entries and

\[
U := \prod_{i=1}^{n} \left(-\frac{1}{2}, \frac{1}{2}\right).
\]

If

1. \( C \subset MU \) (or equivalently \( C \subset MU \cap \mathbb{Z}^n \)),
2. \( \lim_{k \to \infty} M^k U = \mathbb{R}^n \),

then for every \( x \in \mathbb{Z}^n \), a nonnegative integer \( l \) and points \( a_0, a_1, \ldots, a_{l-1} \in D_{l-1} \) and \( a_l \in D \setminus \{0\} \) are uniquely determined by the formula

\[
x = M^l a_l + \cdots + M a_1 + a_0.
\]

**Remark 2.3.** It seems likely that the condition \( \lim_{k \to \infty} M^k U = \mathbb{R}^n \) is equivalent to \( M \) satisfying the condition that every eigenvalue of \( M \) has modulus greater than 1, but this remains as an open question. However, it can be shown that \( \lim_{k \to \infty} M^k U = \mathbb{R}^n \) by diagonalization of the matrix with the diagonal entries being those real eigenvalues if \( M \) has only real eigenvalue greater than 1.

**Proof of Theorem 2.2.** Let \( D_0 = D := MU \cap \mathbb{Z}^n \) be the digit set. Define the sequence of sets \( D_k \), \( k = 0, 1, 2, \ldots \), in \( \mathbb{Z}^n \) inductively as

\[
D_k = \bigcup_{x \in D_{k-1}} \{ Mx + d \mid d \in D \}.
\]

Then we have \( D_0 \subseteq D_1 \subseteq D_2 \subseteq \cdots \). To show \( \lim_{k \to \infty} D_k = \mathbb{Z}^n \), observe that

\[
D_{k+1} = \mathbb{Z}^n \cap \{ M(x + U) \mid x \in D_k \}
\]

for each \( k = 0, 1, 2, \ldots \). See figure 1 on the next page.

By induction, suppose

\[
M^k U \cap \mathbb{Z}^n \subset D_{k+1}.
\]
Then

\[ M^k U \subset \bigcup_{x \in D_{k+1}} (x + U) \]

and

\[ M^{k+1} U \subset \bigcup_{x \in D_{k+1}} M(x + U). \]

Thus

\[ M^{k+1} U \cap \mathbb{Z}^n \subset \left( \bigcup_{x \in D_{k+1}} M(x + U) \right) \cap \mathbb{Z}^n. \]

By (2.3) we have

\[ M^{k+1} U \cap \mathbb{Z}^n \subset D_{k+2}. \]

Since, by our initial hypotheses, \( \lim_{k \to \infty} (M^{k+1} U \cap \mathbb{Z}^n) = \mathbb{Z}^n \), we have \( \lim_{k \to \infty} D_k = \mathbb{Z}^n \). Now consider any \( x \in \mathbb{Z}^n \). If \( x \in D_0 \), we are done.
Otherwise, \( x \notin D_0 \), and so there exists an unique nonnegative integer \( l \) such that \( x \in D_l \) and \( x \notin D_{l-1} \), which ensures that there exists a unique \( x_1 \in D_{l-1} \) and \( a_0 \in D \) such that

\[
x = Mx_1 + a_0.
\]

If \( l > 1 \), by repeating this step \( l \) times we get

\[
x = M^l x_1 + M^{l-1} a_{l-1} + \cdots + Ma_1 + a_0
\]

where \( a_0, a_1, \ldots, a_{l-1} \in D \) and \( x_1 \in D_{l-l} = D_0 = D \). Since \( x \notin D_{l-1} \), we have \( x_1 \neq (0, \ldots, 0) \) in \( \mathbb{Z}^n \). We complete the proof by setting \( a_l = x_l \in D - \{0\} \).

\[
\square
\]

**Remark 2.4.** Since the cardinality of \( C \) is \( 2n + 1 \), \( C \) is a subset of \( D \) and the cardinality of the digit set \( D \) is the Lebesgue measure of \( MU \) which is \( |\text{det} M| = N \), we need an integer \( n \times n \) matrix satisfying \( |\text{det} M| \geq 2n + 1 \) in Theorem 2.2.

### 3. A radix-representation in \( \mathbb{R}^n \)

We say that an \( n \times n \) integer matrix \( M_1 \) is **integrally similar** to another \( n \times n \) matrix \( M_2 \) if there exists some \( Q \in \text{GL}(n, \mathbb{Z}) \) such that \( M_2 = Q M_1 Q^{-1} \). We say that an \( n \times n \) matrix is **integrally reducible** if \( M \) is integrally similar to a matrix

\[
\begin{pmatrix}
A_1 & 0 \\
B & A_2
\end{pmatrix}
\]

where \( A_1, A_2 \) are \( r \times r \) and \( (n-r) \times (n-r) \) matrices respectively for some \( 1 \leq r \leq n-1 \) such that \( |\lambda_{1i}| > 1 \) and \( |\lambda_{2i}| > 1 \) for all eigenvalues \( \lambda_{1i} \) of \( A_1 \) and \( \lambda_{2i} \) of \( A_2 \). We call \( M \) **integrally irreducible** if it is not integrally reducible.

**Theorem 3.1.** Let \( M \) be an \( n \times n \) integer matrix such that

1. \( C \subseteq MU \),
2. \( \lim_{k \to \infty} M^k U = \mathbb{R}^n \),
3. every eigenvalue of \( M \) has modulus greater than \( 1 \),
4. \( M \) is integrally irreducible.

Then we have a radix-representation of \( x \in \mathbb{R}^n \),

\[
x := a_l a_{l-1} \cdots a_2 a_1 a_0 \cdot a_{-1} a_{-2} \cdots,
\]

in the sense that \( x = M^l a_l + \cdots + Ma_1 + a_0 + M^{-1} a_{-1} + M^{-2} a_{-2} + \cdots \)

where \( a_k \in D \) for \( k = l-1, l-2, \ldots, 1, 0, -1, \ldots \) and \( a_l \in D - \{0\} \) for some nonnegative integer \( l \).
Figure 2. $\bigcup\{ T + d \mid d \in MU \cap \mathbb{Z}^2 \}$ with $M = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$

**Corollary 3.2.** Suppose that an $n \times n$ matrix $M$ with integer entries satisfies (3) and (4) in Theorem 3.1. If $D := MU \cap \mathbb{Z}^n$, then the set $T := \{ \sum_{k=1}^{\infty} M^{-k}d_{ik} \mid d_{ik} \in D \}$ is lattice-tiles $\mathbb{R}^n$ with lattice $\mathbb{Z}^n$.

**Proof.** See figure 2 or [1], [7], or [11]. \qed

**Proof of Theorem 3.1.** For $x \in \mathbb{R}^n$, by virtue of Corollary 3.2, we have $x = z + t$ for $z \in \mathbb{Z}^n$ and $t \in T$. With conditions (1) and (2) in Theorem 3.1, we have

$$z = M^l a_l + M^{l-1} a_{l-1} + \cdots + Ma_2 + Ma_1 + a_0 + M^{-1}a_{-1} + M^{-2}a_{-2} + \cdots.$$  

Like a decimal expansion in $\mathbb{R}$ we have a representation for $x$,

$$a_la_{l-1}\cdots a_2 a_1 a_0 \cdot a_{-1} a_{-2} \cdots.$$ \qed

**Example 1.** Consider a $2 \times 2$ matrix $M = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ with $|\det M| = 5$. It is easy to see that $C \subset D$ where $D = \{ (0, 0), (1, 0), (-1, 0), (0, -1) \}$ (in fact $C = D$), $\lim_{k \to \infty} M^k U = \mathbb{R}^n$, and the eigenvalues of $M$ are
\[ \lambda_1 = 2 + i, \lambda_2 = 2 - i \text{ with } |\lambda_i| = \sqrt{5} > 1, \ i = 1, 2. \] If a matrix \( A \) is integrally similar to a matrix \( \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix} \), then \( |\det A| = |\det A_1| \cdot |\det A_2| \). Furthermore, the characteristic polynomial of \( A \) is irreducible over \( \mathbb{Q} \) implies that \( A \) is integrally irreducible. Thus if \( |\det A| = p \), where \( p \) is prime, then \( A \) is integrally irreducible. We have shown that the matrix \( \begin{pmatrix} \frac{2}{-1} & 1 \\ \frac{1}{2} & -1 \end{pmatrix} \) fulfills the conditions (1)–(4) in Theorem 3.1. A point \( \begin{pmatrix} 4 & 6 \\ -6 & 1 \end{pmatrix} \) in \( \mathbb{R}^2 \) can be represented as following:

\[
\begin{pmatrix} 4 & 6 \\ -6 & 1 \end{pmatrix} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix},
\]
in the sense of \( \begin{pmatrix} \frac{2}{-1} & 1 \\ \frac{1}{2} & -1 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{2}{-1} & 1 \\ \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} \frac{2}{-1} & 1 \\ \frac{1}{2} & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.\]

4. Applications and concluding remarks

There is an application of the number system in \( \mathbb{Z}^n \) described in this paper. The Cuntz algebra \( O_n \) is the \( C^* \)-algebra generated by \( N \) elements \( s_1, \ldots, s_N \) satisfying

\[ s_i^* s_j = \delta_{ij} I \quad \text{and} \quad \sum_{i=1}^N s_i s_i^* = I. \tag{4.1} \]

One of the recent developments of the study of \( O_N \) is the representations on Hilbert spaces \( L^2(\mathbb{R}^n) \) or \( L^2(\mathbb{T}^n) \). The infinite nature of \( O_N \) with Cuntz properties fits into a fractal property which gives certain type of representations from wavelet, [3] or [6]. Another type of representations of the Cuntz algebra \( O_N \) is closed related to the study of the endomorphisms of \( B(\mathcal{H}) \), where \( B(\mathcal{H}) \) is the \( C^* \)-algebra of bounded linear operators on a separable, infinite dimensional Hilbert space \( \mathcal{H} \), [3], [4], and [10]. There is a correspondence between endomorphisms of \( B(\mathcal{H}) \) of Powers index \( N \) and representations of \( O_N \) up to unitary action. If the representation of \( O_N \) is irreducible, then the corresponding endomorphism is an ergodic of Powers index \( N \), vice versa. A UHF algebra is a norm separable \( C^* \)-algebra which is the norm closure of an increasing sequence of type \( I_{n_i} \)–factors and so a UHF algebra can be identified with the tensor product matrix algebra, \( \bigotimes_{i=1}^\infty M_{n_i}, \ n_i \in \{2, 3, \ldots\} \). When \( n_i = N \), for all \( i \), a UHF algebra is denoted by \( \text{UHF}_N \) which we can understand as a subalgebra of \( O_N \), where the element \( e_{i_1 j_1} \otimes e_{i_2 j_2} \otimes \cdots \otimes e_{i_k j_k} \otimes I \otimes \cdots \) in \( \text{UHF}_N \) is identified with \( s_{i_1} \cdots s_{i_k} s_{j_1}^* \cdots s_{j_k}^* \) in \( O_N \). Thus a representation of \( \text{UHF}_N \) also induces an endomorphism on \( B(\mathcal{H}) \). If the image of
UHF$_N$ under representation is weakly dense in $B(\mathcal{H})$, the corresponding endomorphism is shift of Powers index $N$ [4] and [10]. However, the representation of $O_N$ and UHF$_N$ are famous examples whose representations are bad [2], [3], [4], and [8] among many others. The representation $\pi$ of $O_N$, and UHF$_N$ satisfying

$$\pi(s_i)(e_x) \in \{e_x : x \in \mathbb{Z}^n\},$$

is related with tight frame in wavelet, [6], and endomorphism on the Hilbert space $B(L^2(\mathbb{R}^n))$, was studied in [2] and [3], where $\{e_x : x \in \mathbb{Z}^n\}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$ or $L^2(\mathbb{T}^n)$ where $\mathbb{T} = \mathbb{R}/2\pi \mathbb{Z}$. This kind of representations are predominant in papers, [2], [3], [4], and [8]. The hard part is to find the irreducible subrepresentations. Using our number system in $\mathbb{Z}^n$, we are able to show that the representation $\pi$ of $O_n$ defined by

$$\pi(s_i)e_x = e_{Mx+d_i}$$

with $d_i \in D := MU \cap \mathbb{Z}^n$ is irreducible under conditions in Theorem 2. Under same condition in Theorem 2, if we take a residue set $D$ modulo $MZ^n$ in $\mathbb{Z}^n$, we then have rich representations of $O_N$ and UHF$_N$, but these are not irreducible, in general. We use our number system in $\mathbb{Z}^n$ to show that the representations defined by $\pi(s_i)e_x = e_{Mx+d_i}$ with a residue set $D$ modulo $MZ^n$ in $\mathbb{Z}^n$, decomposed into finite irreducible subrepresentations, see [8].

A number system in $\mathbb{R}^n$, at this moment, is rather novel. When $M$ is an integrally irreducible matrix whose eigenvalues have modulus greater than 1, then the set $\mathbf{T} := \{\sum_{k=1}^{\infty} M^{-k}d_k \mid d_k \in D\}$ has the following tiling property: the Lebesgue measure of $(x + \mathbf{T}) \cap (y + \mathbf{T})$ is 0 or 1 for $x, y \in \mathbb{Z}^n$. See figure 2. Since the set $\mathbf{T}$ is compact, there are cases when $(x + \mathbf{T}) \cap (y + \mathbf{T}) \neq \emptyset$ even though it has Lebesgue measure 0. As a result, the radix-representation in $\mathbb{R}^n$,

$$a_1a_{l-1} \cdots a_2a_1a_0 \cdot a_{-1}a_{-2} \cdots,$$

described in Theorem 3.1, is not unique. If we take a subset $\mathbf{T}$ of the set $\mathbf{T}$ satisfying

$$\bigcup_{x \in \mathbb{Z}^n} (x + \mathbf{T}) = \mathbb{R}^n$$

and

$$(x + \mathbf{T}) \cap (y + \mathbf{T}) = \emptyset \quad \text{if} \quad x \neq y$$

for $x, y \in \mathbb{Z}^n$, the radix-representation $a_1a_{l-1} \cdots a_2a_1a_0 \cdot a_{-1}a_{-2} \cdots$ is unique. For example, in $\mathbb{R}$, this situation can be remedied by setting
\( \hat{T} := T - T_9 \) for
\[
T = \left\{ \sum_{k=1}^{\infty} 10^{-k}d_{ik} \mid d_{ik} \in \{0, 1, \ldots, 9\} \text{ for all } k \right\} = [0, 1],
\]
\( T_9 = \{ t \in T \mid \text{there exists a positive integer } n_0 \)
such that \( d_k = 9 \) for all \( k \geq n_0 \} \).

We do not have a general result in \( \mathbb{R}^n \), but the only case is studied. With the matrix \( M \) and the digit set \( D \) in the above Example 1, we are currently investigating possible unique radix-representations in \( \mathbb{R}^n \) using four \( \hat{T} \)'s defined as follows:
\[
\hat{T} = T - T_i, \quad i = 1, 2, 3, 4,
\]
with
\[
T = \left\{ \sum_{k=1}^{\infty} M^{-k}d_{ik} \mid d_{ik} \in D \right\},
\]
\( T_1 = T \{ (0) \} \cup T \{ (1) \} \),
\( T_2 = T \{ (0) \} \cup T \{ (1) \} \),
\( T_3 = T \{ (0) \} \cup T \{ (1) \} \),
and
\[
T_4 = T \{ (0) \} \cup T \{ (1) \} ,
\]
where
\[
T \{ (0) \} := \left\{ \sum_{k=1}^{\infty} M^{-k}d_{ik} \mid d_{ik} \in \{ (0) \} \right\}
\]
with analogous definitions for \( T \{ (1) \} \), \( T \{ (0) \} \), etc. The resulting structure suggests general results for unique radix-representation in \( \mathbb{R}^n \). But, at least, we have a number system in \( \mathbb{R}^n \) which is similar to the familiar one in \( \mathbb{R} \).

References

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