Nonparametric Inference for Accelerated Life Testing*

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가속화 수명 실험에서의 비모수적 추론

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Abstract

Several statistical methods are introduced to analyze the accelerated failure time data. Most frequently used method is the log-linear approach with parametric assumption. Since the accelerated failure time experiments are exposed to many environmental restrictions, parametric log-linear relationship might not be working properly to analyze the resulting data. The models proposed by Buckley and James(1979) and Stute(1993) could be useful in the situation where parametric log-linear method could not be applicable. Those methods are introduced in accelerated experimental situation under the thermal acceleration and discussed through an illustrated example.

1. Introduction

Accelerated life testing of products under higher stress levels without introducing additional failure modes can provide significant savings of both time and money.

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Often one desires to estimate the relationship between some measures of product performance and one or more stresses, environmental or other independent variables. Typically, performance measurements are obtained at a number of stress conditions. The resulting data are used to estimate relationship between performance and stress. The estimated relationship, which smooths the data, is then used to estimate performance at one or more conditions.

It may happen that the performance data are censored; that is, some performance values at some stress conditions are known only to be above or else below some value. Such data often arise when the dependent (performance) variable is time to failure. The time to failure of each unfailed unit is known only to exceed some value, which represents its survival time at the time of the analysis.

Several approach have been suggested fitting linear regression models to censored data. Nelson and Hahn(1972, 1973) proposed to estimate relationships between stress and product life from the censored data. Later Hahn and Schmee(1979) proposed an iterative least square estimation and showed that it was as efficient as maximum likelihood estimator, but easier to use.

Buckley and James(1979) proposed an extension of least squares for fitting multiple regression models when the response variable is right-censored as in the analysis of survival time data. The Buckley-James method has been shown to have good statistical properties under usual regularity conditions(Lai and Ying, 1991).

Stute(1993) proposed a weighted least square method and adopted a candidate method replacing Cox(1972)'s proportional hazard model in biometric fields(Orbe, Ferreira, Nunez-Anton, 2002).

Buckley and James' estimator and Stute's have been in common in biometric fields. In this paper we focus the generalization of Buckley and James' estimator and Stute's under accelerated life testing. This could be viewed as a nonparametric generalization of Hahn and Schmee(1979)'s method. In section 2, statistical aspects of accelerated life testing is introduced and the need of nonparametric methods are emphasized in life testing field. Current nonparametric methods are modified and discussed in section 3 and an example is illustrated with implications in section 4.

2. Statistical aspects of accelerated life testing

Accelerated life testing is achieved by subjecting the test units to conditions that are more severe than normal to shorten lives. If the results can be extrapolated to the normal conditions, they yield estimates of the life under normal conditions. Correct analysis of data gathered via such accelerated life testing would yield parameters and other information for the product life under use stress conditions.

Statistically speaking, there are two major concerns in accelerated life testing; the life distribution and the relationship between failure times and stress levels. Analysis of accelerated life test data, then, consists of an underlying life distribution that describes the product at different stress levels and a stress-life relationship that quantifies the manner in which life distribution changes across different stress levels.

2.1 Life distribution

The first step in performing an accelerated life test analysis is to choose life distribution. Although it is rarely appropriate, the exponential distribution, because of its simplicity, is very commonly used as the underlying life distribution. The Weibull and log-normal distributions, which require much more involved calculation, are more appropriate for most uses.

Whenever we assume that the data follow a specific distribution, we also assume risk. If the assumption is invalid, then the confidence levels of the confidence intervals or the hypothesis tests will be incorrect. The consequences of assuming the wrong distribution may prove very costly.

2.2 Life-stress relationship

When discussing the life of manufactured goods generally, the expression "$\theta°$ C rule" can be used. This expression can be used as in the "10°C rule" to mean that a 10°C rise in the ambient temperature cuts life in half, a 20°C rise in ambient temperature cuts in life in one quarter, etc.

The Arrhenius model is widely used for acceleration of temperature-related stress. The Arrhenius life-stress relationship is given by:

$$L(V) = Ce^{\frac{B}{V}},$$

where

L represents a quantifiable life measure like median life or B(10) life,

V represents the stress level (formulated for temperature and temperature values in absolute units like degrees Kelvin,

C is one of the model parameters to be determined, (C > 0),

B is another model parameter to be determined.

The parameter B can be replaced by :

$$B = \frac{E_A}{K},$$

where

E_A represents activation energy,

K represents Boltzman's constant (8.623 × 10^{-5} eVK^{-1}.

3. Log-linear regression with censored data

3.1 Schmee and Hahn estimator

Assume the standard simple regression model within the region of interest between the stress x and the average time to failure μ_x for some device on life test at that stress, i.e.,

(1) $\mu_x = \beta_0 + \beta_1 x$,

where β_0 and β_1 are unknown parameters. One or more units are tested at each of several stresses. The resulting data consist of the failure times on the failed units or "run-outs."

More formally, the linear regression model (1) is

(2) $t_i = \beta_0 + \beta_1 x + \varepsilon_i, \quad (i = 1, \cdots, n)$

where x is a given stress and the ε_is are independently and identically distributed with normal distribution function, mean zero and variance σ^2. The response variable t_is are sometimes log-transformed. One could use other known distribution of ε_i such as extreme value distribution.

Since we usually only observe $y_i = \min(t_i, c_i)$ due to some restricted experimental situations, where c_is are censoring times, the usual least square approach is not applicable.

In many accelerated experiments, various censoring patterns occurred. Mostly both type I and type III censoring happen simultaneously. The simplest case is type I censoring, which is usually assumed through the experiment. However, if we consider certain kind of failure mode, then type III censored data should be considered together.

Hahn and Schmee(1979) assumed the case of Type I censored data. Let c_x denote the censoring time for a particular run-out at stress x. Then, using the well-known properties of the truncated normal distribution, expected value μ_x^* of the failure time for this unit is

(3) $\mu_x^* = \mu_x + \sigma\phi(z)/[1 - \Phi(z)]$

where

(4) $z = (c_x - \mu_x)/\sigma$

and $\phi(z)$ and $\Phi(z)$ denote the ordinate at z and the area to the left of z of a standard normal distribution, respectively. For this situation, Hahn and Schmee proposed an iterative least square procedure; i.e., first, estimate the regression coefficient β_0 and β_1 by treating censored observation as if they were uncensored. Secondly, estimate equation (3) and replace it with the censored observations. Then obtain a revised least square estimators using the replaced observations. Repeat these procedures until convergence is achieved. The variance of the estimated regression coefficients is also obtained by standard regression method. They asserted that their method is very comparable to maximum likelihood estimation and easier to obtain the estimators.

3.2 Buckley and James estimator

Buckley and James(1979) considered the same form of linear regression model (2) without assuming normality; i.e., ε_is are distributed with an unspecified distribution function F, mean zero and finite variance. Buckley and James(1979) define the pseudo random variable

(5) $y_i^* = y_i\delta_i + E(t_i | t_i > y_i)(1-\delta_i)$,

where $\delta_i = I(t_i \le c_i)$, the censoring indicator. It can be shown that $E(y_i^*) = E(t_i)$. The idea then is to replace y_i for censored observations with y_i^*. To do this we need to estimate the quantity $E(t_i | t_i > y_i)$ for such y_i. Now

(6) $E(t_i | t_i > y_i) =$

$$\int_{y_i - \beta_0 - \beta_1 x}^{\infty} \varepsilon \frac{dF}{1 - F(y_i - \beta_0 - \beta_1 x)} .$$

In order to obtain the estimator of $E(t_i | t_i > y_i)$, we need an estimator of F. The usual estimator of F is Kaplan and Meier(1958)'s estimator; i.e.,

$$\widehat{F}(t) = 1 - \prod_{j:\, y_{(j)} \le t} \left(\frac{n_j - d_j}{n_j} \right)^{\delta_{(j)}},$$

where $n_j = \#\{i: y_i \ge y_{(j)}\}$ and d_j is the number of failures at $y_{(j)}$ with ordered failure times $y_{(1)} \le y_{(2)} \le \cdots \le y_{(n)}$ and the corresponding $\delta_{(j)}$.

One can estimate (6) like

(7) $y_i^* = y_i\delta_i +$

$$\left(\widehat{\beta}_1 x_i + \frac{\sum_{e_j > e_i} w_j e_j}{1 - \widehat{F}(e_i)} \right)(1-\delta_i),$$

where w_js are steps of Kaplan and Meier's estimator like

(8) $$w_j = \frac{\left(\frac{n_j - d_j}{n_j} \right)^{\delta_{(j)}}}{\prod_{i=1}^{j-1} \left(\frac{n_i - d_i}{n_i} \right)^{\delta_{(i)}}}$$

with the residual $e_i = y_i - \widehat{\beta}_1 x$.

Once we observe y_i^*, then a reasonable estimate of β_1 would be

$$\widehat{\beta}_1 = \frac{\sum_i (x_i - \bar{x}) y_i^*(\widehat{\beta}_1)}{\sum_i (x_i - \bar{x})^2} .$$

Replacing y_i^* by their estimates and taking into account that estimates depend on β_1, we need iterations. Iterations based on these are then performed until the sequence of $\widehat{\beta}_1$ meets a convergence criteria, for instance the difference of successive two estimators are less than 0.001. After we obtain the convergent $\widehat{\beta}_1$, then we can have $\widehat{\beta}_0 = \bar{y}^* - \widehat{\beta}_1 \bar{x}$.

In this method of fitting regression model to censored data, the censored observations are replaced selectively by the values predicted by the currently fitted model where all the observations are treated as if they were uncensored. However, the predicted value for the i-th censored observation is often less than the censoring time for that observation, which is not admissible. Chatterjee and McLeish(1986) suggested to retain c_i like

(9) $y_i^* = \begin{cases} c_i, & \text{if } c_i > y_i^* \\ y_i^*, & \text{otherwise} \end{cases}$.

The Buckley and James method can be viewed as a nonparametric analogue of a normal theory technique due to Hahn and Schmee(1979).

Buckley and James gave the variance estimator of $\hat{\beta}_1$ without mathematical justification by

$$var(\hat{\beta}_1) = \frac{\widehat{\sigma^2_{BJ}}}{\sum_i (x_i - \overline{x}_u)^2},$$

where

$$\hat{\sigma}^2_{BJ} = \frac{1}{(n_u - 2)} \sum_{i=1}^{n} \left(\delta_i e_i - n_u^{-1} \sum_{j=1}^{n} \delta_j e_j \right)^2.$$

and n_u is the number of uncensored observations. The variance estimator of $\hat{\beta}_0$ is following by the standard regression method.

Weissfeld and Schneider(1986) proposed the alternate of σ^2 like

$$\hat{\sigma}^2_{WS} = \frac{n_u}{(n_u - 2)n} \sum_{i=1}^{n} \left(\delta_i e_i^2 + (1 - \delta_i) \sum_{j=i}^{n} w_j e_j^2 \right)^2.$$

This estimator, unlike $\hat{\sigma}^2_{BJ}$, uses the information in both the censored and uncensored observations. Weissfeld and Schneider showed that $\hat{\sigma}^2_{WS}$ tended to have a smaller mean squared error than $\hat{\sigma}^2_{BJ}$.

3.3 Stute estimator

Stute(1993) proposed the weighted least square estimators by minimizing

$$\sum_{i=1}^{n} w_i (y_{(i)} - \beta_0 - \beta_1 x_i)^2,$$

where w_is are defined in (8) and x_i is the corresponding value of $y_{(i)}$. The weighted least square estimators are easily obtained ;

$$\hat{\beta}_1^* = \frac{\sum_i w_i (x_i - \overline{x}) y_{(i)}}{\sum_i w_i (x_i - \overline{x})^2},$$

$$\hat{\beta}_0^* = \overline{y} - \hat{\beta}_1^* \overline{x}.$$

Stute's estimator can be viewed as the generalized least square estimator such as

$$\hat{\underline{\beta}}^* = (X^T W X)^{-1} X^T W \underline{y}_{(\cdot)},$$

where X is the design matrix, X^T is the transpose of the matrix X, W is a diagonal matrix with the weights w_j given in (8) and $\underline{y}_{(\cdot)} = (y_{(1)}, \cdots, y_{(n)})^T$.

Stute(1993) studied the consistency of this estimator and its asymptotic normality. The variance of regression coefficient $\hat{\underline{\beta}}^*$ is

(10) $var(\hat{\underline{\beta}}^*) = \sigma^2 (\Sigma^{-1} \Sigma_0 \Sigma^{-1})$,

where $\Sigma = X^T W X$ and $\Sigma_0 = X^T W^2 X$. The estimated variance of σ^2 can be obtained like $\hat{\sigma}^2_{BJ}$.

Stute also showed that his estimator outperforms the Buckley and James estimator in his simulation study where censoring occurs in the same pattern of life times.

Buckley and James's estimator and Stute's allow the estimation without assuming any distribution for failure times. Furthermore, those two methods are based on solid statistical background such as least square method. Therefore, they are interesting alternatives to the well-known parametric approach.

4. Illustrated Example

Table 1 gives the results of temperature accelerated life tests on electrical insulation in 40 motorettes, given by Hahn and Schmee(1979). Ten motorettes were tested at each of four temperatures. Testing was terminated at different times at each temperature, resulting in a total 17 failed units and 23 censored ones. The model used to analyze the data assumes that:

(1) for any temperature, the distribution of time to failure is lognormal or unknown;

(2) The standard deviation σ is constant;

(3) the mean of the logarithm of the time to failure μ_x holds the Arrhenius relationship to the stress levels.

An Arrhenius-lognormal model is fitted to the data in this example. That is, the data are assumed to have the lognormal (base e) distribution, and the location parameter of the lognormal distribution is assumed to depend on the centigrade temperature through the Arrhenius relationship (1) where $x = \frac{11604}{{}^\circ C + 273.2}$ is reciprocal absolute temperature. The graphic output consists of the probability plots of the data and the distribution fit lines.

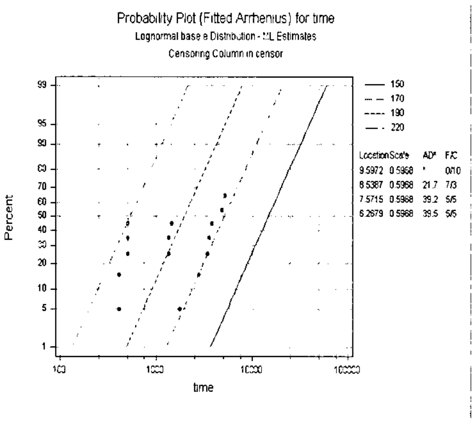
The usual accelerated life testing analysis by the common commercial software like MINITAB under the assumptions (1) through (3) produces the following output by maximum likelihood estimation.

[Table 1] Life data (unit : hrs)

150。C	170。C	190。C	220。C
8064+	1764	408	408
8064+	2772	408	408
8064+	3444	1344	504
8064+	3542	1344	504
8064+	3780	1440	504
8064+	4860	1680+	528+
8064+	5196	1680+	528+
8064+	5448+	1680+	528+
8064+	5448+	1680+	528+
8064+	5448+	1680+	528+

(+ : censored data)

Regression Table			
Predictor	DF	Coef	Std Err
INTERCPT	1	-13.858	2.180
X	1	0.85527	0.08663
SCALE	1	0.5968	0.1090



[Figure 1] Probability Plot

From the plot one can easily find out that the slope of the estimated line is a bit biased. Probably a bit steeper line would be more appropriate. If an Arrhenius-Weibull model is tried, the similar plot would be shown. The parametric assumption induces this phenomenon. One can often see this kind of happening in practice. More reliable approach would be needed, which is not affected much by the assumed parametric distribution.

Hahn and Schmee's method shows a pretty similar result as MINITAB output. It's because they also assume lognormal distribution. It is worth to note that the standard errors of the estimates are smaller than their maximum likelihood estimators.

Regression Table			
Predictor	DF	Coef	Std Err
INTERCPT	1	-13.397	1.3270
X	1	0.8342	0.052
SCALE	1	0.4706	0.051

Buckley and James' method shows a bit different results. The slope estimator is steeper than maximum likelihood estimator and Hahn and Schmee's method. However, the scale estimator and its standard error is slightly larger. It's because this dos not assume a particular life distribution.

Regression Table			
Predictor	DF	Coef	Std Err
INTERCPT	1	-15.6736	0.79310
X	1	0.9195	0.03256
SCALE	1	0.6258	0.09076

Stute's method shows a bit slower slope with moderate scale estimator comparing to other methods. Stute mentioned that his slope estimator has smaller mean squared errors than Buckley and James estimator in his simulation study and it worked this example again.

Regression Table

Predictor	DF	Coef	Std Err
INTERCPT	1	-12.3399	0.80504
X	1	0.7777	0.03146
SCALE	1	0.6066	0.08443

All mentioned methods have their own benefits and shortcomings. But nonparametric methods like Buckley and James' method and Stute's seem to be superior to parametric methods like maximum likelihood estimator and Hahn and Schmee's estimator since they yield quite similar results with moderate standard errors without assuming particular life distribution. It is also worthwhile to note that the calculation of Stute's estimator is the simplest, meanwhile all other methods need iterations with proper convergence criterion.

5. Conclusion

Accelerated testing, when properly modeled and analyzed, yields desired information on product life or performance under normal use. Analysis of accelerated life test data consists of an underlying life distribution and a stress-life relationship.

The life distribution enables assessment of product reliability at any desired stress level. However, if the assumed life distribution is invalid, then the statistical inferences based on the estimation would be incorrect and the consequences cause huge cost.

Some interesting nonparametric statistical methods were presented in this paper. Buckley and James's method and Stute's, which are relatively well introduced in biometric fields, have been rarely used in accelerated life testing field. However, since they both are based on well-known linear regression model theory, practitioners would apply easily in the data analysis. This is probably the first paper to apply those methods in accelerated life testing field. The combination of the current method and proposed methods here gives engineers the opportunity of performing data analysis in accelerated life testing area. Larger sample sizes with formal experimental design should be explored for sound results of both the accelerated life test and analysis and more theoretical studies of further applications would be needed.

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