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APPROXIMATION OF THE SOLUTION OF STOCHASTIC EVOLUTION EQUATION WITH FRACTIONAL BROWNIAN MOTION†

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ABSTRACT

We study the approximation of the solution of linear stochastic evolution equations driven by infinite-dimensional fractional Brownian motion with Hurst parameter $H > 1/2$ through discretization of space and time. The rate of convergence of an approximation for Euler scheme is established.

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1. INTRODUCTION

When the statistical inference is based on the solution of Stochastic Partial Differential Equation (SPDE) driven by a noise process, we cannot expect the solution of an equation to be observed at all space and time. Hence it is assumed that we have only a finite dimensional projection of the solution and sampling instants over a specified time. In order to practically use these equations, we need to rely on the method of numerical approximations. In this context we investigate the approximation by space and time discretization of linear stochastic evolution equations driven by fractional Brownian motion (FBM) with Hurst parameter $H > 1/2$. There have been the studies of the approximation of evolution equations driven by Brownian motion. Greksch and Kloeden (1996) study the approximation of parabolic SPDE's with a real-valued Wiener process by using eigenfunctions. Gyöngy (1998, 1999) and Shardlow (1999) apply finite differences to approximate the mild solution of parabolic SPDE's driven by space time

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white noise. Hausenblas (2003) investigates the accuracy of approximation of quasi-linear evolution equations through space and time discretization.

For $0 < H < 1$, a real-valued FBM $(\beta^H(t))$, $t \in [0,T]$, with Hurst parameter H is a centered Gaussian random field with the covariance function given by

$$E[\beta^H(s)\beta^H(t)] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad s,t \in [0,T]. \tag{1.1}$$

Some properties of this process were studied by Mandelbrot and Van Ness (1968). Because of the self-similar and long-range dependence properties of FBM, this process has been used to describe the model for financial market and telecommunication traffic. For these practical applications, it is necessary to develop stochastic calculus with respect to FBM because FBM ($H \neq 1/2$) is not semimartingale. Recently, there have been a series of works on stochastic calculus and stochastic differential equations with respect to a finite dimensional FBM (see, *e.g.* Decreusefond and Üstünel, 1999; Coutin and Qian, 2000; Duncan *et al.*, 2000). The question of infinite dimensional equations has been considered very recently (see, *e.g.* Duncan *et al.*, 2002; Grecksch and Anh, 1999; Tindel *et al.*, 2003; Maslowski and Nualart, 2003). In particular, Duncan *et al.* (2002) consider a linear stochastic evolution equation in a Hilbert space driven by cylindrical FBM with Hurst parameter $H > 1/2$. Existence and uniqueness of mild solutions, continuity of the sample paths and regularity of the solution are examined.

In this paper we investigate the rate of convergence for an approximation of the evolution equation considered by Duncan *et al.* (2002). Our result is based on the method of the approximation given in Hausenblas (2003) to treat the evolution equations driven by the Hilbert-valued Brownian motion with the nuclear covariance operator.

2. EVOLUTION EQUATION AND APPROXIMATION

Let V be a separable Hilbert space. Assume that $A : \mathrm{Dom}(A) \supset V \to V$ is the infinitesimal generator of an analytic semigroup $T_A(t)$ of negative type, $t \geq 0$ and $\Phi \in \mathcal{L}(V,V)$. We study an approximation of Stochastic Evolution Equation driven by V-valued FBM $(B^H(t))$, $t \geq 0$, with Hurst parameter $H > 1/2$:

$$\begin{cases} dX(t) &= AX(t)dt + \Phi dB^H(t), \quad t \in [0,T], \\ X(0) &= x_0 \in V. \end{cases} \tag{2.1}$$

The solution of the initial value problem (2.1) will be given by the mild solution, *i.e.*, for $t \in [0,T]$,

$$X(t) = T_A(t)x + \int_0^t T_A(t-s)\Phi dB_s^H. \tag{2.2}$$

Here for the definition of Wiener integral with respect to FBM with $H > 1/2$, see Duncan *et al.* (2002). We denote by V_δ, $\delta \geq 0$, the domain of the fractional power $(-A)^\delta$ equipped with the norm $\|x\|_\delta = \|(-A)^\delta x\|$ for $x \in \mathrm{Dom}((-A)^\delta)$. Let $\mathcal{L}_2(V,V)$ be the class of Hilbert-Schmidt linear operators from V to V.

ASSUMPTION 2.1. It will be assumed that there exists $\theta > 0$ such that for $\delta \in [\rho, \gamma]$

$$\int_0^T \int_0^T u^{-(\theta+\delta)} v^{-(\theta+\delta)} \big\|T_A(u)\Phi\big\|_{\mathcal{L}_2} \cdot \big\|T_A(v)\Phi\big\|_{\mathcal{L}_2} \cdot |u-v|^{2H-2} du dv < \infty,$$

where $\|\cdot\|_{\mathcal{L}_2}$ is the Hilbert-Schmidt norm.

By Proposition 3.3 of Duncan *et al.* (2002), we have

THEOREM 2.1. *Under Assumption* 2.1, *there exists a unique mild solution* $(X(t))$, $t \in [0,T]$, *belonging to*

$$\mathcal{L}^2([0,T] \times \Omega;\ \mathrm{Dom}((-A)^{\gamma+\theta})) \cap C([0,T];\ \mathrm{Dom}((-A)^\gamma))$$

for $X(0) = x_0 \in \mathrm{Dom}((-A)^\gamma)$.

Let us denote (A_n, V_n), $n = 1, 2, \ldots$, an approximation of (A, V). One method for the approximation is the method of moments (see, *e.g.* Hausenblass, 2003 or Harrington, 1993): For $x \in V$, we can write $x = \sum_{i=1}^\infty x_i \psi_i$, where $\{\psi_i : i = 1, 2, \ldots\}$ forms a complete set of basis in V. The approximation can be done by only taking a finite number of basis. Let V_n be the d_n-dimensional subspace of V spanned by $\{\psi_i : 1 \leq i \leq d_n\}$. Then approximation is $x^n := \sum_{i=1}^{d_n} x_i \psi_i$. Let $Ax^n = y_n$. By taking inner product with a set of testing function $\{\zeta_i : 1 \leq i \leq d_n\}$, we have $\sum_{i=1}^{d_n} x_i \langle A\psi_i, \zeta_j \rangle = \langle y_n, \zeta_j \rangle$ for $1 \leq j \leq d_n$. The approximation operator A_n can be defined by the matrix $A_n = (\langle A\psi_i, \zeta_j \rangle)_{d_n \times d_n}$.

If $P_n : V \to V_n$ and $D_n : V_n \to V$ are projection and embedding operator respectively, then $A_n = P_n A D_n$, which is a bounded linear operator on V_n. Set $\tilde{A}_n = D_n A_n P_n$. Then $\tilde{A}_n$ is a bounded linear operator on V. Throughout this paper, we assume the following conditions:

ASSUMPTION 2.2. For every $n \geq 1$, $P_n : V \to V_n$ and $D_n : V_n \to V$ are bounded operators such that

(1) $\|P_n\| \leq C_1$, $\|D_n\| \leq C_2$, where C_1 and C_2 are independent of n.

(2) $P_n D_n = I_n$ where I_n is the identity operator on V_n.

(3) There exists a function $\varphi_\delta : \mathbb{N} = \{1, 2, \ldots\} \to [0, 1]$, $\delta \in (0, \gamma]$, such that $\lim_{n\to\infty} \varphi_\delta(n) = 0$ and

(i) $\|(I - D_n P_n)x\| \leq \varphi_\delta(n)\|x\|_\delta$,

(ii) $\|A(I - D_n P_n)x\| \leq \varphi_\delta(n)\|A_n\| \cdot \|x\|_\delta$ for all $x \in \mathrm{Dom}((-A)^\delta)$.

(4) A_n is a bounded operator and there exists some $M < \infty$ and $\omega \in \mathbb{R}$ such that

$$\|e^{tA_n}\| \leq Me^{\omega t} \quad \text{for} \quad t \geq 0 \text{ and } n \geq 1.$$

Our approximation is given by

$$\begin{cases} dX_n(t) &= A_n X_n(t)dt + \Phi_n dP_n B^H(t), \quad t \in [0, T], \\ X_n(0) &= P_n x_0 \in V, \end{cases} \tag{2.3}$$

where Φ_n is a bounded linear operator defined on V_n such that $\Phi_n = P_n \Phi D_n$. Note that $P_n B^H(t)$ is a d_n-dimensional fractional Brownian motion. Now we consider the explicit Euler scheme for the discretization of time t with the space discretization (A_n, V_n), $n = 1, 2, \ldots$ Let $\{\tau_n, n \geq 1\}$ be the sequence of the time step sizes corresponding to the space V_n for time discretization. If we denote by $Y_n(k)$ the approximation of $X_n(k\tau_n)$, then

$$\begin{cases} Y_n(k+1) - Y_n(k) = \tau_n A_n Y_n(k) + \Phi_n \Delta_k B_n^H, \\ Y_n(0) = P_n x_0 \in V, \end{cases}$$

where $\Delta_k B_n^H = B_n^H((k+1)\tau_n) - B_n^H(k\tau_n)$. For every integer $n \geq 1$, we construct the approximation:

$$Y_n(k) = (I + \tau_n A_n)^k P_n x_0 + \sum_{l=0}^{k-1} (I + \tau_n A_n)^{k-l-1} \Phi_n \Delta_l B_n^H. \tag{2.4}$$

Also we assume the following condition:

ASSUMPTION 2.3. The bounded operator A_n and $\{\tau_n\}$ are such that for some $M < \infty$ and $\tilde{\omega} \in \mathbb{R}$ we have

$$\|(I + \tau_n \tilde{A}_n)^k\| \leq Me^{\tilde{\omega}\tau_n k} \quad \text{for } n \geq 1 \text{ and } k \geq 0,$$

moreover $\lim_{n\to\infty} \tau_n = 0$.

3. MAIN THEOREM

In this section, we investigate the rate of convergence for the approximation (2.4) of the solution given in (2.1). First we introduce the following notations: $[t]^+(\tau_n) = (k+1)\tau_n$ and $[t]^-(\tau_n) = k\tau_n$ if $k\tau_n \le t < (k+1)\tau_n$. Also write $m(t) = [t]^-(\tau_n)/\tau_n$.

THEOREM 3.1. *It will be assumed that*

$$\left\|(-A)^{-\theta+\gamma}\Phi\right\|_{\mathcal{L}_2} < \infty, \tag{3.1}$$

$0 < \theta < H$ *and* $\theta < \gamma - \rho$. *For arbitrary* $\epsilon > 0$, *the error bound of the approximation is given at* $t = k\tau_n$ *by*

$$\begin{aligned}
&E\|X(k\tau_n) - D_nY_n(k)\|_\rho^2 \\
&\le C_1\kappa^2(n)\left\{\frac{1}{\epsilon^2}\varphi^2_{\gamma-\epsilon}(n) + ([t]^-(\tau_n))^{-2\rho}\varphi^2_\gamma(n)\right\} + C_2\tau_n^{2\min(1,\gamma-\rho)} \\
&\quad + C_3\kappa^{2\theta}(n)\varphi^2_{\gamma-\rho}(n) + C_4\tau_n^{2\min(1,\gamma-\rho-\theta)} + C_5\tau_n^2\left\|(-A_n)^{\max(0,1+\rho+\theta-\gamma)}\right\|^2,
\end{aligned}$$

where $\kappa(n) = \|A_n\|\cdot\|A_n^{-1}\|$.

PROOF. Equation (2.4) can be written as

$$Y_n(m(t)) = (I+\tau_nA_n)^{m(t)}P_nx_0 + \int_0^{[t]^-(\tau_n)} (I+\tau_nA_n)^{m(t)-m(s)-1}\Phi_n dB_n^H(s). \tag{3.2}$$

Hence

$$X(k\tau_n) - D_nY_n(k) := \sum_{i=1}^5 I_n^i(t),$$

where

$$\begin{aligned}
I_n^1(t) &= T_A\big([t]^-(\tau_n)\big)x_0 - D_n(I+\tau_nA_n)^{m(t)}P_nx_0, \\
I_n^2(t) &= \int_0^{[t]^-(\tau_n)} T_A\big([t]^-(\tau_n)-s\big)[\Phi - D_n\Phi_nP_n]dB^H(s), \\
I_n^3(t) &= \int_0^{[t]^-(\tau_n)} \Big[T_A\big([t]^-(\tau_n)-s\big) - D_nT_{A_n}\big([t]^-(\tau_n)-s\big)P_n\Big] \\
&\qquad \times D_n\Phi_nP_ndB^H(s), \\
I_n^4(t) &= \int_0^{[t]^-(\tau_n)} D_nT_{A_n}\big([t]^-(\tau_n)-[s]^-(\tau_n)-\tau_n\big) \\
&\qquad \times \Big[T_{A_n}\big([s]^-(\tau_n)+\tau_n-s\big) - I\Big]P_nD_n\Phi_nP_ndB^H(s),
\end{aligned}$$

$$I_n^5(t) = \int_0^{[t]^-(\tau_n)} D_n\Big[T_{A_n}\big([t]^-(\tau_n) - [s]^-(\tau_n) - \tau_n\big) - (I + \tau_n A_n)^{m(t)-m(s)-1}\Big] P_n D_n \Phi_n P_n dB^H(s).$$

We give the five separate proofs: $I_n^i(t)$ for $i = 1, \ldots, 5$. Throughout this proof, the notation $C(H)$ denotes a generic constant which does not depend on n.
(i) $I_n^1(t)$ term: From Hausenblas (2003), it follows that for fixed $\epsilon > 0$,

$$\big\|I_n^1(t)\big\|_\rho^2 \le C_1\kappa^2(n)\left\{\frac{1}{\epsilon^2}\varphi_{\gamma-\epsilon}^2(n) + ([t]^-(\tau_n))^{-2\rho}\varphi_\gamma^2(n)\right\}\|x_0\|_\gamma^2 + C_2\tau_n^{2\min(1,\gamma-\rho)}\|x_0\|_{\min(1+\rho,\gamma)}. \tag{3.3}$$

(ii) $I_n^2(t)$ term: By $D_n\Phi_n = \Phi D_n$ and definition of stochastic integral with respect to FBM with $H > 1/2$,

$$\begin{aligned}
&E\big\|I_n^2(t)\big\|_\rho^2 \\
&= E\left\|\int_0^{[t]^-(\tau_n)} (-A)^\rho T_A\big([t]^-(\tau_n) - s\big)\left[\Phi - D_n\Phi_n P_n\right] dB^H(s)\right\|^2 \\
&= C(H)\sum_{l=1}^\infty \int_0^{[t]^-(\tau_n)}\int_0^{[t]^-(\tau_n)} \Big\langle (-A)^\rho T_A\big([t]^-(\tau_n) - u\big) \\
&\quad \times\Phi[I - D_nP_n]e_l, (-A)^\rho T_A\big([t]^-(\tau_n) - v\big)\Phi[I - D_nP_n]e_l\Big\rangle_V \\
&\quad \times|u - v|^{2H-2}dudv \\
&\le C(H)\sum_{l=1}^\infty \big\|(-A)^{\rho-\theta}\Phi[I - D_nP_n]e_l\big\|^2 \\
&\quad \times\int_0^{[t]^-(\tau_n)} \big\|(-A)^\theta T_A\big([t]^-(\tau_n) - u\big)\big\| \cdot \big\|(-A)^\theta T_A\big([t]^-(\tau_n) - v\big)\big\| \\
&\quad \times|u - v|^{2H-2}dudv.
\end{aligned} \tag{3.4}$$

Using Theorem 2.6.13 of Pazy (1983), the last term in (3.4) becomes

$$\begin{aligned}
&E\big\|I_n^2(t)\big\|_\rho^2 \\
&= C(H)\sum_{l=1}^\infty \big\|(-A)^{\rho-\theta}\Phi[I - D_nP_n]e_l\big\|^2 \\
&\quad \times\int_0^{[t]^-(\tau_n)}\int_0^{[t]^-(\tau_n)} \big([t]^-(\tau_n) - u\big)^{-\theta}\big([t]^-(\tau_n) - v\big)^{-\theta}|u - v|^{2H-2}dudv
\end{aligned}$$

$$\leq C(H)\sum_{l=1}^{\infty}\left\|(-A)^{\rho-\theta}\Phi[I-D_nP_n]e_l\right\|^2\int_0^{[t]^-(\tau_n)}u^{-2\theta+2H-1}du. \tag{3.5}$$

Now Remark 4.2 in Hasenblas (2003) gives

$$\begin{aligned}
&\left\|(-A)^{\rho-\theta}\Phi[I-D_nP_n]e_l\right\|^2\\
&\leq \left\|\left((-A)^{-\theta}-(-A_n)^{-\theta}\right)\Phi e_l\right\|_\rho^2+\left\|(-A_n)^{-\theta}\Phi[I-D_nP_n]e_l\right\|_\rho^2\\
&\quad+\left\|\left((-A)^{-\theta}-(-A_n)^{-\theta}\right)D_n\Phi P_ne_l\right\|_\rho^2\\
&\leq \kappa^{2\theta}(n)\varphi^2_{\gamma-\rho}(n)\left\|(-A)^{-(\theta-\gamma)}\Phi e_l\right\|.
\end{aligned} \tag{3.6}$$

For $\theta < H$, we have

$$\int_0^{[t]^-(\tau_n)}u^{-2\theta+2H-1}du<\infty. \tag{3.7}$$

From (3.6) and (3.7), it follows that

$$E\left\|I_n^2(t)\right\|_\rho^2\leq C(H)\kappa^{2\theta}(n)\varphi^2_{\gamma-\rho}(n)\left\|(-A)^{-(\theta-\gamma)}\Phi\right\|_{\mathcal{L}_2}. \tag{3.8}$$

(iii) $I_n^3(t)$ term: Since $D_nA_n^k = A^kD_n$ for $k = 1, 2, \ldots$, we have

$$D_ne^{tA_n}P_n=D_n\sum_{k=1}^{\infty}\frac{t^k}{k!}A_n^kP_n=\sum_{k=1}^{\infty}\frac{t^k}{k!}A^kD_nP_n=e^{tA}D_nP_n. \tag{3.9}$$

From (3.9) and $P_nD_n = I$, we get

$$\begin{aligned}
E\left\|I_n^3(t)\right\|_\rho^2&=\left\|\int_0^{[t]^-(\tau_n)}(-A)^\rho T_A\left([t]^-(\tau_n)-s\right)(I-D_nP_n)D_n\Phi_nP_ndB^H(s)\right\|^2\\
&=0.
\end{aligned} \tag{3.10}$$

(iv) $I_n^4(t)$ term: By the definition of stochastic integrals with respect to FBM and $(-A)^\rho D_n = D_n(-A_n)^\rho$, we have

$$\begin{aligned}
E\left\|I_n^4(t)\right\|_\rho^2=\bigg\|\int_0^{[t]^-(\tau_n)}&(-A)^\rho e^{\left([t]^-(\tau_n)-[s]^-(\tau_n)-\tau_n\right)\tilde{A}_n}\\
&\times\left[e^{\left([s]^-(\tau_n)+\tau_n-s\right)\tilde{A}_n}-I\right]D_n\Phi_nP_ndB^H(s)\bigg\|^2
\end{aligned}$$

$$= C(H)\sum_{l=1}^{\infty}\int_0^{[t]^-(\tau_n)}\int_0^{[t]^-(\tau_n)}\Big\langle(-\tilde{A}_n)^{\rho}e^{\left([t]^-(\tau_n)-[u]^-(\tau_n)-\tau_n\right)\tilde{A}_n}$$

$$\times\Big[e^{\left([u]^-(\tau_n)+\tau_n-u\right)\tilde{A}_n}-I\Big]D_n\Phi_nP_ne_l,\ \ (-\tilde{A}_n)^{\rho}e^{\left([t]^-(\tau_n)-[v]^-(\tau_n)-\tau_n\right)\tilde{A}_n}$$

$$\times\Big[e^{\left([v]^-(\tau_n)+\tau_n-v\right)\tilde{A}_n}-I\Big]D_n\Phi_nP_ne_l\Big\rangle|u-v|^{2H-2}dudv$$

$$\leq C(H)\sum_{l=1}^{\infty}\sum_{i=1}^{m(t)-1}\sum_{j=1}^{m(t)-1}\Big\|e^{\left(m(t)-(i+1)\right)\tau_n\tilde{A}_n}\Big\|\cdot\Big\|e^{\left(m(t)-(j+1)\right)\tau_n\tilde{A}_n}\Big\|$$

$$\times\int_0^{\tau_n}\int_0^{\tau_n}\Big\|(-\tilde{A}_n)^{\rho}[e^{u\tilde{A}_n}-I]D_n\Phi_nP_ne_l\Big\|\cdot\Big\|(-\tilde{A}_n)^{\rho}[e^{v\tilde{A}_n}-I]D_n\Phi_nP_ne_l\Big\|$$

$$\times\Big|\big((i+1)\tau_n-u\big)-\big((j+1)\tau_n-v\big)\Big|^{2H-2}dudv. \tag{3.11}$$

Since

$$(T_A(t)-I)x=\int_0^t AT(s)xds,$$

the term (3.11) can be bounded by

$$E\big\|I_n^4(t)\big\|_{\rho}^2$$

$$\leq C(H)\sum_{l=1}^{\infty}\sum_{i=1}^{m(t)-1}\sum_{j=1}^{m(t)-1}\Big\|e^{\left(m(t)-(i+1)\right)\tau_n\tilde{A}_n}\Big\|\cdot\Big\|e^{\left(m(t)-(j+1)\right)\tau_n\tilde{A}_n}\Big\|$$

$$\times\Big\{\int_0^{\tau_n}\int_0^{\tau_n}\int_0^{u}\Big\|(-\tilde{A}_n)^{1+\rho+\theta-\gamma}e^{s\tilde{A}_n}\Big\|ds\int_0^{v}\Big\|(-\tilde{A}_n)^{1+\rho+\theta-\gamma}e^{s\tilde{A}_n}\Big\|ds$$

$$\times\Big|\big((i+1)\tau_n-u\big)-\big((j+1)\tau_n-v\big)\Big|^{2H-2}dudv\Big\}$$

$$\times\big\|(-\tilde{A}_n)^{-\theta+\gamma}D_n\Phi_nP_ne_l\big\|^2. \tag{3.12}$$

For $0<\gamma-\rho-\theta<1$, the expression (3.12) becomes

$$E\big\|I_n^4(t)\big\|_{\rho}^2$$

$$\leq C(H)\sum_{l=1}^{\infty}\big\|(-\tilde{A}_n)^{-\theta+\gamma}D_n\Phi_nP_ne_l\big\|^2\sum_{i=1}^{m(t)-1}\sum_{j=1}^{m(t)-1}\int_0^{\tau_n}\int_0^{\tau_n}u^{\gamma-\rho-\theta}$$

$$\times u^{\gamma-\rho-\theta}\Big|\big((i+1)\tau_n-u\big)-\big((j+1)\tau_n-v\big)\Big|^{2H-2}dudv$$

$$\le C(H)\tau_n^{2(\gamma-\rho-\theta)}\sum_{l=1}^{\infty}\big\|(-\tilde{A}_n)^{-\theta+\gamma}D_n\Phi_nP_ne_l\big\|^2$$

$$\times\sum_{i=1}^{m(t)-1}\sum_{j=1}^{m(t)-1}\int_0^{\tau_n}\int_0^{\tau_n}\Big|\big((i+1)\tau_n-u\big)-\big((j+1)\tau_n-v\big)\Big|^{2H-2}dudv$$

$$\le C(H)\tau_n^{2(\gamma-\rho-\theta)}\sum_{l=1}^{\infty}\big\|(-\tilde{A}_n)^{-\theta+\gamma}D_n\Phi_nP_ne_l\big\|^2$$

$$\times\sum_{i=1}^{m(t)-1}\sum_{j=1}^{m(t)-1}\int_0^{\tau_n}\int_0^{\tau_n}\Big|\big((i+1)\tau_n-u\big)-\big((j+1)\tau_n-v\big)\Big|^{2H-2}dudv$$

$$\le C(H)\tau_n^{2(\gamma-\rho-\theta)}\sum_{l=1}^{\infty}\big\|(-\tilde{A}_n)^{-\theta+\gamma}D_n\Phi_nP_ne_l\big\|^2\int_0^{[t]^-(\tau_n)}\int_0^{u}v^{2H-2}dudv$$

$$\le C_3\tau_n^{2(\gamma-\rho-\theta)}\sum_{l=1}^{\infty}\big\|(-\tilde{A}_n)^{-\theta+\gamma}D_n\Phi_nP_ne_l\big\|^2. \tag{3.13}$$

For $\gamma-\rho-\theta>1$, we get

$$E\big\|I_n^4(t)\big\|_\rho^2\le C(H)\tau_n^2\sum_{l=1}^{\infty}\big\|(-\tilde{A}_n)^{1+\rho-\theta}D_n\Phi_nP_ne_l\big\|^2. \tag{3.14}$$

Combining (3.6), (3.13) and (3.14), we obtain

$$E\big\|I_n^4(t)\big\|_\rho\le C(H)\tau_n^{2\min(1,\gamma-\rho-\theta)}\big\|(-A)^{-\theta+\min(\gamma,1+\rho)}\Phi\big\|_{\mathcal{L}_2}. \tag{3.15}$$

(v) $I_n^5(t)$ term: By the definition of stochastic integrals with respect to FBM with $H>1/2$,

$$E\big\|I_n^5(t)\big\|_\rho^2$$

$$=\sum_{l=1}^{\infty}\int_0^{[t]^-(\tau_n)}\int_0^{[t]^-(\tau_n)}\Big\langle(-A)^\rho\Big[e^{\left([t]^-(\tau_n)-[u]^-(\tau_n)-\tau_n\right)\tilde{A}_n}$$

$$-(I+\tau_n\tilde{A}_n)^{m(t)-m(u)-1}\Big]D_n\Phi_nP_ne_l,\ (-A)^\rho\Big[e^{\left([t]^-(\tau_n)-[v]^-(\tau_n)-\tau_n\right)\tilde{A}_n}$$

$$-(I+\tau_n\tilde{A}_n)^{m(t)-m(v)-1}\Big]D_n\Phi_nP_ne_l\Big\rangle|u-v|^{2H-2}dudv$$

$$\le\sum_{l=1}^{\infty}\sum_{i=1}^{m(t)-1}\sum_{j=1}^{m(t)-1}\int_{i\tau_n}^{(i+1)\tau_n}\int_{j\tau_n}^{(j+1)\tau_n}$$

$$\times \left\| \left[e^{\left(m(t)-i-1\right)\tau_n \tilde{A}_n} - (I+\tau_n \tilde{A}_n)^{m(t)-i-1} \right] D_n \Phi_n P_n e_l \right\|_\rho$$
$$\times \left\| \left[e^{\left(m(t)-j-1\right)\tau_n \tilde{A}_n} - (I+\tau_n \tilde{A}_n)^{m(t)-j-1} \right] D_n \Phi_n P_n e_l \right\|_\rho$$
$$\times |u-v|^{2H-2} du dv. \tag{3.16}$$

Let us define $\Psi_k = (I+\tau_n \tilde{A}_n)^k - T_{\tilde{A}_n}(k\tau_n)$ for $k \geq 1$. By induction, we have

$$\Psi_k x = \sum_{i=0}^{k-1} T_{\tilde{A}_n}(k-i-1)(I+\tau_n \tilde{A}_n)^i \Psi_1 x.$$

First note that

$$(-\tilde{A}_n)^{-1}\left[I+\tau_n \tilde{A}_n - T_{\tilde{A}_n}(\tau_n)\right]x = \int_0^{\tau_n} \int_0^s (-\tilde{A}_n) T_{\tilde{A}_n}(u) x du.$$

From this, we get

$$\begin{aligned}
&\left\|(-\tilde{A}_n)\Psi_1 x\right\|_\rho \\
&\leq \int_0^{\tau_n} \int_0^s \left\|(-\tilde{A}_n)^{\max(0,1+\rho+\theta-\gamma)} T_{\tilde{A}_n}(u)\right\| du \cdot \left\|(-\tilde{A}_n)^{-\theta} x\right\|_{\min(\gamma,1+\rho+\theta)} \\
&\leq \tau_n^2 \left\|(-\tilde{A}_n)^{\max(0,1+\rho+\theta-\gamma)}\right\| \cdot \left\|(-\tilde{A}_n)^{-\theta} x\right\|_{\min(\gamma,1+\rho+\theta)}.
\end{aligned} \tag{3.17}$$

From (3.17) and Hausenblas (2003), we get

$$\left\|\Psi_k x\right\|_\rho \leq C\tau_n \left\|(-\tilde{A}_n)^{\max(0,1+\rho+\theta-\gamma)}\right\| \cdot \left\|(-\tilde{A}_n)^{-\theta} x\right\|_{\min(\gamma,1+\rho+\theta)}. \tag{3.18}$$

From (3.18), it follows that

$$\begin{aligned}
&E\left\|I_n^5(t)\right\|_\rho^2 \\
&\leq \sum_{l=1}^{\infty} \sum_{i=1}^{m(t)-1} \sum_{j=1}^{m(t)-1} \left\|\Psi_i D_n \Phi_n P_n e_l\right\|_\rho \cdot \left\|\Psi_j D_n \Phi_n P_n e_l\right\|_\rho \\
&\qquad \times \int_{i\tau_n}^{(i+1)\tau_n} \int_{j\tau_n}^{(j+1)\tau_n} |u-v|^{2H-2} du dv \\
&\leq C\tau_n^2 \left\|(-\tilde{A}_n)^{\max(0,1+\rho+\theta-\gamma)}\right\|^2 \sum_{l=1}^{\infty} \left\|(-\tilde{A}_n)^{-\theta} D_n \Phi_n P_n e_l\right\|_{\min(\gamma,1+\rho+\theta)}^2 \\
&\qquad \times \int_0^{[t]^-(\tau_n)} \int_0^{[t]^-(\tau_n)} |u-v|^{2H-2} du dv \\
&\leq C(H)\tau_n^2 \left\|(-A_n)^{\max(0,1+\rho+\theta-\gamma)}\right\|^2 \sum_{l=1}^{\infty} \left\|(-\tilde{A}_n)^{-\theta} D_n \Phi_n P_n e_l\right\|_{\min(\gamma,1+\rho+\theta)}^2
\end{aligned}$$

$$\leq C(H)\tau_n^2 \left\|(-A_n)^{\max(0,1+\rho+\theta-\gamma)}\right\|^2 \cdot \left\|(-A)^{-\theta+\min(\gamma,1+\rho+\theta)}\Phi\right\|_{\mathcal{L}_2}.$$

□

REMARK 3.1. The result of Theorem 3.1 gives the behavior of the mean square error of approximation when (V_n, A_n) converges to (V, A) or the time discretization is refined, *i.e.*, τ_n tends to 0. Thus if $\tau_n = \tau$ is fixed and the space is refined, the approximation $Y_n(k)$ will not converge to the exact solution X. But if we refine time discretization τ_n with fixed space discretization $(V_n, A_n) = (V_m, A_m)$ for some $m \in \mathbb{N}$, the approximation $Y_n(k)$ will tend to the solution of Equation (2.3). Also we can refine the time and space discretization simultaneously.

4. EXAMPLE

Let $V = L^2([0,1])$ and $A = \Delta$ with Dirichlet boundary. We consider the stochastic evolution equation:

$$\begin{cases} du(t) = \Delta u(t)dt + (I-\Delta)^{-\alpha}dB^H(t), & t \in [0,T], \\ u(0) = u_0 \in H_0^{2\gamma}. \end{cases} \tag{4.1}$$

Note that if $\alpha > 1/4$, then $(I-\Delta)^{-\alpha} \in \mathcal{L}_2(V,V)$. Obviously, the operator $(-\Delta)$ is a nonnegative self-adjoint operator on $L^2([0,1])$ and $\mathrm{Dom}((-\Delta)) = W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$. It is obvious that $e_i = \sin(\pi i x)$, $i = 1, 2, \ldots$ are the eigenfunctions and $\lambda_i = \pi i^2$ are the eigenvalues of the operator Δ.

Let P_n be the orthogonal projection onto the space $\mathrm{span}\{e_1, \ldots, e_n\}$. Then D_n is defined by $D_n a = \sum_{i=1}^n a_i e_i$. From these we obtain

$$\begin{aligned} \left\|(I - D_nP_n)x\right\|^2 &\leq \frac{1}{(\pi(n+1))^{2\gamma}} \sum_{i=n+1}^{\infty} \langle(-\Delta)^\gamma x, e_i\rangle^2 \\ &\leq \frac{1}{(\pi(n+1))^{2\gamma}} \left\|x\right\|_\gamma^2, \\ \left\|\Delta(I - D_nP_n)x\right\|^2 &\leq \sum_{i=n+1}^{\infty} (\pi(n+1))^{4-2\gamma} \langle(-\Delta)^\gamma x, e_i\rangle^2 \\ &\leq (\pi(n+1))^{4-2\gamma} \left\|x\right\|_\gamma^2. \end{aligned}$$

If $\varphi_\gamma(n) = 1/(\pi(n+1))^\gamma$, then

$$\left\|(I - D_nP_n)x\right\| \leq \varphi_\gamma(n)\left\|x\right\|_\gamma \quad \text{and} \quad \left\|\Delta(I - D_nP_n)x\right\| \leq \varphi_\gamma(n)\left\|\Delta_n\right\|\left\|x\right\|_\gamma,$$

where Δ_n is a diagonal matrix, diag$[-\lambda_1, \ldots, -\lambda_n]$. Since the semigroup generated by Δ_n is given by $e^{t\Delta_n} = \text{diag}[e^{-t\lambda_1}, \ldots, e^{-t\lambda_n}]$, we have that for all $n \geq 1$, $\|e^{t\Delta_n}\| = \sup\{|e^{t\Delta_n}a| : |a| \leq 1, a \in \mathbb{R}^n\} \leq e^{-t\lambda_1}$. Hence Δ_n satisfies (4) of Assumption 2.2. It is obvious that we may choose τ_n such that Assumption 2.3 is fulfilled since

$$\|(I + \tau_n\Delta_n)^k\| = \sup\left\{\left(\sum_{i=1}^{n}(1 - \tau_n\lambda_i)^{2k}a_i^2\right)^{1/2} : |a| \leq 1, \ a \in \mathbb{R}^n\right\}.$$

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