ON n-TUPLES OF TENSOR PRODUCTS OF p-HYPONORMAL OPERATORS

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ABSTRACT. The operator $A \in L(H_i)$, the Banach algebra of bounded linear operators on the complex infinite dimensional Hilbert space $H_i$, is said to be $p$-hyponormal if $(A^*A)^p \geq (AA^*)^p$ for $p \in (0, 1]$. Let $\hat{H} = H_1 \otimes \cdots \otimes H_n$ denote the completion of $H_1 \otimes \cdots \otimes H_n$ with respect to some crossnorm. Let $I_i$ be the identity operator on $H_i$. Letting $T_i := I_1 \otimes \cdots \otimes I_{i-1} \otimes A_i \otimes \cdots \otimes I_n$ on $\hat{H}$, where each $A_i$ is $p$-hyponormal, it is proved that the commuting $n$-tuple $T = (T_1, \ldots, T_n)$ satisfies Bishop's condition ($\beta$) and that if $T$ is Weyl then there exists a non-singular commuting $n$-tuple $S$ such that $T = S + F$ for some $n$-tuple $F$ of compact operators.

1. INTRODUCTION

Let $L(H)$ denote the Banach algebra of bounded linear operators acting on a complex infinite dimensional Hilbert space $H$. Let $T = (T_1, \ldots, T_n)$ denote a commuting $n$-tuple of operators in $L(H)$. Recall (Curto [2], Taylor [11]) that $T$ is said to be non-singular if the Koszul complex for $T$, denoted by $K(T, H)$, is exact at every stage. Also, $T$ is said to be Fredholm if the Koszul complex $K(T, H)$ is Fredholm, i.e., all homology spaces of $K(T, H)$ are finite dimensional. In this case the index of $T$, denoted $\text{ind}(T)$, is defined as the Euler characteristic of $K(T, H)$, i.e., as the alternating sum of dimensions of all homology spaces of $K(T, H)$. If $T \in L(H)$ is Fredholm with index zero, then we say that $T$ is Weyl. We shall write $\sigma_T(T)$, $\sigma_{Te}(T)$, and $\sigma_{Tw}(T)$ for the Taylor spectrum, the Taylor essential spectrum, and Taylor-Weyl spectrum of $T$, respectively: thus,

$$\sigma_T(T) = \{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : T - \lambda \text{ is singular} \},$$

$$\sigma_{Te}(T) = \{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : T - \lambda \text{ is not Fredholm} \}.$$
and

$$\sigma_{Tw}(T) = \{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : T - \lambda \text{ is not Weyl} \}.$$  

For any open polydisk $D \subset \mathbb{C}^n$, let $O(D, \mathcal{H})$ denote the Frechét space of $\mathcal{H}$-valued analytic functions on $D$. Then we say (Eschmeier & Putinar [6]) that a commuting $n$-tuple $T$ has the single valued extension property, shortened to SVEP, if the Koszul complex $K(T - \lambda, O(D, \mathcal{H}))$ is exact in positive degrees and $T$ has Bishop’s condition $(\beta)$ if it has the SVEP and its Koszul complex has also separated homology in degree zero. Obviously, the following implication holds:

Bishop’s condition $(\beta) \implies$ the SVEP.

For more details, see (Eschmeier & Putinar [6]).

We shall write

$$\sigma_p(T) = \{ \lambda \in \mathbb{C}^n : \text{there exists a non-zero vector } x \in \mathcal{H} \text{ such that } x \in \bigcap \ker(T_i - \lambda_i) \}$$

for the eigenvalues of $T$,

$$p_{00}(T) = \sigma_T(T) \setminus \{ \sigma_{Te}(T) \cup \text{acc}(\sigma(T)) \}$$

for the Riesz points of $\sigma_T(T)$, and $\text{iso}\sigma_T(T)$ for all isolated points of $\sigma_T(T)$, respectively. Recall (Aluthge [1], Duggal [4, 5], Jeon & Duggal [8], Yingbin & Zikun [14]) that an operator $T \in L(\mathcal{H})$ is said to be $p$-hyponormal if

$$(T^*T)^p - (TT^*)^p \geq 0 \text{ for } p \in (0, 1].$$

If $p = 1$, $T$ is just hyponormal. We shall denote the class of $p$-hyponormal operators by $\Sigma(p)$; $\Sigma(p_i(p))$ shall denote the class of those $p$-hyponormal operators for which the partial isometry $U$ in the polar decomposition $T = U|T|$ is unitary.

Throughout this paper, for complex infinite dimensional Hilbert spaces $\mathcal{H}_i$ ($1 \leq i \leq n$), we let $\hat{\mathcal{H}} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ denote the completion of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ with respect to some crossnorm. Let $I_i$ be the identity operator on $\mathcal{H}_i$. For $A_i \in L(\mathcal{H}_i)$, let

$$T_i := I_1 \otimes \cdots \otimes I_{i-1} \otimes A_i \otimes \cdots \otimes I_n \text{ on } \hat{\mathcal{H}}.$$  

Then $T = (T_1, \ldots, T_n)$ is a commuting (in fact, doubly commuting) $n$-tuple of operators on $\hat{\mathcal{H}}$. 
2. RESULTS

Theorem 1. Let $A_i \in \mathcal{H}(p)$ and let $T = (T_1, \ldots, T_n)$ be the n-tuple of operators

\[ T_i := I_1 \otimes \cdots \otimes I_{i-1} \otimes A_i \otimes \cdots \otimes I_n \text{ on } \hat{\mathcal{H}}. \]

Then there exist an n-tuple $S = (S_1, \ldots, S_n)$ with

\[ S_i := I_1 \otimes \cdots \otimes I_{i-1} \otimes B_i \otimes \cdots \otimes I_n \text{ on } \hat{\mathcal{H}} \text{ for some } B_i \in \mathcal{H}(1), \]

a quasiaffinity $X$ and an injection $Y$ such that both $T$ and $S$ have Bishop's condition (β) and

\[ XT = SX \text{ and } TY = YS. \]

Proof. Given an $A_i \in \mathcal{H}(p)$, we decompose $A_i$ into its normal and pure parts by

\[ A_i = A_{i0} \oplus A_{i1} \text{ with respect to the decomposition } \mathcal{H}_i = \mathcal{H}_{i0} \oplus \mathcal{H}_{i1}. \]

Then $A_{i1} \in \mathcal{H}(p)$. Let $A_{i1}$ have the polar decomposition $A_{i1} = U_{i1}|A_{i1}|$ where $U_{i1}$ is an isometry. Define the Aluthge transform $\tilde{A}_{i1}$ of $A_{i1}$ by $\tilde{A}_{i1} = |A_{i1}|^{1/2}U_{i1}|A_{i1}|^{1/2}$. Let $\tilde{A}_{i1}$ have the polar decomposition $\tilde{A}_{i1} = V_{i1}|\tilde{A}_{i1}|$ where $V_{i1}$ is (also) an isometry. Again, define the Aluthge transform $\tilde{A}_{i1}$ of $\tilde{A}_{i1}$ by $\tilde{A}_{i1} = |\tilde{A}_{i1}|^{1/2}V_{i1}|\tilde{A}_{i1}|^{1/2}$. Then $\tilde{A}_{i1} \in \mathcal{H}(1)$ by Aluthge [1]. Let $X_{i1} = |\tilde{A}_{i1}|^{1/2}|A_{i1}|^{1/2}$ and $Y_{i1} = U_{i1}|A_{i1}|^{1/2}V_{i1}|\tilde{A}_{i1}|^{1/2}$. Then $X_{i1}$ is a quasiaffinity and $Y_{i1}$ is an injection such that

\[ X_{i1}A_{i1} = \tilde{A}_{i1}X_{i1} \text{ and } A_{i1}Y_{i1} = Y_{i1}\tilde{A}_{i1}. \]

Let $B_i = A_{i0} \oplus \tilde{A}_{i1}$. Then $B_i \in \mathcal{H}(1)$. Defining the quasiaffinity $X_i$ by $X_i = I_{\mathcal{H}_{i0}} \oplus X_{i1}$ and the injection $Y_i$ by $Y_i = I_{\mathcal{H}_{i0}} \oplus Y_{i1}$, it follows that

\[ X_iA_i = B_iX_i \text{ and } A_iY_i = Y_iB_i. \]

Considering the tensor products of operators $X$ and $Y$,

\[ X := X_1 \otimes \cdots \otimes X_n \text{ and } Y := Y_1 \otimes \cdots \otimes Y_n, \]

it is then seen that $X$ is a quasiaffinity, $Y$ is an injection and

\[ XT = SX \text{ and } TY = YS. \]

Recall from Duggal [4, Theorem 3] that if $B_1, B_2 \in L(\mathcal{H}_i)$, then

\[ B_1 \otimes B_2 \in \mathcal{H}(p) \text{ if and only if } B_1, B_2 \in \mathcal{H}(p). \] (2.1)

Hence it follows from a finite induction argument that

\[ T_i \in \mathcal{H}(p) \text{ if and only if } A_i \in \mathcal{H}(p) \text{ for all } i = 1, \ldots, n. \] (2.2)
Since each $T_i$ has Bishop's condition ($\beta$) (see Duggal [5, Theorem 1], Yingbin & Zikun [14, Theorem 7]), it follows from Wolff [13, Corollary 2.2] that the $n$-tuple $T = (T_1, \ldots, T_n)$ has Bishop's condition ($\beta$). Similarly, the $n$-tuple $S = (S_1, \ldots, S_n)$ also has Bishop's condition ($\beta$).

\[ \sigma_\ast(T) = \sigma_\ast(S), \text{ where } \sigma_\ast \text{ stands for either of } \sigma_T, \sigma_{Te}, \sigma_{Tw}. \]  

(2.3)

**Corollary 2.** We first notice that both $X$ and $Y$ constructed in proof of Theorem 1 are quasiaffinities when each $A_i$ belongs to $\mathcal{H}l(p)$.

**Proof.** We first notice that $X$ and $Y$ constructed in proof of Theorem 1 become both quasiaffinities when each $A_i$ belongs to $\mathcal{H}l(p)$. Thus $T$ and $S$ are (jointly) quasisimilar $n$-tuples. Since $T$ and $S$ have Bishop's condition ($\beta$) by Theorem 1, Putinar [9, Theorem 1] implies (2.3). This completes the proof.

Fredholm $n$-tuples enjoy most of the properties single Fredholm operators possess (see Curto [3]). It is well known that a Fredholm operator of index zero (i.e., Weyl operator) can be perturbed by a compact operator to an invertible operator. Thus one may ask if this property holds in several variables (cf. Curto [2, Problem 3]). As it turns out, this perturbation property fails in several variables (see Gelca [7] for an example). Despite the failure of this property for the general case, the following result gives a positive answer to the question in case of tensor products considered here.

**Theorem 3.** Let $A_i \in \mathcal{H}(p)$ and let $T = (T_1, \ldots, T_n)$ be an $n$-tuple of operators

\[ T_i := I_1 \otimes \cdots \otimes I_{i-1} \otimes A_i \otimes \cdots \otimes I_n \text{ on } \mathcal{H}. \]

If $T$ is Weyl and singular, then there exists a non-singular commuting $n$-tuple $S = (S_1, \ldots, S_n)$ such that $T = S + F$ for some $n$-tuple of compact operators $F_i$ ($i = 1, \ldots, n$).

**Proof.** Since $T$ has Bishop's condition ($\beta$), if $T$ is Weyl and singular, then Putinar [10, Theorem 1] implies $0 \in p_{00}(T)$. Let $f$ be the characteristic function of $0 \in \text{iso}\sigma_T(T)$. Since $f$ is analytic in a neighborhood of $\sigma_T(T)$, Taylor [11, Theorem 4.8] implies the existence of an idempotent $P_0 = f(T) \in L(\mathcal{H})$ such that $P_0 T_i = T_i P_0$, $T_i$ is quasinilpotent on ran$P_0$, and

\[ 0 \notin \sigma_T(T|_{\ker P_0}). \]  

(2.4)
Since the restriction of a $p$-hyponormal operator to an invariant subspace is again $p$-hyponormal Uchiyama ([12, Lemma 4]) and $p$-hyponormal operators are normaloid, we see that $T_i|_{\text{ran}P_0} = 0$. Then the fact that $0 \in \rho_{p0}(T)$ implies that the subspace $\text{ran}P_0$ is finite dimensional, and so $P_0$ is a compact operator on $\mathcal{H}$. Considering $F = (P_0, \ldots, P_0)$ and $S = T - F = (T_1 - P_0, \ldots, T_n - P_0)$, it now follows that $S$ is a commuting $n$-tuple. This by [Curto [3] p. 39] implies that

$$\sigma_T(S) = \sigma_T((T - F)|_{\text{ran}P_0}) \cup \sigma_T((T - F)|_{\text{ker}P_0}).$$

Obviously, $0 \notin \sigma_T((T - F)|_{\text{ran}P_0})$ and by (2.4)

$$0 \notin \sigma_T((T - F)|_{\text{ker}P_0}) = \sigma_T(T|_{\text{ker}P_0}).$$

Thus $0 \notin \sigma_T(S)$, i.e., $S = T - F$ is non-singular, and hence $T = S + F$. \hfill \Box

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REFERENCES


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