A VSR $\bar{X}$ Chart with Multi-state VSS and 2-state VSI Scheme

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Abstract

Variable sampling interval (VSI) control charts vary the sampling interval according to value of the control statistic while the sample size is fixed. It is known that control charts with 2-state VSI scheme, which uses only two sampling intervals, give good statistical properties. Variable sample size (VSS) control charts vary the sample size according to value of the control statistic while the sampling interval is fixed. In the VSS scheme no optimal results are known for the number of sample sizes. It is also known that the variable sampling rate (VSR) $\bar{X}$ control chart with 2-state VSS and 2-state VSI scheme leads to large improvements in performance over the fixed sampling rate (FSR) $\bar{X}$ chart, but the optimal number of states for sample size is not known. In this paper, the VSR $\bar{X}$ charts with multi-state VSS and 2-state VSI scheme are designed and compared to 2-state VSS and 2-state VSI scheme. The multi-state VSS scheme is considered to achieve an additional improvement by switching from the 2-state VSS scheme. On the other hand, the multi-state VSI scheme is not considered because the 2-state scheme is known to be optimal. The 3-state VSS scheme improves substantially the sensitivity of the $\bar{X}$ chart especially for small and moderate mean shifts.

1. Introduction

Control charts have been widely used to monitor processes in detecting changes in the process that may result in lower-quality process output. A typical example of the control chart is the Shewhart $\bar{X}$ chart. $\bar{X}$ charts are shown to be less efficient in monitoring small

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shifts in the process parameters compared to large ones despite its easiness of application. Many studies have been done to improve the efficiency of the $\overline{X}$ chart by modifying the procedure.

A recent modification of the $\overline{X}$ chart is to use the variable sampling interval (VSI) scheme which allows the time intervals between samples to vary depending on the previous value of the control statistic. Another modification of the $\overline{X}$ chart is to use the variable sample size (VSS) scheme which allows the sample size used at each sampling point to vary depending on the previous value of the control statistic. Variable sampling rate (VSR) control charts allow both the sample size and the sampling interval to vary depending on the previous value of the control statistic. VSR $\overline{X}$ charts were studied by Prabhu et al. (1994), Costa (1997), and Park and Reynolds (1999).

Prabhu et al. (1994) studied the VSR $\overline{X}$ chart with 2-state VSS and 2-state VSI scheme. 2-state VSS (or VSI) scheme represents the VSS (or VSI) scheme where only two possible sample sizes (or sampling intervals) are considered. They showed that it is more efficient in detecting shifts in the process mean than the traditional FSR $\overline{X}$ chart or the $\overline{X}$ chart with 2-state VSI scheme or the $\overline{X}$ chart with 2-state VSS scheme. Zimmer et al. (1998) proposed the $\overline{X}$ chart with 3-state VSS scheme and compared its statistical performance to that with 2-state VSS scheme. In this paper, we propose a VSR $\overline{X}$ chart with g-state VSS (g=3, 4) and 2-state VSI scheme, and compare its performance to that with 2-state VSS and 2-state VSI scheme.

2. Description of the VSR $\overline{X}$ Chart

Let $\{X_{t1}, X_{t2}, \cdots, X_{tn}\}$ be a random sample obtained at time t from a normal distribution with mean, μ_t, and standard deviation, σ, and the objective is to detect shifts in μ_t from a target value μ_0. Let N_t be the sample size at the t-th sampling time and H_t be the interval between sampling times $t-1$ and t. Then the standardized sample mean at sampling time t is obtained as $Z_t = \sqrt{N_t}\left[(\overline{X}_t - \mu_0)/\sigma\right]$, and the monitoring procedure is to signal if $|Z_t| \geq c$ for a control limit c. The values of N_t and H_t for the sample t are determined according to the value of the previous statistic, Z_{t-1}.

It is known that the VSS $\overline{X}$ chart outperforms the traditional FSR $\overline{X}$ chart, but no optimal results are known for the choice of number of different sample sizes. Usually two sample sizes are considered for simplicity.

It has been shown that using only two sampling intervals is the optimal scheme in VSI control charts (see, for example, Reynolds and Arnold, 1989; and Runger and Montgomery, 1993). In this paper we also consider only two sampling intervals, h_1 and h_2, for the VSI scheme. Then the sampling interval H_t can be represented as

$$H_t = \begin{cases} h_1 & \text{if } |Z_{t-1}| < c_I \\ h_2 & \text{if } c_I \leq |Z_{t-1}| < c \end{cases},$$

where $h_2 < h_1$, and c_I denotes the threshold limit to switch between the two sampling intervals. In many applications in which the VSR chart with 2-state VSI scheme is to be used it may be practically convenient to restrict c_I equal to one of the threshold limits for sample size. It has been shown that this restriction leads to only a negligible amount of loss in the performance of the VSR $\overline{X}$ chart with 2-state VSS and 2-state VSI scheme (see, for example, Park and Reynolds, 1999).

In this section we describe the multi-state VSS scheme when the VSI scheme is fixed as the 2-state scheme and the threshold limit c_I for sampling interval is restricted to be equal to one of the threshold limits for sample size.

2.1 The 2-state VSS and 2-state VSI Scheme

In the 2-state VSS scheme, we consider two sample sizes, n_1 and n_2, and divide the continuation region of the $\overline{X}$ chart, $(-c, c)$, into two subregions. The sample size N_t for the 2-state VSS scheme is determined according to the previous control statistic as follows.

$$N_t = \begin{cases} n_1 & \text{if } |Z_{t-1}| < c_{S1} \\ n_2 & \text{if } c_{S1} \leq |Z_{t-1}| < c \end{cases},$$

where $n_1 < n_2$, and c_{S1} denotes the threshold limit to switch between the two sample sizes.

When the same threshold limit is used, the sample size N_t and the sampling interval H_t can be represented as follows.

$$(N_t, H_t) = \begin{cases} (n_1, h_1) & \text{if } |Z_{t-1}| < c_{S1}\,(= c_I) \\ (n_2, h_2) & \text{if } c_{S1}\,(= c_I) \leq |Z_{t-1}| < c \end{cases}.$$

2.2 The 3-state VSS and 2-state VSI Scheme

Zimmer et al. (1998) showed that the $\overline{X}$ chart with 3-state VSS scheme is more effective in reducing the time to detect a small shift in the process mean than that with 2-state VSS scheme. We consider three sample sizes, n_1, n_2, and n_3, which are based on three subregions.

Then the sample size N_t can be represented as

$$N_t = \begin{cases} n_1 & \text{if } |Z_{t-1}| < c_{S1} \\ n_2 & \text{if } c_{S1} \le |Z_{t-1}| < c_{S2} \\ n_3 & \text{if } c_{S2} \le |Z_{t-1}| < c \end{cases},$$

where $n_1 < n_2 < n_3$, and c_{S1} and c_{S2} denote the threshold limits to switch among the three sample sizes.

We restrict the threshold limit for sampling interval, c_I, so that $c_I = c_{S1}$ or $c_I = c_{S2}$ for convenience. According to the restriction, the sample size N_t and the sampling interval H_t for the VSR $\bar{X}$ chart with 3-state VSS scheme and 2-state VSI scheme can be represented as follows. For the case that $c_I = c_{S1}$,

$$(N_t, H_t) = \begin{cases} (n_1, h_1) & \text{if } |Z_{t-1}| < c_{S1} \\ (n_2, h_2) & \text{if } c_{S1} \le |Z_{t-1}| < c_{S2} \\ (n_3, h_2) & \text{if } c_{S2} \le |Z_{t-1}| < c \end{cases}.$$

For the case that $c_I = c_{S2}$,

$$(N_t, H_t) = \begin{cases} (n_1, h_1) & \text{if } |Z_{t-1}| < c_{S1} \\ (n_2, h_1) & \text{if } c_{S1} \le |Z_{t-1}| < c_{S2} \\ (n_3, h_2) & \text{if } c_{S2} \le |Z_{t-1}| < c \end{cases}.$$

2.3 The 4-state VSS and 2-state VSI Scheme

The 4-state VSS scheme has three threshold limits, which divide the continuation region of the control chart into four subregions. The sample size N_t can be represented as

$$N_t = \begin{cases} n_1 & \text{if } |Z_{t-1}| < c_{S1} \\ n_2 & \text{if } c_{S1} \le |Z_{t-1}| < c_{S2} \\ n_3 & \text{if } c_{S2} \le |Z_{t-1}| < c_{S3} \\ n_4 & \text{if } c_{S3} \le |Z_{t-1}| < c \end{cases},$$

where $n_1 < n_2 < n_3 < n_4$, and c_{S1}, c_{S2}, and c_{S3} denote the threshold limits to switch among the four sample sizes.

Although there are four threshold limits, c_I, c_{S1}, c_{S2}, and c_{S3}, we restrict the choice of threshold limits so that $c_I = c_{S1}$ or $c_I = c_{S2}$ or $c_I = c_{S3}$. According to the restriction, the sample size N_t and the sampling interval H_t for the VSR $\bar{X}$ chart with 4-state VSS scheme and 2-state VSI scheme can be represented as follows. For the case that $c_I = c_{S1}$,

$$(N_t, H_t) = \begin{cases} (n_1, h_1) & \text{if } |Z_{t-1}| < c_{S1} \\ (n_2, h_2) & \text{if } c_{S1} \le |Z_{t-1}| < c_{S2} \\ (n_3, h_2) & \text{if } c_{S2} \le |Z_{t-1}| < c_{S3} \\ (n_4, h_2) & \text{if } c_{S3} \le |Z_{t-1}| < c \end{cases}.$$

For the case that $c_I = c_{S2}$,

$$(N_t, H_t) = \begin{cases} (n_1, h_1) & \text{if } |Z_{t-1}| < c_{S1} \\ (n_2, h_1) & \text{if } c_{S1} \le |Z_{t-1}| < c_{S2} \\ (n_3, h_2) & \text{if } c_{S2} \le |Z_{t-1}| < c_{S3} \\ (n_4, h_2) & \text{if } c_{S3} \le |Z_{t-1}| < c \end{cases}.$$

For the case that $c_I = c_{S3}$,

$$(N_t, H_t) = \begin{cases} (n_1, h_1) & \text{if } |Z_{t-1}| < c_{S1} \\ (n_2, h_1) & \text{if } c_{S1} \le |Z_{t-1}| < c_{S2} \\ (n_3, h_1) & \text{if } c_{S2} \le |Z_{t-1}| < c_{S3} \\ (n_4, h_2) & \text{if } c_{S3} \le |Z_{t-1}| < c \end{cases}.$$

3. Performance Measures of the VSR $\bar{X}$ Chart

The statistical performance of a VSR control chart can be evaluated by the average time to signal (ATS). The ATS computed for a in-control period is a measure of the false-alarm rate, and the ATS computed for an out-of-control period is a measure of the chart's ability to detect a shift to $\mu_t = \mu_0 + \delta\sigma$ if the process starts out with $\mu_t = \mu_0 + \delta\sigma$. In many situations, however, the process may start with $\mu_t = \mu_0$ and then shift from μ_0 at some unknown time in the future. When the shift occurs some time after monitoring has started, the effect of the shift to the value of the control statistics at the time of the shift can be modeled by the steady-state distribution of the control statistic. The ATS calculated under the steady-state distribution is denoted as the steady-state ATS (SSATS). General methods for computing the SSATS of VSR control charts are given in Reynolds and Arnold (2001).

Let $SSATS_\delta$ denote the SSATS when the process mean has shifted from μ_0 to $\mu_0 + \delta\sigma$. The $SSATS_\delta$ can be calculated by using a Markov chain approach, where subregions of the continuation region generated by the threshold limits for sample size correspond to transient states and the region outside the continuation region corresponds to an absorbing state. The Markov chain, for VSR $\overline{X}$ charts with g-state VSS (g=2, 3, 4) and 2-state VSI scheme, has g+1 states corresponding to the g+1 regions as follows. For $1 \le i \le g+1$,

$$I_i = (-c_i, -c_{i-1}] \cup [c_{i-1}, c_i),$$

where $c_0 = 0$, $c_i = c_{Si}$ for $1 \le i < g$, $c_g = c$, and $c_{g+1} = \infty$.

The region I_{g+1} corresponds to an absorbing state, and the transient state transition probability matrix for this Markov chain is given by

$$Q_\delta = [q_{ij}^\delta]_{g \times g}$$

where $q_{ij}^\delta = Pr(Z_t \in I_j | Z_{t-1} \in I_i, \mu_t = \mu_0 + \delta\sigma)$ for $i, j = 1, 2, \cdots, g$. For given δ, the transition probability, q_{ij}^δ, can be expressed as

$$q_{ij}^\delta = \begin{cases} \Phi(c_1 - \sqrt{n_i}\delta) - \Phi(-c_1 - \sqrt{n_i}\delta) & \text{if } j = 1 \\ \Phi(c_j - \sqrt{n_i}\delta) - \Phi(c_{j-1} - \sqrt{n_i}\delta) & \\ \quad + \Phi(-c_{j-1} - \sqrt{n_i}\delta) - \Phi(-c_j - \sqrt{n_i}\delta) & \\ \quad \text{if } j = 2, \cdots, g & \end{cases}.$$

Using properties of the Markov chain and results in Reynolds and Arnold (2001), the $SSATS_\delta$ can be obtained as

$$SSATS_\delta = s'\{[I - Q_\delta]^{-1} - \frac{1}{2} I\}h \ ,$$

where $s' = (s_1, s_2, \cdots, s_g)$ is a vector of initial probabilities, I is a $g \times g$ identity matrix, and h is a vector of sampling intervals defined as

$$h' = \begin{cases} (h_1, h_2) & \text{if } g = 2 \\ (h_1, h_2, h_2) & \text{if } g = 3 \text{ and } c_I = c_{S1} \\ (h_1, h_1, h_2) & \text{if } g = 3 \text{ and } c_I = c_{S2} \\ (h_1, h_2, h_2, h_2) & \text{if } g = 4 \text{ and } c_I = c_{S1} \\ (h_1, h_1, h_2, h_2) & \text{if } g = 4 \text{ and } c_I = c_{S2} \\ (h_1, h_1, h_1, h_2) & \text{if } g = 4 \text{ and } c_I = c_{S3} \end{cases}.$$

The initial probabilities are used as

$$s_i = q_{ii}^0/(q_{i1}^0 + q_{i2}^0 + \cdots + q_{ig}^0),\ i = 1, 2, \cdots, g$$

(see, for example, Prabhu et al., 1994; and Reynolds and Arnold, 2001).

4. Determination of Chart Parameters

In order to compare the statistical performance of the FSR $\overline{X}$ chart and the VSR $\overline{X}$ charts, one must match their in-control performances. This can be accomplished by designing the VSR chart such that the average sample size, the average sampling interval, and ATS when $\mu = \mu_0$ are equal to the fixed sample size (n_0), the fixed sampling interval (h_0), and the fixed constant (A_0), respectively. This ensures that the average number of items sampled per unit time while the process is in control are identical, and the false alarm rate of the compared charts are equal. Hence, the VSR $\overline{X}$ chart with g-state VSS $(g = 2, 3, 4)$ and 2-state VSI scheme is designed to satisfy the following three constraints.

$(C1)\ E[N_t \| Z_{t-1}| < c, \delta = 0] = n_0,$

$(C2)\ E[H_t \| Z_{t-1}| < c, \delta = 0] = h_0,$

$(C3)\ ATS_0 = A_0,$

where $\mathrm{ATS}_0 = \mathrm{s}'[\mathrm{I} - \mathrm{Q}_0]^{-1}\mathrm{h}.$

Since constraint (C2) can be expressed as

$$\frac{h_1\{2\Phi(c_I) - 1\} + h_2\{2(\Phi(c) - \Phi(c_I))\}}{2\Phi(c) - 1} = h_0\ ,$$

where $\Phi(\cdot)$ denotes the standard normal distribution function, the threshold limit for the sampling interval, c_I, is obtained as

$$c_I = \Phi^{-1}\left[\frac{(h_1 - h_0) - 2\Phi(c)(h_2 - h_0)}{2(h_1 - h_2)}\right]. \quad (1)$$

By using $ATS_0 = h_0/[2\{1 - \Phi(c)\}]$ in the FSR $\overline{X}$ chart, constraint (C3) determines the control limit as

$$c = \Phi^{-1}[1 - h_0/(2A_0)]. \quad (2)$$

Some of chart parameters are determined by constriants (C1), (C2), and (C3), and the remaining parameters can be obtained by minimizing for given $SSATS_\delta$ for given δ, n_0, h_0, and A_0. For calculation of $SSATS_\delta$, it is assumed that a possible range of values for the sample sizes is from 1 to 50, that for the sampling interval is from 0.1 to 5.0, and that for the thresholds limits is from 0.1 to c.

4.1 The 2-state VSS and 2-state VSI Scheme

In the 2-state VSS scheme, constraint (C1) can be expressed as

$$\frac{n_1\{2\Phi(c_{S1}) - 1\} + n_2\{2(\Phi(c) - \Phi(c_{S1}))\}}{2\Phi(c) - 1} = n_0.$$

Thus the threshold limit for the sample sizes, c_{S1}, is obtained as

$$c_{S1} = \Phi^{-1}\left[\frac{(n_1 - n_0) - 2\Phi(c)(n_2 - n_0)}{2(n_1 - n_2)}\right]. \quad (3)$$

In designing an VSR $\bar{X}$ chart, the objective is to find the set of seven chart parameters $\{n_1, n_2, h_1, h_2, c_I, c_{S1}, c\}$ which minimizes $SSATS_\delta$ with constraints (C1), (C2), and (C3), for fixed values of δ, n_0, h_0, and A_0. Since we restrict $c_I = c_{S1}$, the long sampling interval h_1 can be determined by equating equations (1) and (3) to obtain

$$h_1 = (F_1 + h_2 F_2)\ /(F_2 + n_2 - n_1), \qquad (4)$$

where $F_1 = (n_1 - n_2)\{2\Phi(c)((h_0 - h_2) - h_0\}$ and $F_2 = (n_1 - n_0) - 2\Phi(c)(n_2 - n_0)$. In this case, while the control limit c is fixed as equation (2) and c_I, c_{S1}, and h_1 are determined by equations (1), (3), and (4), respectively, we should find the remaining chart parameters $\{n_1, n_2, h_2\}$ which minimizes $SSATS_\delta$.

4.2 The 3-state VSS and 2-state VSI Scheme

In the 3-state VSS scheme, the threshold limits for the sample sizes, c_{S1} and c_{S2}, are obtained by constraint (C1) as

$$c_{S1} = \Phi^{-1}\left[\frac{(n_1 - n_0) - 2\Phi(c_{S2})(n_2 - n_3)}{2(n_1 - n_2)} - \frac{2\Phi(c)(n_3 - n_0)}{2(n_1 - n_2)}\right], \qquad (5)$$

$$c_{S2} = \Phi^{-1}\left[\frac{(n_1 - n_0) - 2\Phi(c_{S1})(n_1 - n_2)}{2(n_2 - n_3)} - \frac{2\Phi(c)(n_3 - n_0)}{2(n_2 - n_3)}\right]. \qquad (6)$$

When we restrict $c_I = c_{S1}$, the long sampling interval h_1 is determined by equating equations (1) and (5) as

$$h_1 = (F_1 + h_2 F_2)/(F_2 + n_2 - n_1), \qquad (7)$$

where $F_1 = (n_1 - n_2)\{2\Phi(c)(h_0 - h_2) - h_0\}$ and $F_2 = (n_1 - n_0) - 2\Phi(c_{S2})(n_2 - n_3) - 2\Phi(c)(n_3 - n_0)$. In this case, while the control limit c is fixed as equation (2) and c_I, c_{S1}, and h_1 are determined by equations (1), (5), and (7), respectively, we should find the set of remaining chart parameters $\{n_1, n_2, n_3, h_2, c_{S2}\}$ which minimizes $SSATS_\delta$.

When we restrict $c_I = c_{S2}$, h_1 is determined by equating equations (1) and (6) as

$$h_1 = (F_1 + h_2 F_2)/(F_2 + n_3 - n_2),$$

where $F_1 = (n_2 - n_3)\{2\Phi(c)(h_0 - h_2) - h_0\}$ and $F_2 = (n_1 - n_0) - 2\Phi(c_{S1})(n_1 - n_2) - 2\Phi(c)(n_3 - n_0)$. In this case, we should find the set of chart parameters $\{n_1, n_2, n_3, h_2, c_{S1}\}$ which minimizes $SSATS_\delta$.

For the numerical calculations, we select values of δ, n_0, h_0, and A_0 as follows:

$$\delta \in \{0.5, 1.0, 1.5, 2.0, 3.0\},\ n_0 \in \{3, 5\},$$
$$h_0 = 1.0,\ A_0 = 370.4.$$

The optimal chart parameters and $SSATS_\delta$ of the VSR $\bar{X}$ chart with 3-state VSS and 2-state VSI scheme are given Table 1. The entries in the first row are for the restriction that $c_I = c_{S1}$

<Table 1> Optimal chart parameters and the corresponding $SSATS_\delta$ of the VSR $\bar{X}$ chart with 3-state VSS and 2-state VSI scheme for given δ, n_0, h_0 (=1.0), A_0(=370.4)

n_0	δ	n_1	n_2	n_3	h_1	h_2	c_{S1}	c_{S2}	c_I	c	$SSATS_\delta$
3	0.5	1	16	38	1.1	0.1	1.65	2.20	1.65	3	12.10
		1	16	36	1.0	0.1	1.70	2.08	2.08	3	12.66
3	1.0	1	3	10	3.1	0.1	0.39	1.70	0.39	3	1.45
		1	2	8	1.3	0.1	0.50	1.19	1.19	3	1.84
3	1.5	2	3	7	4.2	0.1	0.28	1.90	0.28	3	0.66
		2	3	4	1.7	0.1	0.60	0.75	0.75	3	0.74
3	2.0	2	3	6	4.4	0.1	0.26	1.80	0.26	3	0.54
		2	3	4	1.7	0.1	0.60	0.75	0.75	3	0.56
3	3.0	2	3	4	4.8	0.1	0.24	1.30	0.24	3	0.50
		2	3	4	1.1	0.1	0.10	1.74	1.74	3	0.50
5	0.5	1	9	35	1.4	0.1	0.99	1.90	0.99	3	5.35
		1	11	33	1.1	0.1	1.20	1.75	1.75	3	6.02
5	1.0	3	5	13	4.2	0.1	0.28	1.90	0.28	3	0.82
		3	4	8	1.6	0.1	0.80	0.85	0.85	3	0.98
5	1.5	4	5	10	4.3	0.1	0.27	2.00	0.27	3	0.55
		4	5	6	1.7	0.1	0.60	0.75	0.75	3	0.56
5	2.0	4	5	6	4.8	0.1	0.24	1.30	0.24	3	0.51
		4	5	6	1.3	0.1	0.30	1.18	1.18	3	0.51
5	3.0	4	5	6	4.8	0.1	0.24	1.30	0.24	3	0.50
		4	5	6	1.1	0.1	0.10	1.74	1.74	3	0.50

and the entries in the second row are for the restriction that $c_I = c_{S2}$. Comparing values of $SSATS_\delta$ in Table 1 shows that the 3-state VSS and 2-state VSI scheme with restriction $c_I = c_{S1}$ performs better than that with restriction $c_I = c_{S2}$ for small and moderate shifts.

4.3 The 4-state VSS and 2-state VSI Scheme

In the 4-state VSS scheme, the threshold limits for the sample sizes, c_{S1}, c_{S2}, and c_{S3}, are obtained by constraint (C1) as

$$c_{S1} = \Phi^{-1}\left[\frac{(n_1-n_0)-2\Phi(c_{S2})(n_2-n_3)}{2(n_1-n_2)} - \frac{2\Phi(c_{S3})(n_3-n_4)-2\Phi(c)(n_4-n_0)}{2(n_1-n_2)}\right], \quad (8)$$

$$c_{S2} = \Phi^{-1}\left[\frac{(n_1-n_0)-2\Phi(c_{S1})(n_1-n_2)}{2(n_2-n_3)} - \frac{2\Phi(c_{S3})(n_3-n_4)-2\Phi(c)(n_4-n_0)}{2(n_2-n_3)}\right], \quad (9)$$

$$c_{S3} = \Phi^{-1}\left[\frac{(n_1 - n_0) - 2\Phi(c_{S1})(n_1 - n_2)}{2(n_3 - n_4)} \right. \\ \left. \frac{- 2\Phi(c_{S2})(n_2 - n_3) - 2\Phi(c)(n_4 - n_0)}{2(n_3 - n_4)}\right]. \quad (10)$$

When we restrict $c_I = c_{S1}$, h_1 is determined by equating equations (1) and (8) as

$$h_1 = (F_1 + h_2 F_2)/(F_2 + n_2 - n_1),$$

where $F_1 = (n_1 - n_2)\{2\Phi(c)(h_0 - h_2) - h_0\}$ and $F_2 = (n_1 - n_0) - 2\Phi(c_{S2})(n_2 - n_3) - 2\Phi(c_{S3})(n_3 - n_4) - 2\Phi(c)(n_4 - n_0)$. In this case we should find the set of chart parameters $\{n_1, n_2, n_3, n_4, h_2, c_{S2}, c_{S3}\}$ which minimizes $SSATS_\delta$.

When we restrict $c_I = c_{S2}$, h_1 isdetermined by equating equations (1) and (9) as

$$h_1 = (F_1 + h_2 F_2)/(F_2 + n_3 - n_2),$$

where $F_1 = (n_2 - n_3)\{2\Phi(c)(h_0 - h_2) - h_0\}$ and $F_2 = (n_1 - n_0) - 2\Phi(c_{S1})(n_1 - n_2) - 2\Phi(c_{S3})(n_3 - n_4) - 2\Phi(c)(n_4 - n_0)$. In this case we should find the set of chart parameters $\{n_1, n_2, n_3, n_4, h_2, c_{S1}, c_{S3}\}$ which minimizes $SSATS_\delta$.

When we restrict $c_I = c_{S3}$, h_1 is determined by equating equations (1) and (10) as

$$h_1 = (F_1 + h_2 F_2)/(F_2 + n_4 - n_3),$$

where $F_1 = (n_3 - n_4)\{2\Phi(c)(h_0 - h_2) - h_0\}$ and $F_2 = (n_1 - n_0) - 2\Phi(c_{S1})(n_1 - n_2) - - 2\Phi(c)(n_4 - n_0)$. In this case we should find the set of chart parameters $\{n_1, n_2, n_3, n_4, h_2, c_{S1}, c_{S2}\}$ which minimizes $SSATS_\delta$.

The optimal chart parameters and $SSATS_\delta$ of the VSR $\overline{X}$ chart with 4-state VSS and 2-state VSI scheme are given Table 2. The entries in the first row are for the restriction that $c_I = c_{S1}$, the entries in the second row are for the restriction that $c_I = c_{S2}$, and the entries in the third row are for the restriction that $c_I = c_{S3}$. Comparing values of $SSATS_\delta$ in Table 2 shows that the 4-state VSS and 2-state VSI scheme with restriction $c_I = c_{S1}$ performs better than that with restriction $c_I = c_{S2}$ or $c_I = c_{S3}$ for small shifts. This result is similar to the 3-state VSS and 2-state VSI scheme.

5. Evaluation of VSR $\overline{X}$ Charts

Previous research on the VSR $\overline{X}$ chart has shown that using the 2-state VSS and 2-state VSI scheme can significantly reduce the ATS for detecting shifts in the process mean (see, for example, Prabhu et al., 1994). This section considers the performance for the g-state VSS (g=3, 4) and 2-state VSI scheme compared to the 2-state VSS and 2-state VSI scheme.

<Table 2> Optimal chart parameters and the corresponding $SSATS_\delta$ of the VSR $\overline{X}$ chart with 4-state VSS and 2-state VSI scheme for given δ, n_0, h_0 (=1.0), A_0(=370.4)

n_0	δ	n_1	n_2	n_3	n_4	h_1	h_2	c_{S1}	c_{S2}	c_{S3}	c_I	c	$SSATS_\delta$
3	0.5	1	2	20	39	1.6	0.1	0.85	1.80	2.30	0.85	3	11.52
		1	2	16	38	1.1	0.1	1.60	1.65	2.20	1.65	3	12.10
		1	14	21	37	1.0	0.1	1.70	1.90	2.13	2.13	3	12.61
3	1.0	1	2	5	12	4.8	0.1	0.24	1.10	1.90	0.24	3	1.38
		1	2	3	10	3.7	0.1	0.30	0.32	1.80	0.32	3	1.45
		1	2	3	8	1.2	0.1	0.50	1.00	1.24	1.24	3	1.83
3	1.5	2	3	5	8	4.7	0.1	0.25	1.80	2.30	0.25	3	0.66
		1	2	3	7	4.7	0.1	0.10	0.25	1.80	0.25	3	0.67
		1	2	3	4	2.0	0.1	0.10	0.60	0.62	0.62	3	0.75
3	2.0	2	3	4	7	4.9	0.1	0.24	1.50	2.30	0.24	3	0.54
		1	2	3	7	4.7	0.1	0.10	0.25	1.80	0.25	3	0.55
		1	2	3	4	2.0	0.1	0.10	0.60	0.62	0.62	3	0.57
3	3.0	2	3	4	5	5.0	0.1	0.23	1.40	2.20	0.23	3	0.50
		2	3	4	5	1.1	0.1	0.10	1.74	2.90	1.74	3	0.50
		2	3	4	5	1.0	0.1	0.10	2.00	2.06	2.06	3	0.50
5	0.5	1	2	14	35	4.6	0.1	0.25	1.30	2.00	0.25	3	4.91
		1	2	9	35	1.4	0.1	0.90	1.00	1.90	1.00	3	5.35
		1	7	15	34	1.1	0.1	1.10	1.50	1.79	1.79	3	5.97
5	1.0	3	4	7	14	4.8	0.1	0.24	0.10	1.90	0.24	3	0.80
		3	4	5	12	3.7	0.1	0.30	0.32	1.80	0.32	3	0.82
		2	3	4	8	1.6	0.1	0.10	0.80	0.81	0.81	3	0.98
5	1.5	4	5	7	11	5.0	0.1	0.23	1.90	2.30	0.23	3	0.55
		3	4	5	10	4.7	0.1	0.10	0.25	1.90	0.25	3	0.55
		3	4	5	6	2.0	0.1	0.10	0.60	0.62	0.62	3	0.56
5	2.0	4	5	6	7	5.0	0.1	0.23	1.40	2.20	0.23	3	0.51
		4	5	6	7	1.3	0.1	0.30	1.19	2.70	1.19	3	0.51
		3	4	5	6	1.4	0.1	0.10	0.30	1.00	1.00	3	0.51
5	3.0	4	5	6	7	5.0	0.1	0.23	1.40	2.20	0.23	3	0.50
		4	5	6	7	1.1	0.1	0.10	1.74	2.90	1.74	3	0.50
		4	5	6	7	1.0	0.1	0.10	2.00	2.06	2.06	3	0.50

The scheme with restriction $c_I = c_{S1}$ performed uniformly better than any other restriction on c_I for various values of δ. Thus we compare the schemes only with the case $c_I = c_{S1}$. Table 3 gives the optimal chart parameters and the corresponding $SSATS_\delta$ of the FSR $\overline{X}$ chart and the VSR $\overline{X}$ charts with multi-state VSS and 2-state VSI scheme for given $A_0 = 370.4$. The entries in the first row are for the FSR $\overline{X}$ chart, the entries in the second row are for the VSR $\overline{X}$ chart with 2-state VSS and 2-state VSI scheme, the entries in the third row are for the VSR $\overline{X}$ chart with 3-state VSS and 2-state VSI scheme, and the entries in the fourth row are for the VSR $\overline{X}$ chart with 4-state VSS and 2-state VSI scheme. The entries in the third and fourth rows in Table 3 are the same as the entries of the first row in Table 1 and Table 2 respectively, but those are written again for comparison.

The last column labeled PR denotes the percent reduction in SSATS by switching to the g-state VSS scheme from the 2-state VSS scheme. PR is calculated as

$$PR = \frac{SSATS_{\delta,2} - SSATS_{\delta,g}}{SSATS_{\delta,2}} \times 100,$$

where $SSATS_{\delta,2}$ denotes the value of $SSATS_\delta$ for the 2-state VSS and 2-state VSI scheme, and $SSATS_{\delta,g}$ denotes the value of $SSATS_\delta$ for the g-state VSS (g =3, 4) and 2-state VSI scheme.

The results in Table 3 show that there is a significant improvement in the average time to detect process shifts by using the VSR $\overline{X}$ charts when compared to the FSR $\overline{X}$ chart for small and moderate shifts, and the predominance of the VSR charts decreases as the shift increases. Table 3 also shows that the g-state VSS (g=3, 4) and 2-state VSI scheme is more efficient than the 2-state VSS and 2-state VSI scheme. The percent reduction in $SSATS_\delta$ ranges from a modest 7.2% to a substantial 27.0% for small shifts (δ =0.5 and 1.0). When the amount of shift exceeds one standard deviation, all of the VSR chart schemes perform similarly. However, since one is frequently interested in small shifts, the g-state VSS (g=3, 4) and 2-state VSI scheme may be potentially useful in detecting a very small shift in the process mean.

Note that the optimal value for the short sampling interval h_2 is always the minimum possible sampling interval 0.1. It was shown by Reynolds and Arnold (1989), in the VSI chart, that the optimal short sampling interval is the minimum possible sampling interval. This property is also shown to hold in the VSR $\overline{X}$ charts with multi-state VSS and 2-state VSI scheme.

6. Conclusions

<Table 3> Optimal FSR and VSR $\bar{X}$ chart parameters and the corresponding $SSATS_{\delta}$

δ	n_0 / n_1	n_2	n_3	n_4	h_0 / h_1	h_2	c_{S1}	c_{S2}	c_{S3}	c_I	c	$SSATS_{\delta}$	PR
	n_0				h_0						c		
	n_1	n_2			h_1	h_2	c_{S1}			c_I	c		
	n_1	n_2	n_3		h_1	h_2	c_{S1}	c_{S2}		c_I	c		
	n_1	n_2	n_3	n_4	h_1	h_2	c_{S1}	c_{S2}	c_{S3}	c_I	c		
	3				1.0						3	60.69	
0.5	1	30			1.1	0.1	1.80			1.80	3	13.04	
	1	16	38		1.1	0.1	1.65	2.20		1.65	3	12.10	7.2
	1	2	20	39	1.6	0.1	0.85	1.80	2.30	0.85	3	11.52	11.7
	3				1.0						3	9.76	
1.0	1	7			1.4	0.1	0.96			0.96	3	1.89	
	1	3	10		3.1	0.1	0.39	1.70		0.39	3	1.45	23.3
	1	2	5	12	4.8	0.1	0.24	1.10	1.90	0.24	3	1.38	27.0
	3				1.0						3	2.91	
1.5	2	4			1.9	0.1	0.67			0.67	3	0.74	
	2	3	7		4.2	0.1	0.28	1.90		0.28	3	0.66	10.8
	2	3	5	8	4.7	0.1	0.25	1.80	2.30	0.25	3	0.66	10.8
	3				1.0						3	1.47	
2.0	2	4			1.9	0.1	0.67			0.67	3	0.56	
	2	3	6		4.4	0.1	0.26	1.80		0.26	3	0.54	3.6
	2	3	4	7	4.9	0.1	0.24	1.50	2.30	0.24	3	0.54	3.6
	3				1.0						3	1.01	
3.0	2	4			1.9	0.1	0.67			0.67	3	0.51	
	2	3	4		4.8	0.1	0.24	1.30		0.24	3	0.50	2.0
	2	3	4	5	5.0	0.1	0.23	1.40	2.20	0.23	3	0.50	2.0
	5				1.0						3	33.40	
0.5	1	27			1.2	0.1	1.42			1.42	3	6.42	
	1	9	35		1.4	0.1	0.99	1.90		0.99	3	5.35	16.7
	1	2	14	35	4.6	0.1	0.25	1.30	2.00	0.25	3	4.91	23.5
	5				1.0						3	4.50	
1.0	3	8			1.6	0.1	0.84			0.84	3	0.98	
	3	5	13		4.2	0.1	0.28	1.90		0.28	3	0.82	16.3
	3	4	7	14	4.8	0.1	0.24	1.10	1.90	0.24	3	0.80	18.4
	5				1.0						3	1.57	
1.5	4	6			1.9	0.1	0.67			0.67	3	0.56	
	4	5	10		4.3	0.1	0.27	2.00		0.27	3	0.55	1.8
	4	5	7	11	5.0	0.1	0.23	1.90	2.30	0.23	3	0.55	1.8
	5				1.0						3	1.08	
2.0	4	6			1.9	0.1	0.67			0.67	3	0.51	
	4	5	6		4.8	0.1	0.24	1.30		0.24	3	0.51	0.0
	4	5	6	7	5.0	0.1	0.23	1.40	2.20	0.23	3	0.51	0.0
	5				1.0						3	1.00	
3.0	4	6			1.9	0.1	0.67			0.67	3	0.50	
	4	5	6		4.8	0.1	0.24	1.30		0.24	3	0.50	0.0
	4	5	6	7	5.0	0.1	0.23	1.40	2.20	0.23	3	0.50	0.0

Using the VSR scheme in $\overline{X}$ charts can give significant improvements to the ability in detecting small and moderate shifts in the process mean. Specially, it has been known that the 2-state VSS and 2-state VSI scheme can detect shifts in the process mean much more quickly than the traditional FSR scheme.

In this paper the VSR $\overline{X}$ charts with g -state VSS (g=3, 4) and 2-state VSI scheme is designed and their efficiencies are analyzed. It is seen that these schemes increase the efficiency of the VSR $\overline{X}$ chart moderately or substantially for small shifts when compared to the 2-state VSS and 2-state VSI scheme. Although it may be administratively inconvenient to use a control chart having three or four different sample sizes, the proposed schemes are recommended for applications in which small shifts in the process mean are important to detect quickly.

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