

Mathematical Model of Optimal Payouts under Non-linear Demand Curve*

Chaehwan Won**

Department of Business Administration
Sejong University, 98 Kunja-Dong, Kwangjin-Ku, Seoul, 143-747, Korea

(Received May 2004 ; Revised Oct. 2004 ; Accepted Oct. 2004)

ABSTRACT

In this study, a mathematical model that shows the optimal payout policy is developed. The model is new and unique in the sense that not only continuous-time framework is used, but also both partial differential equation (PDE) and real-option approach are utilized in the derivation of optimal payouts for the first time. In the model building, non-linear demand curve for dividend payouts in the competitive capital markets is assumed.

From the sensitivity analysis using traditional comparative static analysis, some useful managerial implications which are consistent with famous previous studies are derived under realistic conditions. All results in this study, however, are valid under the assumption that the opportunity costs follow geometric Brownian motion, which is widely used in economic science and finance literature.

Keywords: Dividend payout, Geometric Brownian motion, Equilibrium model

1. INTRODUCTION

Previous studies regarding dividend payouts show that dividend policy is irrelevant in all instances regardless of the existence of growth or corporate taxes. It has no effect on shareholder's wealth. Only when personal taxes are introduced do we have a result where dividend payouts matter. For shareholders who pay high taxes on dividends than on capital gains, the preferred dividend payout is zero; they would rather have the company distribute cash payments via the share re-

* The author is grateful for helpful comments and suggestions from the two anonymous referees and the members of Sejong Finance Club, David Jinhyung Cho, Seok-Kyu Lee, Chang-Mo Ahn, Soon-Jae Lee, and Jae-Wook Jung. Remaining error, if any, is mine.

** Email: chwon@sejong.ac.kr

purchase mechanism.

Yet, it is well-known fact that corporations do pay dividends in reality. Since there appear to be strong cross-sectional regularities in dividend payout, there may be optimal dividend policy which results from a trade-off between the costs and benefits of paying dividends as Rozeff (1982) suggests. The list of possible costs includes tax advantages of receiving income in the form of dividends rather than capital gains and the cost of raising external capital if dividends are paid out. On the other hand, the possible benefits of dividend payouts are higher perceived corporate value because of the signaling content of dividend, the lower agency costs of external equity, and the ability of dividend payments to help complete markets. In other words, there is the possibility that we can derive the optimal payout policy of an individual firm under some conditions imposed by the financing and investment policy of the firm. As Brealey and Myers (1991) point out, however, the dividend policy of a firm still remains one of the most controversial subjects in the modern finance theory. They argue that three major theories about determination of the optimal dividend policy are competing as follows: First, Hansen *et al.* (1994) argue that agency cost of dividends is one of the major factors affecting decision making of payouts. According to this hypothesis, dividend payouts can serve as a way to reduce agency costs. By applying dividends equal to the amount of "surplus" cash flow, a firm can reduce management's ability to squander the firm's resources. Since dispersion of ownership among shareholders is a basic measure of agency costs, it would be expected that firms with high dispersed ownership would have high dividends. Second, Bhattacharya (1979), Miller and Rock (1985), and Nissim and Ziv (2001) suggest 'information content hypothesis' that dividends serve to signal to shareholders the firm's current and future performance. Third, Lowellen *et al.* (1978) propose 'clientele-effect hypothesis' that those individuals in high tax brackets are likely to prefer either no or low dividends, and vice versa. In addition, Gordon (1959, 1962) and Lintner (1962), Modigliani and Brennan (1971), Litzenberger and Ramaswami (1979), Black and Scholes (1974), Miller and Scholes (1978, 1982), Hess (1983), Eades *et al.* (1984), Benartzi *et al.* (1997), DeAngelo *et al.* (1996) and Nissim and Ziv (2001), Shiller (1981), Marsh and Merton (1987), Hakansson (1982) and others argue about the relevancy of dividend policy.

As we discussed briefly, most of the previous studies (with few exceptions, such as Shiller (1981), Marsh and Merton (1987), Hakansson (1982) on both normative and positive models of dividends) focused on the micro behavior of individual firms. In other words, since we could not predict accurately the future behavior of a firm in dividend payout without a theoretical model, the main purpose

of this study is to develop a mathematical model regarding dividends. Also, even though the foregoing studies appear to support MM proposition from the viewpoint of the micro behavior of individual firms, they do not necessarily rule out the possibility that there may exist an aggregate equilibrium supply of dividends. This point motivates this study. Thus, this study is different from previous studies in that it uses an aggregate dividend model similar to Marsh and Merton (1987). In addition, in this study, continuous-time stochastic model is utilized to describe the behavior of aggregate dividends and other key variables affecting dividend policy. The model specified here has a similar form to the real option models, which are widely used in the valuation of real assets nowadays.¹ Therefore, the basic motivation of this study can be summarized as follows: First, in order to explain the relevancy of dividend payouts in the sense that the payouts can increase a firm value, we build a mathematical model. Second, to understand the aggregate dividend behavior of all firms, we apply the equilibrium framework rather than simple descriptive analysis, such as regression analysis. Third, to understand how the optimal dividend policy of a firm should be changed after the change in at least one of the exogenous variables, we exploit the sensitivity analysis. All results in this study, however, are valid under the assumption that the opportunity costs follow geometric Brownian motion, which is widely used in economic science and finance literature.

This paper is organized as follows: In Section 2, we introduce some assumptions necessary for the model building and describe capital market settings. Then, we develop a mathematical model using equilibrium framework in Section 3, and we derive useful managerial implications via sensitivity analyses in Section 4. In Section 5, we conclude the paper with both discussion about the results and future direction of the research.

2. ASSUMPTIONS AND MARKET DESCRIPTION

2.1 Assumptions

To build new model we assume as follow: Capital market is perfectly competitive. Since there exists sufficiently large number of firms issuing equity in the capital

¹ It is well-known fact that the real-option valuation model is one of the hottest in the current financial economics. In this sense, Dixit (1989) and Grenadier (1995) models are very helpful to understand the framework of this study.

market and, actually, the entry and exit to the capital market is almost free, this assumption seems reasonable in the market. The investment decision of a firm is assumed to be independent of the dividend policy. This assumption is directly from the results of previous studies, such as Miller and Modigliani (1961), Fama (1974), and Miller and Scholes (1978). If this assumption does not hold, dividend policy might be a function of investment decision, leading to a more complicated model. Since firms can issue equity and borrow debt for the fund of investment freely in the capital market by the first assumption, this assumption does make sense. We suppose that stock prices follow geometric Brownian motion (hereafter, GBM), indicating that future stock prices conditional on the present price will follow log normal distribution. Needless to say, this stochastic specification is widely used without debate in the financial literature. This assumption is very important to derive the behavior of aggregate dividends in the market. The functional equation to represent the stochastic dynamics (GBM: Geometric Brownian Motion) is as follows.

$$dS = \mu_S S dt + \sigma_S S dZ_S \quad (1)$$

where S is instantaneous price of an asset,² μ_S is instantaneous conditional expected change in S , t is time dimension, σ_S is instantaneous conditional standard deviation of S , and dZ_S is the increment of a standard Wiener process in S . As we can see from Hull (2000, 277-282), stock indices can be considered that they follow continuous stochastic process since each index includes large number of individual stocks paying dividends. Payment of dividends is assumed to incur an 'opportunity cost'. This opportunity cost is due to the fact that the fund distributed as dividends would be invested in other projects with, at least, riskless rate of return, if it were not used as dividends.³ In this study, since the cash flow from any arbitrary investment is random walk, the behavior of an opportunity cost is assumed to follow GBM as follows⁴;

² This is a market index, such as S&P 500 or DJIA.

³ Eades, Hess, and Kim (1984) discuss this issue very well, even though they do not use the term 'opportunity cost' explicitly. They argue in pp.1630-1632 that 'we also expect interest rates on cash-equivalent securities to affect the level of dividend capturing. Corporations usually invest excess cash in short-term and low-risk securities... The lost tax premium of debt, or higher before-tax rate of return, is an opportunity cost of dividend capturing...'

⁴ Like other areas, normal distribution is widely used in finance area also. However, the problem of using simple normal distribution is that the values can fall below zero which is the case of negative NPV. If the investment was predicted to have negative expected

$$dC = \mu_C C dt + \sigma_C C dZ_C \tag{2}$$

where C is instantaneous opportunity cost, μ_C is instantaneous conditional expected change in C , σ_C is instantaneous conditional standard deviation of C , dZ_C is the increment of a standard Wiener process in C . In addition, it is reasonable to think that the cost be a function of dividend and opportunity cost of capital; namely, $C = f(D, r)$, where D is a dividend and r is the opportunity cost of capital. We can consider r as a riskless rate, such as T-bill rate as we can see in the Eades et al. (1984), because corporations usually invest excess cash in short-term and low-risk (almost riskless) securities. They argue that corporate investment in financial securities is typically a temporary use of funds that have a more permanent home (e.g., payment of dividends or investment in real assets)(p.1631). The capital structure for an average firm⁵ is composed of only debt and common equity. This assumption is necessary to find equilibrium cost of capital in the market.

2.2 Non-linear Demand and Supply function

Generally speaking, the value of a firm is a function of many variables, such as cash flows, cost of capital, capital structure, dividend, tax rates, and others. However, since, if we introduce all those variables into a model, the model might be very complex and closed-form solution can hardly be obtained, this study focuses on the ‘marginal effect’⁶ of dividend on the firm value. Since a stock price is a function of an opportunity cost and dividend, a firm value can be expressed as a function of cost and dividend as $V(C, D)$. However, to distinguish a group of firms

NPV, there is no reason that the project should be undertaken. Therefore, in reality, log-normal distribution is preferred to normal distribution to describe the cash flows from an investment. If the conditional distribution of future cash flows given current level of cash flow follows lognormal distribution, we call it GBM. So, this is the reason that most of the literature uses GBM as an appropriate stochastic process for cash flows of arbitrary investment. McDonald and Siegel (1986), Pindyck (1991), Capozza D. and Yuming Li (1994), Dixit and Pindyck (1994), Grenadier (1995), Abel *et al.* (1994), Lang *et al.* (1996), Barnett and Sakellaris (1998), Parrino and Weisbach (1999), Erickson and Whited (2000), Schwartz and Zozaya (2000), Boyle and Guthrie (2003), and Hennessy (2004) are typical examples utilizing GBM in their papers.

⁵ Even though firms can use different kinds of assets, such as preferred stocks, warrants, and others, for the necessary supply of funds, we assume that most of the firms (so, an average firm) have the capital structure of common stocks, and debt.

⁶ Here, marginal effect means that only the change in dividend policy is considered hold-

that do not pay dividends currently from a group of firms that pay dividends currently, two definitions in the firm value will be used as follows;⁷

$V(C, D)$ = the value of a firm in the group which is waiting for the optimal moment to pay a dividend,

$W(C, D)$ = the value of a firm in the group which is currently paying dividends (3)

According to previous studies, it is a very common fact that the dividend announcement affects stock price, even though the underlying reason for the price change is somewhat controversial (i.e., some argue signaling or information effect and others tax effect) according to Litzenberger and Ramaswamy (1982) and Hess (1982). Thus, it is reasonable to posit that a dividend itself has some effect on the stock price. In addition, as Litzenberger and Ramaswamy (1982) and other researchers argue, dividends are non-linearly associated with stock prices. Thus, in this study, the following inverse demand function is used to describe the demand function for the dividends;

$$S(t) = Q(t)^{-\alpha} * D(t), \quad (4)$$

where $S(t)$ is the stock index at time t , $Q(t)$ is the aggregate supply of equity at time t in the market, $D(t)$ is the aggregate supply of dividends at time t in the market, and α is assumed a constant elasticity⁸ of the demand for and equity. In the equation (4), we can think that dividends have multiplicative (i.e., non-linear) demand shock on stock indices. In addition, the supply of dividends depends on competitive firms that may pay dividends at any time at the expense of 'opportunity costs'. Since while dividend-paying firms can make additional capital gains by paying dividends, they also should pay opportunity costs, the net cash flows at each time will be the difference between additional capital gains and opportunity costs.

Therefore, the interaction between competitive, value maximizing supplies of and derived demand for dividends results in an equilibrium determination of optimal aggregate dividend level in the market. Before we derive the value of a firm, it is necessary to mention the reason that the continuous-time, not discrete,

ing other policy variables like a capital structure and an investment strategy constant.

⁷ This notation is similar to those used in Dixit (1989) and in Grenadier (1995).

⁸ In this model, price elasticity of demand is assumed to be constant. This means that the demand for dividends is constant regardless of the level of market index. This seems reasonable, because those who want dividends buy some equity without worrying about the level of market index.

model can be used in the dividend study. If we focus on the individual firm, it is truly evident that the dividend payment is discrete event occurring at specific dividend-payment dates (e.g., since most firms pay dividend each quarter, dividends occur only four times a year). But, if we consider all firms in the markets, it is also natural that we use continuous occurrence of dividends around the year. Thus, as we can use a continuous dividend-yield model which is frequently adopted in the pricing of options (e.g., options in stock index) with dividend payment⁹, we can use a continuous-time model for the aggregate dividend, because aggregate equity produce almost continuous flows of dividend payments.

3. NEW MODEL OF DIVIDEND DYNAMICS

Under given assumptions and the demand and supply mechanism described above, the value of a firm can be derived by applying a stochastic calculus. From equation (4), the inverse demand function can be rewritten as follows;¹⁰

$$D = S * Q^\alpha. \quad (5)$$

By applying Ito's Lemma to equation (5), we can derive the following Lemma 1:

$$\textbf{Lemma 1.} \quad dD = \mu_S D dt + \sigma_S D dZ_S + (\alpha D/Q) dQ \quad (6)$$

$$\begin{aligned} \textbf{Proof.} \quad dD &= D_S dS + D_Q dQ + \frac{1}{2} [D_{SS} (dS)^2 + 2D_{SQ} dS dQ + D_{QQ} (dQ)^2] \\ &= Q^\alpha [\mu_S S dt + \sigma_S S dZ_S] + \alpha \left(\frac{D}{Q}\right) dQ \quad (\text{from equation (1) and (5)}) \\ &= \mu_S (SQ^\alpha) dt + \sigma_S (SQ^\alpha) dZ_S + \left(\alpha \frac{S}{Q}\right) (Q^\alpha) dQ \\ &= \mu_S D dt + \sigma_S D dZ_S + (\alpha D/Q) dQ \quad (\text{from equation (5)}). \quad \textit{Q.E.D.} \end{aligned}$$

In the traditional option-pricing literature, contingent claims are priced through arbitrage arguments. However, such an approach requires assumptions about free tradability (or liquidity) of the underlying asset. The dividend itself is not freely tradable, even though equity with dividend has liquidity, and much trans-

⁹ In the Black-Scholes European option pricing equation with dividend payment, dividend yield is considered by multiplying e^{-yt} , where y is a dividend yield, to the asset price in the equilibrium setting.

¹⁰ For the notational convenience, t (time dimension) will be dropped hereafter.

action cost is necessary to buy dividend through equity. In addition short sale is impossible for dividend. Thus, such arbitrage arguments are not applicable to the dividend case. Clearly, in the absence of a freely traded underlying security (e.g., typically, the value of a piece of land on which a call option is written), an equilibrium approach may be used, even though an appropriate equilibrium model must be chosen. In this study, continuous-time equilibrium model similar to Dixit (1989) is applied.

Consider the instantaneous return on $W(C, D)$ over a region in which a dividend policy is unchanging. By Ito's Lemma, the instantaneous change in W is obtained as following Lemma 2:

$$\text{Lemma 2. } dW = \left[\frac{1}{2} \sigma_C^2 C^2 W_{CC} + \rho \sigma_C \sigma_S CDW_{CD} + \frac{1}{2} \sigma_S^2 D^2 W_{DD} + \mu_C CW_C + \mu_S DW_D \right] dt + \sigma_C CW_C dZ_C + \sigma_S DW_D dZ_S, \quad (7)$$

where ρ is the correlation coefficient between dZ_C and dZ_S .

$$\begin{aligned} \text{Proof. } dW &= W_C dC + W_D dD + \frac{1}{2} [W_{CC} dC^2 + 2W_{CD} dC dD + W_{DD} dD^2] \quad (\text{by Lemma 1}) \\ &= W_C [\mu_C C dt + \sigma_C C dZ_C] + W_D [\mu_S D dt + \sigma_S D dZ_S + (\alpha D/Q) dQ] \\ &\quad + \frac{1}{2} W_{CC} \sigma_C^2 C^2 dt + W_{CD} [\mu_C C dt + \sigma_C C dZ_C] [\mu_S D dt + \sigma_S D dZ_S \\ &\quad + (\alpha/Q) dQ] + \frac{1}{2} W_{DD} \sigma_S^2 D^2 dt \\ &= \frac{1}{2} \sigma_C^2 C^2 W_{CC} + \rho \sigma_C \sigma_S CDW_{CD} + \frac{1}{2} \sigma_S^2 D^2 W_{DD} + \mu_C CW_C + \mu_S CW_D \Big] dt \\ &\quad + \sigma_C CW_C dZ_C + \sigma_S DW_D dZ_S. \quad \text{Q.E.D.} \end{aligned}$$

In addition to the capital gains earned on the asset, dW , the asset also yields additional cash flows due to dividend payment of $E\left(\frac{dS}{S}\right)D$ and¹¹ an opportunity cost of $E\left(\frac{dC}{C}\right)D$, indicating that the additional net capital gain due to dividend payment is $E\left[\frac{dS}{S} - \frac{dC}{C}\right]D$ where E is an expectation operator. But, from equations (1) and (2),

¹¹ Since the expected total capital gain earned from the investment in equity is $E[dS/S](N \times S)$, where N is total number of stocks outstanding, the capital gain which we can get by investing the amount of D in the stocks with same expected return is $E(dS/S)D$.

$$\begin{aligned}
 E\left(\frac{dS}{S}\right) &= E(\mu_S dt + \sigma_S dZ_S) = \mu_S dt, \text{ and} \\
 E\left(\frac{dC}{C}\right) &= E(\mu_C dt + \sigma_C dZ_C) = \mu_C dt. \text{ Thus,} \\
 E\left(\frac{dS}{S} - \frac{dC}{C}\right) D &= (\mu_S - \mu_C) D dt \tag{8}
 \end{aligned}$$

Therefore, the total expected return on W per unit time is, from Lemma 2 in equation (7) and (8),

$$\begin{aligned}
 kwWdt &= E(dW) + E\left(\frac{dS}{S} - \frac{dC}{C}\right) D \\
 &= \left[\frac{1}{2}\sigma_C^2 C^2 W_{CC} + \rho\sigma_C\sigma_S CDW_{CD} + \frac{1}{2}\sigma_S^2 D^2 W_{DD} + \mu_C CW_C\right. \\
 &\quad \left. + \mu_S DW_D\right]dt + (\mu_S - \mu_C)Ddt
 \end{aligned}$$

Simplifying above equation yields the following equilibrium partial differential equation (PDE);¹²

$$0 = \frac{1}{2}\sigma_C^2 C^2 W_{CC} + \rho\sigma_C\sigma_S^2 D^2 W_{DD} + \mu_C CW_C + \mu_S DW_D - kwW + (\mu_S - \mu_C)D \tag{9}$$

With the same reasoning as $W(C, D)$, PDE for the $V(C, D)$ can be derived as follows;¹³

$$0 = \frac{1}{2}\sigma_C^2 C^2 V_{CC} + \rho\sigma_C + \sigma_S CDV_{CD} + \frac{1}{2}\sigma_S^2 C^2 V_{DD} + \mu_C CV_C + \mu_S DV_D - kvV \tag{10}$$

To solve two partial differential equations (9) and (10), we should know additional boundary conditions. Since the values of two kinds of firms, $V(C, D)$ and $W(C, D)$, are connected by individual firm behavior and equilibrium conditions, it is necessary to consider individual firm behavior. A firm currently not paying dividends will decide to pay dividend and opportunity cost, $C(t)$, when it is individually optimal. Solving this problem necessitates two types of condition. That is, the value matching condition is $V(C^*, D^*) = W(C^*, D^*) - C^*$, where C^* and D^* de-

¹² kw is the cost of capital for the firm which has capital structure in assumption. In the equilibrium, it is determined in the market as follows; $kw = k_d(1-\tau)[B/(B+E)] + k_e[B/(B+E)]$, where k_d and k_e are the cost of debt and the cost of equity, respectively, τ is a corporate tax rate, and B and E denote the total amount of debt and equity, respectively. We can apply the CAPM to derive the equilibrium rates of k_d and k_e .

¹³ kv is equal to the kw if a dividend policy is independent of the capital structure of a firm.

note values of $C(t)$ and $D(t)$ which trigger to pay dividends; and smooth-pasting or high-contact conditions are $V_C(C^*, D^*) = W_C(C^*, D^*) - 1$ and $V_D(C^*, D^*) = W_D(C^*, D^*)$.¹⁴ These conditions guarantee that the non-dividend paying firm chooses to pay dividends at the optimal point, meaning that these optimality conditions determine the trigger values that maximize the value of the firm.

Up to this point, individual optimal behavior has been described and used to derive the PDE for firm values. To find a closed-form solution to the partial differential equations (9) and (10), a competitive equilibrium can be utilized. In a competitive equilibrium, as we can see in the first assumption, there is no economic profit, because free entry and exit eat away all excess profits. In other words, if there is an excess profit, this induces immediate new entry, and the profit will disappear right away. Thus, the free-entry condition in competitive equilibrium leads the following additional boundary condition;

$$W(C^*, D^*) = C^* \quad (11)$$

Since all firms are solving the same optimization problem and will each choose the same optimal policy, this condition ensures that there can be no excess profit accruing to entry to the dividend market. Thus, this condition ensures that $V(C, D) \equiv 0$; i.e., the marginal value of a firm¹⁵ which does not pay dividend is always zero, meaning that the (additional or marginal) firm value is considered to be zero when the firm does not pay dividend, *ceteris paribus*. Therefore, combining partial differential equations (9) and (10) and the boundary conditions results in following PDE in equilibrium;

$$\begin{aligned} 0 = & \frac{1}{2} \sigma_C^2 C^2 W_{CC} + \rho \sigma_C \sigma_S CD W_{CD} + \frac{1}{2} \sigma_S^2 D^2 W_{DD} + \mu_C C W_C \\ & + \mu_S D W_D - k_W W + (\mu_S - \mu_C) D \end{aligned} \quad (12)$$

subject to

$$W(C^*, D^*) = C^*, W_D(C^*, D^*) = 0, W_C(C^*, D^*) = 0, W(C^*, 0) = 0, \quad (13)$$

$$W(\infty, D) = [(\mu_S - \mu_C)D] / k_W, \quad (14)$$

¹⁴ For details of high-contact conditions, see Merton (1973).

¹⁵ The concept of marginal value of a firm' is consistent with the concept 'marginal effect' that we introduce in the Section 2. It should be emphasized that we focus only on the additional (marginal) effect of change in dividend on the firm value in this study. That is, $V(C, D) \equiv 0$ means that the firm value is equivalent to zero when the firm does not pay dividends, *ceteris paribus*.

where conditions (13) are from value-matching, smooth-pasting, and competitive equilibrium conditions, (14) is regularity condition., $W(C, 0) = 0$ is the same as explained in the footnote 18. But, the condition $W(\infty, D) = [(\mu_S - \mu_C)D] / k_W$ ensures that, for very large opportunity costs, there is almost no new entry to the dividend market. Thus, the value of a firm that is currently paying dividend is equivalent to the perpetuity value of additional capital gain due to dividend payout.

The solution to PDE (12) subject to (13) and (14) can be derived like following Proposition 1;

Proposition 1. $W(C, D) = (-)^{\gamma} \gamma^{\beta} \left(\frac{\beta}{\beta-1}\right)^{-\beta} \left(\frac{1}{\beta-1}\right) C^{1-\beta} D^{\beta} + \gamma D$ (15)

$$\text{where}^{16} \beta \equiv \left(\frac{1}{2}\sigma_S^2\right) \left(\frac{1}{2}\sigma_S^2 - \rho\sigma_C\sigma_S + \frac{1}{2}\sigma_S^2 - \mu_S\right) + \left(\frac{1}{2}\sigma_S^2\right) \left[(\rho\sigma_C\sigma_S + \mu_S - \frac{1}{2}\sigma_C^2 - \frac{1}{2}\sigma_S^2)^2 - 2\sigma_S^2(\mu_C - k_W)\right]$$

$$\gamma \equiv (\mu_C - \mu_D) / (\rho\sigma_C\sigma_S - \frac{1}{2}\sigma_C^2 + \mu_S + \mu_C - k_W).$$

Proof. To solve PDE (12) subjects to boundary conditions (13) and (14), new variables are defined as follows for the convenience;

$$X \equiv \frac{D}{C} \quad \text{and} \quad Y \equiv \frac{1}{C} W(C, D) \quad (16)$$

Since $W = CY$, $W_C = Y$,

$$W_{CC} = \frac{\partial Y}{\partial C} = \frac{\partial Y}{\partial X} \frac{\partial X}{\partial C} = Y_X \left(-\frac{D}{C^2}\right), \text{ (by a chain rule)}$$

$$W_{CD} = \frac{\partial Y}{\partial D} = \frac{\partial Y}{\partial X} \frac{\partial X}{\partial D} = Y_X \left(\frac{1}{C}\right)$$

$$W_D = C \frac{\partial Y}{\partial D} = C \frac{\partial Y}{\partial X} \frac{\partial X}{\partial D} = Y_X C \left(\frac{1}{C}\right) = Y_X$$

$$W_{DD} = \frac{\partial Y}{\partial D} = \frac{\partial Y}{\partial X} \frac{\partial X}{\partial D} = Y_{XX} \left(\frac{1}{C}\right)$$

Then, equation (12) becomes

$$0 = \frac{1}{2}\sigma_S^2 D^2 \frac{1}{C} Y_{XX} + \rho\sigma_C\sigma_S CD \frac{1}{C} Y_X + \frac{1}{2}\sigma_C^2 C^2 \left(-\right) \frac{D}{C^2} Y_X + \mu_S D Y_X + \mu_C C Y - k_W C Y + (\mu_S - \mu_C) D$$

¹⁶ If $\frac{1}{2}\sigma_S^2 + \rho\sigma_C\sigma_S + \mu_S < \frac{1}{2}\sigma_C^2$, then $\beta > 1$; otherwise $\beta < 1$.

Dividing each side by C yields

$$0 = \frac{1}{2}\sigma_S^2 D^2 \frac{1}{C^2} Y_{XX} + (\rho\sigma_C\sigma_S + \mu_S - \frac{1}{2}\sigma_C^2) \frac{D}{C} Y_X + (\mu_C - k_W)Y + (\mu_S - \mu_C) \frac{D}{C}. \quad (17)$$

Substituting equation (16) into equation (17) yields the following differential equation,

$$0 = \frac{1}{2}\sigma_S^2 X^2 Y_{XX} + (\rho\sigma_C\sigma_S - \frac{1}{2}\sigma_C^2 + \mu_S)XY_X + (\mu_C - k_W)Y + (\mu_S - \mu_C)X \quad (18)$$

subjects to

$$Y(0) = 0, \quad Y(X^*) = (1/C^*)W(C^*D^*) = (1/C^*)C^* = 1, \quad Y_X(X^*) = 0, \quad (19)$$

where $X^* = D^*/C^*$.

The PDE (18) subjects to boundary conditions (19) can be transformed to the following simplified form.

$$aX^2Y_{XX} + bXY_X + cY = Xd + e \quad (20)$$

where $a = \frac{1}{2}\sigma_S^2$, $b = \rho\sigma_C\sigma_S - \frac{1}{2}\sigma_C^2 + \mu_S$, $c = \mu_C - k_W$, $d = -(\mu_S - \mu_C)$, and $e = 0$.

Since we know the solution to PDE (20), we can obtain the following solution to PDE (18) as follows;

$$Y(X) = A_1 X^\beta + \frac{X(\mu_S - \mu_C)}{\rho\sigma_C\sigma_S - \frac{1}{2}\sigma_C^2 + \mu_S + \mu_C - k_W}.$$

$$\beta = \left(\frac{1}{\sigma_S^2} \right) \left(\frac{1}{2}\sigma_S^2 - \rho\sigma_C\sigma_S + \frac{1}{2}\sigma_C^2 - \mu_S + [(\rho\sigma_C\sigma_S - \frac{1}{2}\sigma_C^2 + \mu_S - \frac{1}{2}\sigma_S^2)^2 - 2\sigma_S^2(\mu_C - k_W)]^{\frac{1}{2}} \right)$$

By applying boundary conditions (19), we can find

$$X^* = \frac{\beta}{\lambda(\beta-1)} \quad \text{and} \quad A_1 = (-)\left(\frac{\gamma}{\beta}\right)^\beta \left(\frac{1}{\beta-1}\right)^{1-\beta}$$

where β and γ are defined as equation (15).

As a result, we can derive

$$Y(X) = (\gamma / \beta)^\beta [1/(\beta - 1)]^{1-\beta} X^\beta + \gamma X, \beta \text{ and } X^* = \beta / [\gamma(\beta - 1)] \equiv D^* / C^* \cdot \beta$$

But, from equation (16), since $W(C, D) = CY(X)$,

$$W(C, D) = (-)^\beta \gamma^\beta \left(\frac{\beta}{\beta - 1}\right)^{-\beta} \left(\frac{1}{\beta - 1}\right) C^{1-\beta} D^\beta + \gamma D. \quad Q.E.D.$$

Similarly, $V(C, D)$ can be derived as following Proposition 2

Proposition 2. $V(C, D) = (-)^\beta \gamma_V^\beta \left(\frac{\beta}{\beta - 1}\right)^{-\beta} \left(\frac{1}{\beta - 1}\right) C^{1-\beta} D^\beta.$ (21)

where β is the same as the case of $W(C, D)$ and γ_V is slightly different from γ in the sense that kw in γ should be replaced by kv .

It is interesting to note that only differences between $V(C, D)$ and $W(C, D)$ are γD term in equation (15) and kw and kv terms in γ . This might have very important implication for an optimal dividend, similar to many other real options literature, such as Dixit (1989), Pindyck (1991), McDonald and Siegel (1985), and Brennan and Schwartz (1985). That is, since the denominator of γ may represent the risk-adjusted discount rate¹⁷, γD is perpetuity value of the additional capital gain made when the firm pays dividends forever. Therefore, the first part in equation (15) can be interpreted as “the value of an option to eliminate dividend”. Similarly, the value $V(C, D)$ in equation (21) can be interpreted as “the value of an option to start paying dividend”.

In addition, the optimal aggregate dividend can be expressed as a function of C^* as follows;

$$D^* = C^* \frac{\beta}{\gamma(\beta - 1)} \tag{22}$$

¹⁷ If the denominator of γ is simply kw , it is evident that γD is perpetuity value of additional capital gains earned from dividend payment. Unfortunately, the denominator is $\rho\sigma_C\sigma_S - \frac{1}{2}\sigma_C + \mu_S + \mu_C - kw$. Since σ_C^2 is less than σ_C , $\rho\sigma_C\sigma_S - \frac{1}{2}\sigma_C^2$ should be positive. Thus, $\mu_S + \mu_C - kw$ determines the sign of the denominator of γ . In addition, the denominator of γ reflects the risk in C and S . So, we define it as a ‘risk-adjusted’ discount rate.

To find a closed form solution for D , it is necessary to estimate the value of an exogenous variable C^* . Although various functional forms are possible for C , linear relationship, like $C = A + (1 + r)D$ (where A is constant), is assumed in this study, because extra funds can be invested, at least, in the riskless rates. This means that if we invest extra funds in more risky assets, C can be increased and 'A' should be positive. Thus,

$$D^* = A \frac{\beta / [\gamma(\beta - 1)]}{1 - \frac{\beta(1 + \gamma)}{\gamma(\beta - 1)}} \quad (23)$$

Now that we express $W(C, D)$ and D as exogenous state variables ($\mu_C, \mu_S, \sigma_C, \sigma_S, \rho, \gamma, B$ (debt), Q, S and A), the remaining thing to do is to estimate those variables. It should be done in the future research with empirical test of the model derived in this study.

4. SENSITIVITY ANALYSIS

Using partial derivatives, we can execute sensitivity analyses. In other words, by applying comparative static analysis which is widely used traditionally in economic science, we can see the effect of changes in the key variables on $W(C, D)$ or D^* .

To investigate the effect of change in D^* on $W(C, D)$, partial derivative can be used as follows;

$$\frac{\partial W(C, D)}{\partial D} = (-)\gamma^\beta \left[\frac{\beta}{\beta - 1} \right]^{-\beta} \left[\frac{1}{\beta - 1} \right] C^{1-\beta} \beta D^{\beta-1} + \gamma. \quad (24)$$

In equation (24), the sign of $\partial W(C, D) / \partial D$ is dependent on the values β and γ , i.e., the state variables $\mu_C, \mu_S, \sigma_C, \sigma_S, \rho$ and k . If $\frac{1}{2}\sigma_S^2 + \rho\sigma_C\sigma_S + \mu_S = \frac{1}{2}\sigma_C^2$ and $\mu_C = \mu_S$, then $\partial W(C, D) / \partial D = 0$ supporting the Miller and Modigliani (1961) proposition (i.e., irrelevancy of a dividend on the firm value). Particularly, since the condition of $\mu_C = \mu_S$ means that additional expected capital gain is equal to the expected opportunity cost of capital, this implies $V(C, D) = W(C, D)$. That is, a dividend is irrelevant. If $\mu_C \neq \mu_S$, a dividend policy affects a firm value in the equilibrium. Therefore, the appropriate estimation of exogenous state variables is

critical to determine the relevancy of dividend policy.

Similarly, we can draw various managerial implications from following sensitivity analyses:

$$\partial D/\partial \tau > 0, \text{ if } \frac{1}{2}\sigma_S^2 + \rho\sigma_C\sigma_S + \mu_S < \frac{1}{2}\sigma_C^2 \text{ and } \mu_C < \mu_S, \text{ and}$$

$$\partial D/\partial \tau \leq 0, \text{ if } \frac{1}{2}\sigma_S^2 + \rho\sigma_C\sigma_S + \mu_S \geq \frac{1}{2}\sigma_C^2 \text{ and } \mu_C \geq \mu_S.$$

The first relationship implies that, under a given condition (actually, this condition is more realistic than the latter), if tax rate is increased, then dividend payment triggering non-dividend-paying firms to enter the dividend market should be increased. This means that the higher corporate tax rate, the more dividends should be paid out. This fact seems reasonable, because firms had better pay out dividends as much as possible to increase debt to maximize tax shields if corporate tax rate is high.

$$\frac{\partial D}{\partial B} > 0, \text{ if } \frac{1}{2}\sigma_S^2 + \rho\sigma_C\sigma_S + \mu_S < \frac{1}{2}\sigma_C^2 \text{ and } \mu_C < \mu_S,$$

$$\frac{\partial D}{\partial B} < 0, \text{ if } \frac{1}{2}\sigma_S^2 + \rho\sigma_C\sigma_S + \mu_S > \frac{1}{2}\sigma_C^2 \text{ and } \mu_C > \mu_S.$$

This is an expected result, because the higher tax rate increases both B and D. The fact that B and D have a positive relationship gives bondholders an extremely important implication. That is, if a firm increases a dividend payout, it signals also a bad news to the bondholders, because the probability that the firm may increase the debt is high. This fact is very consistent with the common sense that a dividend payment induces an agency problem, because bondholders are concerned about the increase in dividend payment to shareholders at the expense of them.

To compare this model with the analysis of Eades et al. (1994), the impact of riskless rate on the dividend capturing is investigated by using that $\partial D/\partial r > 0$. This is easily derived from the equation (23). According to Eades et al. (1994), T-bill rates are negatively related with dividend capturing (p.1632), contrary to this study. They use the relationship between business cycle and dividend capturing to explain their result. But, this study uses 'opportunity cost' concept as follows; if r goes up, the opportunity cost will also go up, indicating that the triggering point of D should be increased. The reason of the existence of the difference between two studies might be partly because their model focuses on the individual firm behavior, while this model is aggregate model, partly because of some misspecifi-

cation of the model.

Finally, the effect of a cost of capital (WACC; k_w) on D is investigated. As already analyzed in the 'second' above,

$$\frac{\partial D}{\partial k_w} < 0, \text{ if } \frac{1}{2}\sigma_S^2 + \rho\sigma_C\sigma_S + \mu_S < \frac{1}{2}\sigma_C^2 \text{ and } \mu_C < \mu_S, \text{ and}$$

$$\frac{\partial D}{\partial k_w} > 0, \text{ if } \frac{1}{2}\sigma_S^2 + \rho\sigma_C\sigma_S + \mu_S > \frac{1}{2}\sigma_C^2 \text{ and } \mu_C > \mu_S.$$

This result has two important implications; first, a dividend policy is independent of a cost of capital only if $\frac{1}{2}\sigma_S^2 + \rho\sigma_C\sigma_S + \mu_S = \frac{1}{2}\sigma_C^2$ and $\mu_C = \mu_S$, which are not realistic conditions, and, otherwise, it is not independent; second, a cost of capital is negatively related with a dividend, which can be explained by the fact that the increase in the k_w requires higher cost of equity and debt, indicating the decrease in the dividend due to the decrease in equity.

5. CONCLUSION

In this paper, it was shown that a continuous time and aggregate model can be applied in the study of a dividend policy. Combining an individual firm behavior and an aggregate market behavior yields the equilibrium-optimal dividend policy in the perfectly competitive market setting. The major results of this study are as follows; dividend irrelevance theory can be valid only when very specific and unrealistic conditions are met; a corporate tax rate, a debt policy, a cost of capital can affect the dividend policy under specific conditions. All results in this study, however, are valid under the assumption that the opportunity costs follow geometric Brownian motion, which is widely used in economic science and finance literature.

Even though this approach produces some important implications for the dividend policy, some problems should be solved in the future research. Since some assumptions are strong, it is more desirable to relax them to yield more realistic results. Since optimal dividend policy is critically dependent on the exogenous variables, the appropriate estimation methods should be developed for the variables. Empirical research must be done to make sure the model is valid to be applied in the real world.

REFERENCES

- [1] Ackert, L. F. and B. F. Smith, Stock volatility, ordinary dividends, and other cash flows to share-holders, *Journal of Finance* 48 (Sept. 1993), 1147-1160.
- [2] Benartzi, Shlomo, Rini Michaely, and Richard Thaler, "Do changes in dividends signal the future or the past?," *Journal of Finance* 52 (1997), 1007-1034.
- [3] Bhattacharya, S., "Imperfect Information, dividend policy, and the 'Bird-in-the-Hand' fallacy," *Bell Journal of Economics and Management Science*, (1979), 259-270.
- [4] Black, F. and M. S. Scholes, "The effects of dividend yield and dividend policy on common stock prices and returns," *Journal of Financial Economics* 1 (May 1974), 1-22.
- [5] Boyd, J. H. and R. Jagannathan, "Ex-dividend price behavior of common stocks," *Review of Financial Studies* 7, 4 (1994), 711-741.
- [6] Brealey, R. A. and S. C. Myers, *Principles of Corporate Finance*, 4th ed., N.Y.; McGraw-Hill, 1991.
- [7] Brennan, M. J., "A note on Dividend Irrelevance and the Gordon valuation model," *Journal of Finance* 26 (Dec. 1971), 1115-1122.
- [8] Brennan, M. J. and E. Schwartz, "Evaluating natural resource investments," *Journal of Business* 58 (1985), 135-157.
- [9] Capozza D. and Yuming Li, "The Intensity and Timing of Investment: The Case of Land," *American Economic Review* 84 (1994), 889-904.
- [10] De Angelo, H. and L. De Angelo, "Dividend policy and financial distress: An empirical investigation of troubled NYSE firms," *Journal of Finance* 45 (Dec. 1990), 1415-1431.
- [11] De Angelo, H., L. De Angelo and Douglas J. Skinner, "Reversal of fortune: Dividend signaling and the disappearance of sustained earnings growth," *Journal of Financial Economics* 40 (1996), 341-371.
- [12] Dhillon, U. S. and H. Johnson, "The effect of Dividend changes on stock and bond prices," *Journal of Finance* 49 (March 1994), 281-289.
- [13] Dixit, A., "Entry and exit decisions under uncertainty", *Journal of Political Economy* 97 (1989), 620-638.
- [14] Dixit, A. and R. Pindyck, *Investment under Uncertainty*, Princeton, NJ., 1994.
- [15] Eades, K. M., P. J. Hess and E. H. Kim, "On interpreting security returns during the ex-dividend period," *Journal of Financial Economics* 13, 1 (1984), 3-34.

- [16] Eades, K. M, P. J. Hess and E. H. Kim, "Time-series variation in dividend pricing," *Journal of Finance* 49 (Dec. 1994), 1671-1638.
- [17] Easterbrook, F. H., "Two agency cost explanations of dividends," *American Economics Review* 74, 4 (1984), 650-659.
- [18] Fama, E., "The empirical relationship between the dividend investment decision of firms," *American Economic Review* 64, 3 (1974), 304-318.
- [19] Gordon, M. J., "dividends, earnings and stock prices," *Review of Economics and Statistics* 41 (May 1959), 99-105.
- [20] Gordon, M. J., "The savings, investment and valuation of corporation," *Review of Economics and Statistics* 44 (Feb 1962), 37-51.
- [21] Grenadier, S. R., "Valuing lease contracts: A real-options approach," *Journal of Financial Economics* 38 (1995), 297-331.
- [22] Hakansson, N., "To pay or not to pay dividends," *Journal of Finance* 37 (May 1982), 415-428.
- [23] Hansen, Robert S., Raman Kumar and Dilip K. Shome, "Dividend policy and corporate monitoring: Evidence from the regulated electric utility industry," *Financial Management* 23 (Spring 1994), 16-22.
- [24] Hess, P. J., "The ex-dividend day behavior of stock returns: Further evidence on tax effects," *Journal of Finance* 37 (May 1982), 445-456.
- [25] Hess, P. J., "Tests for tax effects in the pricing of financial asserts," *Journal of Business* 56, 4 (1983), 537-554.
- [26] Hull, John C., *Options, Futures, and Other Derivatives*, Prentice-Hall, NJ.
- John, K. and A. Kalay, 1982, "Costly contracting and optimal payout constraints," *Journal of Finance* 37 (May 2000), 457-479.
- [27] Jarrow, Robert and Andrew Rudd, *Option Pricing*, Homewood, IL.: Irwin., 1983.
- [28] Kao, C. and C. Wu, "Tests of dividend signaling using the Marsh-Merton model: A generalized friction approach," *Journal of Business* 67, 1 (1994), 45-68.
- [29] Kester, W. Carl, "Today's Options for Tomorrow's Growth," *Harvard Business Review* (1984), 153-160.
- [30] Lintner, J., "Dividends, earnings, leverage, stock prices and the supply of capital to corporations," *Review of Economics and Statistics* 64, 3 (1962), 243-269.
- [31] Litzenberger, R. H. and K. Ramaswamy, "The effect of personal taxes and dividends on capital asset prices," *Journal of Financial Economics* 7 (1979), 163-195.
- [32] Litzenberger, R. H. and K. Ramaswamy, "The effects of dividends on common stock prices: Tax effects or information effects?," *Journal of Finance* 37

- (May 1982), 429-443.
- [33] Lowellen, W., K. L. Stanley, R. C. Lease and G. G. Schlarbaum, "Some direct evidence on the dividend clientele phenomenon," *Journal of Finance* 33 (Dec. 1978).
 - [34] Marsh, T. A. and R. C. Merton, "Dividend Behavior for the Aggregate stock market," *Journal of Business* 60, 1 (1987), 1-40.
 - [35] McDonald, R. L. and D. R. Siegel, "Investment and the valuation of firms there is an option to shut down," *International Economics Review* 26 (June 1985), 331-349.
 - [36] _____, "The Value of Waiting Investment," *Quarterly Journal of Economics*, (1986), 707-727.
 - [37] Merton, R. C., "Theory of rational option pricing," *Bell Journal of Economics and Management Science* 4 (Spring 1973), 141-183.
 - [38] Miller, M. H. and F. Modigliani, "Dividend policy, growth and valuation of shares," *Journal of Business* 34 (Oct. 1961), 411-433.
 - [39] Miller, M. H. and K. Rock, "Dividend policy under asymmetric information," *Journal of Finance* 40 (Sept. 1985), 1031-1051.
 - [40] Miller, M. H. and M. S. Scholes, "Dividends and taxes," *Journal of Financial Economics* 6 (1978), 633-664.
 - [41] Miller, M. H. and M. S. Scholes, "Dividends and taxes: Some empirical evidences," *Journal of Political Economy* 90 (1982), 1118-1142.
 - [42] Nissim, Doron, and Amir Ziv, "Dividend changes and future profitability," *Journal of Finance* 56 (Dec. 2001), 2111-2133.
 - [43] Pindyck, R. S., "Irreversibility, uncertainty, and investment," *Journal of Economic Literature* 29 (1991), 1110-1148.
 - [44] Rozeff, Robert S., "Growth. Beta and agency costs as determinants of dividend payout ratios," *Journal of Financial Research* (Fall 1982), 249-259.
 - [45] Rubinstein, Mark, "Derivative Assets Analysis," *Journal of Economic Perspectives* 1 (1987), 73-94.
 - [46] Schwartz, E. and C. Zozaya, "Valuation of Investment Technology Investments as Real Options," Working paper, University of California, LA., 2000.
 - [47] Shiller, R. J., "Do stock prices move too much to be justified by subsequent changes in dividends?," *American Economic Review* 71, 3 (1981), 421-436.
 - [48] Shimko, D. C., *Finance in Continuous Time: A Primer*, Miami, Kolb Publishing Co., 1992.