Positive Interest Rate Model in the Presence of Jumps

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Abstract

HJM representation of the term structure of interest rates sometimes produces the negative interest rates with positive probability. This paper shows that the condition of positive interest rates can be derived from the jump diffusion process, if a proper positive martingale process with the compensated jump process is chosen. As in Flesaker and Hughston, the condition is incorporated into the bond price process.

Keywords : Positive interest rate model, Wiener processes, Poisson processes, Risk-neutral measure

I. Introduction

The term structure of interest rates modelling usually focuses on two approaches. One, the short rate models describes the state variables or the short rate process directly and expresses the bond price or the derivatives. The affine class such as Vasicek, CIR. and Duffie and Kan (1994) are included in this category.

Another modelling to interest rate, forward rate model, specifies the dynamics of the movement of the yield directly. The approach is so called "HJM"model, and Ho-Lee, Babbs (1990) and Jamshidian (1992) follow the approach.

One disadvantage of these frameworks is the possibility of negative interest rates. This is the results of the model tractability and implementation purposes. Flesaker and Hughston (1996) and Jim and Glasserman (2001) introduced a new framework for pricing interest rate derivatives with an absence of negative interest rates.

Those papers follow the pricing kernel method, However, obtaining the positive interest rate model, the pricing kernel process should follow a specific form which is the restriction of the term structure.

The purpose of this short paper is to apply their approach to the presence of jumps. As in the approach of Flesaker and Hughston, we seek to incorporate the condition of positive

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interest rates into the bond price process. This paper also shows that the condition of positive interest rates can be derived from the jump diffusion process, if a proper positive martingale process with the compensated jump process is chosen.

II. Positive Interest Rate Model

In this section, we review on the framework of positive interest rate model, which is mainly based on Flesaker and Hughston (1996, hereafter, FH). The bond price process P_{tT} of the bond maturing at T is assumed to be adapted to the filtration ξ_T. The bond market is assumed to include a numeraire security B_t. This security is also adapted to the filtration ξ_T.

The probability space (Ω, P, ξ) represents the economy. Following FH, we introduce some notation and assumptions. (1) Let P_{ab} be the value of a discount bond at time a that matures at time b. (2) P_{ab} is assumed to be differential in b. (3) We consider a family of bond price processes P_{ab} for which $0 \leq a \leq b \leq T$, where T is the fixed terminal date. From Harrison and Pliska (1981), there exists a unique pricing measure with respect to the ratio $\dfrac{P_{ab}}{P_{aT}}$, which is a martingale in a for any bond in the given family. Following FH, we denote this martingale by N_{ab}:

$$N_{ab} = \frac{P_{ab}}{P_{aT}}, \quad P_{aa} = 1.$$ (1)

Then, using some algebra, we obtain

$$P_{ab} = \frac{N_{ab}}{N_{aa}}, \quad N_{aT} = 1.$$ (2)

For any c such that $0 \leq a \leq b \leq c \leq T$,

$$\frac{P_{ac}}{P_{ab}} = \frac{N_{ac}}{N_{ab}}.$$ (3)

Positive interest rates require the condition $\dfrac{P_{ac}}{P_{ab}} < 1$. Since the discount bond price is a decreasing function with respect to the time to maturity, the bond price with the shorter maturity has a larger value than with the longer maturity. Eventually, the bond price converge the face value 1. From (3), this condition implies that $\dfrac{N_{ac}}{N_{ab}} < 1$ or $\dfrac{\partial N_{ab}}{\partial b} < 0$.

Hence, there exists a positive martingale M_{ab}, in a, subject to $M_{0b} = 1$ such that

$$\frac{\partial N_{ab}}{\partial b} = \frac{\partial N_{0b}}{\partial b} M_{ab}$$ (4)

satisfying the boundary condition $N_{aT} = 1$ and the initial condition $N_{0b} = \dfrac{P_{0b}}{P_{0T}}$. The solution of the differential equation (4) is given by:

$$N_{ab} = 1 - \frac{\int_b^T \partial_s P_{0s} M_{as} ds}{P_{0T}}. \tag{5}$$

where $\partial_s P_{0s} = \dfrac{\partial P_{0s}}{\partial s}$.

Substituting (5) into (2), FH obtain the following formula:

$$P_{ab} = \frac{P_{0T} - \int_b^T \partial_s P_{0s} M_{as} ds}{P_{0T} - \int_a^T \partial_s P_{0s} M_{as} ds}. \tag{6}$$

Since the choice of T is arbitrary, following FH, we take the limit as T goes to infinity to simplify our analysis. Then equation (6) has a natural expression for large T. For the bond price process we obtain:

$$P_{ab} = \frac{\int_b^\infty \partial_s P_{0s} M_{as} ds}{\int_a^\infty \partial_s P_{0s} M_{as} ds}. \tag{7}$$

Following FH, to specify the bond price process (7), we require two conditions. The first is the initial discount function, or

(i) $P_{00} = 1$, $0 < P_{0b} \le 1$ for all $b \ge 0$

(ii)   There exists $\partial_b P_{0b}$ for all $b \ge 0$, and $\partial_b P_{0b}$ is negative.

The second condition is the family of positive martingales, or

(iii) For $0 \le a \le b \le s$, $M_{as} = E_a(M_{bs})$, $M_{as} > 0$, $M_{0s} = 1$, $\lim_{s \to \infty} M_{as} = 1$.

Conditions (i) and (ii) imply that all rates exist and are positive initially. Condition (iii) requires that the process M_{as} should be a positive martingale.

Instantaneous forward rates $f_{ab} = -\dfrac{\partial \ln P_{ab}}{\partial b}$ are given by

$$f_{ab} = \frac{\partial_b P_{0b} M_{ab}}{\int_b^\infty \partial_s P_{0s} M_{as} ds}. \tag{8}$$

Since the numerator and denominator of (8) are both negative according to the conditions (i), (ii) and (iii), the forward rates are positive. Similarly, the short rate $r_a = f_{aa}$ is also positive and is given by

$$\gamma_a = \frac{\partial_a P_{0a} M_{aa}}{\int_a^\infty \partial_s P_{0s} M_{as} ds}. \tag{9}$$

We incorporate the spot rates positivity property (9) into the drift, volatility and the distribution of jump size of the discount bond process. First, as in the previous section, we define a positive martingale M_{as}. By the martingale representation theorem, we can express M_{as} as

$$\frac{dM_{as}}{M_{as}} = \sigma_{as} dz_a + \gamma_{as}(dQ - \lambda da), \tag{10}$$

where σ_{as} and γ_{as} are adapted processes, and z_a and Q are one-dimensional Wiener and Poisson processes, respectively. We posit the following proposition for the discount bond price process which guarantees positive interest rate :

Proposition : With the expression for the short rate (9) and for the positive martingale process (10), for fixed b, the discount bond process P_{ab}, which guarantees positive interest rates has the following stochastic process

$$\frac{dP_{ab}}{P_{ab}} = (\gamma_a - V_a \Pi_{ab} - \lambda \Phi_{ab}) da + \Pi_{ab} dz_a + \Phi_{ab}(1 - \Gamma_a) dQ, \tag{11}$$

where

$$V_{ab} = \frac{\int_b^\infty \partial_s P_{0s} M_{as} \sigma_{as} ds}{\int_b^\infty \partial_s P_{0s} M_{as} ds}, \qquad \Gamma_{ab} = \frac{\int_b^\infty \partial_s P_{0s} M_{as} \gamma_{as} ds}{\int_b^\infty \partial_s P_{0s} M_{as} ds} \tag{12}$$

and where $\Gamma_{aa} = \Gamma_a$, $V_{aa} = V_a$, $\Phi_{ab} = \Gamma_{ab} - \Gamma_a$, and $\Pi_{ab} = V_{ab} - V_a$.

Proof : First, we substitute the positive martingale process M_{as} (10) into equation (7). To obtain a stochastic process for a discount bond process, we use the following Itô identity:

$$d(\frac{X}{Y}) = \frac{dX}{Y} - \frac{XdY}{Y^2} + \frac{XdY^2}{Y^3} - \frac{dXdY}{Y^2}$$

or

$$\frac{d(\frac{X}{Y})}{\frac{X}{Y}} = \frac{dX}{X} - \frac{dY}{Y} + (\frac{dY}{Y})^2 - \frac{dXdY}{XY}. \tag{13}$$

If we set :

$$X_{ab} = \int_b^\infty \partial_s P_{0s} M_{as} ds, \qquad Y_a = \int_a^\infty \partial_s P_{0s} M_{as} ds, \quad \text{and} \quad \text{from (7)}$$

$$P_{ab} = \frac{X_{ab}}{Y_a}$$

then, for fixed b,

$$dX_{ab} = \int_b^\infty \partial_s P_{0s} dM_{as} ds, \quad dY_a = \int_a^\infty \partial_s P_{0s} dM_{as} ds - \partial_a P_{0a} dM_{as} da.$$

For convenience, we omit the subscripts a or ab in the X_{ab} and Y_a hereafter.

Now using $dz dQ = 0$, $dz^2 = dt$, $dz dt = 0$, $dQ^2 = dQ$, $dt^2 = 0$, and $dQ dt = 0$, and defining

$$V_{ab} = \frac{\int_b^\infty \partial_s P_{0s} M_{as} \sigma_{as} ds}{\int_b^\infty \partial_s P_{0s} M_{as} ds}, \qquad \Gamma_{ab} = \frac{\int_b^\infty \partial_s P_{0s} M_{as} \gamma_{as} ds}{\int_b^\infty \partial_s P_{0s} M_{as} ds},$$

we obtain:

$$\frac{dX}{X} = V_{ab} dz_a + \Gamma_{ab}(dQ - \lambda da),$$

$$\frac{dY}{Y} = V_a dz_a - r_a da + \Gamma_a(dQ - \lambda da),$$

$$(\frac{dY}{Y})^2 = V_a^2 da + \Gamma_a^2 dQ,$$

and

$$\frac{dX}{X} \frac{dY}{Y} = V_{ab} V_a da + \Gamma_{ab} \Gamma_a dQ.$$

If we substitute these four equations into (13), we obtain (11). ∎

Next, as in FH, we briefly discuss an appropriate change of measure to move from the terminal measure to the risk neutral measure.

As in Babbs and Webber (1994), in the risk neutral measure, we want to find z_a^0 and λ^0 such that $dz_a^0 = dz_a - V_a da$ and $\lambda^0 = (1 - \eta_a)\lambda$, where θ and η are the market prices of risk. The η_a are determined to ensure that the numeraire security B_a satisfies the following condition:

$$\rho_a = \frac{P_{ab} B_a}{P_{0b}}, \tag{14}$$

where ρ_a is the Radon-Nikodym derivative and B_a is the money market account. Since

$$N_{00} = (P_{0T})^{-1}, \quad N_{0b} = 1 - (P_{0T})^{-1} \int_b^T \partial_s P_{0s} M_{as} ds,$$

Letting T go infinite, we can find $P_{0b} = -\int_b^T \partial_s P_{0s} M_{as} ds$. Then

$$B_a = \frac{\rho_a}{-\int_a^\infty \partial_a P_{0s} M_{as} ds} \tag{15}$$

$$B_a = \rho_a P_a, \tag{16}$$

where

$$P_a = \frac{1}{-\int_a^\infty \partial_a P_{0s} M_{as} ds}. $$

An advantage of the expressions (15) and (16) is that the money market account B_a is identified as ratio of a martingale process ρ_a and the discount factor P_a. As explained in FH, the process P_a is viewed as an absolute numeraire. Following the same procedure in Proposition taking $X=1$ and $Y = -\int_a^\infty \partial_a P_{0s} M_{as} ds$, the relevant stochastic process for P_a is given by

$$\frac{dP_a}{P_a} = (r_a + V_a^2 + \lambda \Gamma_a) da - V_a dz_a + (\Gamma_a^2 - \Gamma_a) dQ. \tag{17}$$

The price process P_a as the natural numeraire has the property that the ratio of any bond price to this numeraire is a martingale under the risk-neutral measure. As in the discount bond price process, which guarantees positive interest rates (11), V_a may be identified as the volatility of the bond price. Similarly, Γ_a can be obtained as the jump distribution of the bond price.

III. The Pricing of Contingent Claims

To price the contingent claims in the terminal or T-probability measure, we define C_a to be the random payout of an interest rate derivative at time a. Then, for $t \le a \le T$, the conditional expectation, $E[\frac{C_a}{P_{aT}} \mid \xi_t]$ is a martingale under T-measure. The present value of the derivative C_0 can be expressed as

$$C_0 = P_{0T} E[\frac{C_a}{P_{aT}}]. \tag{18}$$

Setting $b = T$ in (6), equation (18) becomes :

$$C_0 = E[(P_{0T} - \int_a^T \partial_s P_{0s} M_{as} ds) * C_a]. \tag{19}$$

In a similar way as in (7), (19) can be represented as

$$C_0 = E[-(\int_a^\infty \partial_s P_{0s} M_{as} ds) * C_a],\qquad(20)$$

where the positive martingale process M_{as} is expressed as :

$$\frac{dM_{as}}{M_{as}} = \sigma_{as} dz_a + \gamma_{as}(dQ - \lambda da).\qquad(21)$$

IV. Conclusion

We have extended the approach of FH to the presence of jumps in interest rates in this paper. Following FH, we have incorporated the positive interest rates property into the discount bond price, but in a different way from that of Miltersen (1994). Miltersen incorporated the condition into the forward rate process. As seen in equation (11), the condition of positive interest rates depends on the types of the volatility structures and the distribution of jump sizes of the bond price process.

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