CONVERGENCE AND ALMOST STABILITY OF ISHIKAWA ITERATION METHOD WITH ERRORS FOR STRICTLY HEMI-CONTRACTIVE OPERATORS IN BANACH SPACES

Zeqing Liu, Jeong Sheok Ume*, and Shin Min Kang

Abstract. Let $K$ be a nonempty convex subset of an arbitrary Banach space $X$ and $T : K \to K$ be a uniformly continuous strictly hemi-contractive operator with bounded range. We prove that certain Ishikawa iteration scheme with errors both converges strongly to a unique fixed point of $T$ and is almost $T$-stable on $K$. We also establish similar convergence and almost stability results for strictly hemi-contractive operator $T : K \to K$, where $K$ is a nonempty convex subset of arbitrary uniformly smooth Banach space $X$. The convergence results presented in this paper extend, improve and unify the corresponding results in Chang [1], Chang, Cho, Lee & Kang [2], Chidume [3, 4, 5, 6, 7, 8], Chidume & Osilike [9, 10, 11, 12], Liu [19], Schu [25], Tan & Xu [26], Xu [28], Zhou [29], Zhou & Jia [30] and others.

1. Introduction

Let $X$ be an arbitrary Banach, $X^*$ be its dual space and $(x, f)$ be the generalized duality pairing between $x \in X$ and $f \in X^*$. The mapping $J : X \to 2^{X^*}$ defined by

$$J(x) = \{ f \in X^* : \text{Re}(x, f) = \|x\|^2 = \|f\|^2 \}, \quad x \in X,$$

is called the normalized duality mapping. It is known that $X$ is uniformly smooth if and only if $X^*$ is uniformly convex. The symbols $D(T)$, $R(T)$ and $F(T)$ denote the domain, the range and the set of fixed points of an operator $T$, respectively.

Definition 1.1 (Chidume & Osilike [9], Weng [27]). Let $X$ be an arbitrary normed linear space and $T : D(T) \subseteq X \to X$ be an operator.

Received by the editors June 3, 2004 and, in revised form, November 17, 2004.

2000 Mathematics Subject Classification. 47H05, 47H06, 47H10, 47H14.

Key words and phrases. Strictly hemi-contractive operator, local strongly pseudocontractive operator, strongly pseudocontractive operator, uniformly continuous operator, Ishikawa iteration method with errors, fixed point, almost stability, Banach space, uniformly smooth Banach space.

*This research was financially supported by Changwon National University in 2004.

(i) $T$ is said to be strongly pseudocontractive if there exists $t > 1$ such that for each $x, y \in D(T)$ and $r > 0$

$$
\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|. \tag{1.1}
$$

(ii) $T$ is said to be local strongly pseudocontractive if for each $x \in D(T)$ there exists $t_x > 1$ such that for all $y \in D(T)$ and $r > 0$

$$
\|x - y\| \leq \|(1 + r)(x - y) - rt_x(Tx - Ty)\|. \tag{1.2}
$$

(iii) $T$ is said to be strictly hemi-contractive if $F(T) \neq \emptyset$ and if there exists $t > 1$ such that for all $x \in D(T)$, $q \in F(T)$ and $r > 0$,

$$
\|x - q\| \leq \|(1 + r)(x - q) - rt(Tx - q)\|. \tag{1.3}
$$

Clearly, each strongly pseudocontractive operator is local strongly pseudocontractive.

Let $K$ be a nonempty convex subset of an arbitrary normed linear space $X$ and $T : K \to K$ be an operator. Assume that $x_0 \in K$ and $x_{n+1} = f(T, x_n)$ defines an iteration scheme which produces a sequence $\{x_n\}_{n=0}^{\infty} \subset K$. Suppose, furthermore, that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $q \in F(T) \neq \emptyset$. Let $\{y_n\}_{n=0}^{\infty}$ be any bounded sequence in $K$ and put $\varepsilon_n = \|y_{n+1} - f(T, y_n)\|$. 

**Definition 1.2.** (i) The iteration scheme $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = f(T, x_n)$ is said to be $T$-stable on $K$ if $\lim_{n \to \infty} \varepsilon_n = 0$ implies that $\lim_{n \to \infty} y_n = q$;

(ii) The iteration scheme $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = f(T, x_n)$ is said to be almost $T$-stable on $K$ if $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ implies $\lim_{n \to \infty} y_n = q$.

It is easy to see that an iteration scheme $\{x_n\}_{n=0}^{\infty}$ which is $T$-stable on $K$ is almost $T$-stable on $K$. Osilike [23] proved that an iteration scheme which is almost $T$-stable on $X$ may fail to be $T$-stable on $X$.

Let us recall the following three iteration processes due to Mann [20], Ishikawa [16] and Xu [28], respectively.

Let $K$ be a nonempty convex subset of an arbitrary normed linear space $X$ and $T : K \to K$ be an operator.

(i) For any given $x_0 \in K$ the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$
x_{n+1} = (1 - a_n)x_n + a_nTy_n, \quad y_n = (1 - b_n)x_n + b_nTx_n, \quad n \geq 0,
$$

is called the Ishikawa iteration sequence, where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are real sequences in $[0, 1]$ satisfying appropriate conditions.
(ii) In particular, if \( b_n = 0 \) for all \( n \geq 0 \), then the sequence \( \{x_n\}_{n=0}^{\infty} \) defined by
\[
x_0 \in K, \ x_{n+1} = (1 - a_n)x_n + a_nTx_n, \ n \geq 0,
\]
is called the \textit{Mann iteration sequence}.

(iii) For any given \( x_0 \in K \) the sequence \( \{x_n\}_{n=0}^{\infty} \) defined by
\[
x_{n+1} = a_n x_n + b_n Ty_n + c_n u_n, \ y_n = a'_n x_n + b'_n Tx_n + c'_n u_n, \ n \geq 0,
\]
where \( \{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty} \) are arbitrary bounded sequences in \( K \) and \( \{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}, \{c_n\}_{n=0}^{\infty}, \{a'_n\}_{n=0}^{\infty}, \{b'_n\}_{n=0}^{\infty}, \{c'_n\}_{n=0}^{\infty} \) are real sequences in \([0, 1]\) such that \( a_n + b_n + c_n = a'_n + b'_n + c'_n = 1 \) for all \( n \geq 0 \), is called the \textit{Ishikawa iteration sequence with errors}.

(iv) If, with the same notations and definitions as in (iii), \( b'_n = c'_n = 0 \) for all \( n \geq 0 \), then the sequence \( \{x_n\}_{n=0}^{\infty} \) now defined by
\[
x_0 \in K, \ x_{n+1} = a_n x_n + b_n Tx_n + c_n u_n, \ n \geq 0,
\]
is called the \textit{Mann iteration sequence with errors}.

It is clear that the Ishikawa and Mann iteration sequences are all special cases of the Ishikawa and Mann iteration sequences with errors, respectively.

Chidume [3] proved that if \( X = L_p \) (or \( l_p \)) for \( p \geq 2 \), \( K \) is a nonempty bounded closed convex subset of \( X \) and \( T : K \to K \) is a Lipschitz strongly pseudocontractive mapping, then the Mann iteration sequence converges strongly to the unique fixed point of \( T \). Afterwards, several authors extended the result of Chidume in various directions (see \textit{e.g.}, Chang [1], Chang, Cho, Lee & Kang [2], Chidume [4, 5, 6, 7, 8], Chidume & Osilike [9, 10, 11, 12], Liu [19], Schu [25], Tan & Xu [26], Xu [28], Zhou [29] and Zhou & Jia [30]). Chidume [4] obtained that the Ishikawa iteration process can be used to approximate the fixed point of the Lipschitz strongly pseudocontractive mapping \( T : K \to K \), where \( K \) is a nonempty bounded closed convex subset of a real uniformly smooth Banach space \( X \). Xu [28] extended the results of Chidume in Chidume [3] and Chidume [4] to both the Ishikawa iteration method with errors and without the Lipschitz assumption. Chidume & Osilike [9] improved the result of Chidume [3] to strictly hemi-contractive mappings and real uniformly smooth Banach spaces. Chidume [9] generalized the results in Chidume [3, 4] and Xu [28] to both real Banach spaces, the Ishikawa iteration method with errors and uniformly continuous strongly pseudocontractive mappings.

A few stability results for certain classes of nonlinear mappings have been established by several authors (see \textit{e.g.}, Harder [13, 14, 15], Osilike [21, 22, 23]). Rhoades
[24] proved that the Mann and Ishikawa iteration methods may exhibit different behaviors for different classes of nonlinear mappings. Harder & Hicks [15] revealed that the importance of investigating the stability of various iteration procedures for various classes of nonlinear mappings. Harder [13] established applications of stability results to first order differential equations. Osilike [21, 22] obtained that certain Mann and Ishikawa iteration methods are $T$-stable on $X$ when $T$ is a Lipschitz strongly pseudocontractive operators in real $q$-uniformly smooth Banach spaces or real Banach spaces, respectively.

Let $K$ be a nonempty closed convex subset of an arbitrary Banach space $X$ and $T : K \to K$ be a uniformly continuous strictly hemi-contractive operator with bounded range. In this paper, we prove that certain Ishikawa iteration scheme with errors both converges strongly to a unique fixed point of $T$ and is almost $T$-stable on $K$. Furthermore, we also establish similar convergence and almost stability results for strictly hemi-contractive operator $T : K \to K$, where $K$ is a nonempty convex subset of arbitrary uniformly smooth Banach space $X$. The convergence results presented in this paper extend, improve and unify the corresponding results in Chang [1], Chang, Cho, Lee & Kang [2], Chidume [3, 4, 5, 6, 7, 8], Chidume & Osilike [9, 10, 11, 12], Liu [19], Schu [25], Tan & Xu [26], Xu [28], Zhou [29], Zhou & Jia [30] and others.

2. Preliminaries

We need the following Lemmas which play crucial roles in the proofs of our results.

**Lemma 2.1** (Kato [17]). Let $X$ be an arbitrary Banach space and $x, y \in X$. Then $\|x\| \leq \|x + ry\|$ for every $r > 0$ if and only if there is $j \in J(x)$ such that $Re(y, j) \geq 0$.

**Lemma 2.2** (Liu [18]). Suppose that $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma\}_{n=0}^{\infty}$ and $\{\omega_n\}_{n=0}^{\infty}$ are nonnegative sequences such that

$$
\alpha_{n+1} \leq (1 - \omega_n)\alpha_n + \beta_n\omega_n + \gamma_n, \quad n \geq 0,
$$

with $\{\omega_n\}_{n=0}^{\infty} \subset [0, 1]$, $\sum_{n=0}^{\infty} \omega_n = \infty$, $\lim_{n \to \infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \gamma_n < \infty$. Then $\lim_{n \to \infty} \alpha_n = 0$.

**Lemma 2.3** (Chidume & Osilike [9]). Let $X$ be a Banach space and $T : D(T) \subseteq X \to X$ be an operator with $F(T) \neq \emptyset$. Then $T$ is strictly hemi-contractive if and only if there exists $t > 1$ such that for each $x \in D(T)$ and $q \in F(T)$, there exists
\[ j \in J(x - q) \text{ satisfying} \]
\[
\text{Re}(x - Tx, j) \geq \left(1 - \frac{1}{t}\right) \|x - q\|^2.
\] (2.1)

**Lemma 2.4.** Let \( X \) be an arbitrary normed linear space and \( T : D(T) \subseteq X \to X \) be an operator.

(i) If \( T \) is a local strongly pseudocontractive operator and \( F(T) \neq \emptyset \), then \( F(T) \) is a singleton and \( T \) is strictly hemi-contractive;

(ii) If \( T \) is strictly hemi-contractive, then \( F(T) \) is a singleton.

**Proof.** Suppose that \( F(T) \neq \emptyset \) and \( T \) is a local strongly pseudocontractive operator. We assert first of all that \( F(T) \) is a singleton. Otherwise there exist distinct elements \( p, q \in F(T) \). Since \( T \) is local strongly pseudocontractive, then there exists \( t_p > 1 \) such that for all \( y \in D(T) \) and \( r > 0 \),
\[
\|p - y\| \leq \|(1 + r)(p - y) - rt_p(Tp - Ty)\|. \tag{2.2}
\]
Set \( y = q \in F(T) \subseteq D(T) \) and \( r = \frac{1}{2(t_p - 1)} \). It follows from (2.2) that
\[
0 < \|p - q\| = |1 + r(1 - t_p)| \cdot \|p - q\| = \frac{1}{2} |p - q|,
\]
which is a contradiction. Hence \( F(T) = \{q\} \) for some \( q \in D(T) \).

Next we show that \( T \) is strictly hemi-contractive. Note that \( T \) is a local strongly pseudocontractive operator and \( F(T) = \{q\} \). Put \( t = t_q \). Then (1.2) ensures that
\[
\|q - y\| \leq \|(1 + r)(q - y) - rt(q - Ty)\|
\]
for all \( y \in D(T) \) and \( r > 0 \). That is, \( T \) is strictly hemi-contractive.

The proof of (ii) now follows exactly as in the first part of the proof of (i). This completes the proof. \( \square \)

**Lemma 2.5** (Chang, Cho, Lee & Kang [2]). Let \( X \) be a Banach space. Then \( X \) is a uniformly smooth if and only if \( J \) is single valued and uniformly continuous on any bounded subset of \( X \).

In the sequel, we shall denote the single valued normalized duality mapping by \( j \).

**Lemma 2.6** (Xu [28]). Let \( X \) be a uniformly smooth Banach space and let \( J : X \to 2^{X^*} \) be the normalized duality mapping. Then
\[
\|x + y\|^2 \leq \|x\|^2 + 2 \text{Re} \langle y, j(x + y) \rangle, \quad x, y \in X.
\]
3. Main results

In this section, $I$ denotes the identity mapping on $X$, $d_n = b_n + c_n$ and $d'_n = b'_n + c'_n$ for all $n \geq 0$ and $k = \frac{t-1}{t} \in (0, 1)$, where $t$ is the constant appearing in (1,3) or (2.1). Our main results are as follows.

**Theorem 3.1.** Let $K$ be a nonempty convex subset of an arbitrary Banach space $X$ and let $T : K \to K$ be a uniformly continuous and strictly hemi-contractive operator with $R(T)$ bounded. Suppose that $\{u_n\}_{n=0}^\infty$ and $\{v_n\}_{n=0}^\infty$ are arbitrary bounded sequences in $K$ and $\{a_n\}_{n=0}^\infty$, $\{b_n\}_{n=0}^\infty$, $\{c_n\}_{n=0}^\infty$, $\{a'_n\}_{n=0}^\infty$, $\{b'_n\}_{n=0}^\infty$, $\{c'_n\}_{n=0}^\infty$ and $\{r_n\}_{n=0}^\infty$ are any sequences in $[0, 1]$ satisfying

\[ a_n + b_n + c_n = a'_n + b'_n + c'_n = 1, \quad n \geq 0; \]
\[ c_n = r_n b_n, \quad n \geq 0; \]
\[ \lim_{n \to \infty} b_n = \lim_{n \to \infty} r_n = \lim_{n \to \infty} b'_n = \lim_{n \to \infty} c'_n = 0; \]
\[ \sum_{n=0}^\infty b_n = \infty. \]

Suppose that $\{x_n\}_{n=0}^\infty$ is the sequence generated from an arbitrary $x_0 \in K$ by

\[ x_n = a'_n x_n + b'_n T x_n + c'_n v_n, \quad x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \quad n \geq 0. \]

Let $\{y_n\}_{n=0}^\infty$ be any bounded sequence in $K$ and define $\{\varepsilon_n\}_{n=0}^\infty$ by

\[ w_n = a'_n y_n + b'_n T y_n + c'_n v_n, \quad \varepsilon_n = \|y_{n+1} - p_n\|, \quad n \geq 0, \]

where $p_n = a_n y_n + b_n T w_n + c_n u_n$. Then there exist nonnegative sequences $\{s_n\}_{n=0}^\infty$ and $\{t_n\}_{n=0}^\infty$ such that

(i) the sequence $\{x_n\}_{n=0}^\infty$ converges strongly to the unique fixed point $q$ of $T$ and

\[ \|x_{n+1} - q\| \leq \left(1 - kb_n\right)\|x_n - q\| + k^{-1} b_n s_n + k^{-1} c_n\|u_n - q\|, \quad n \geq 0; \]

(ii) $\|y_{n+1} - q\| \leq \left(1 - kb_n\right)\|y_n - q\| + k^{-1} b_n t_n + k^{-1} c_n\|u_n - q\| + \varepsilon_n, \quad n \geq 0$;

(iii) $\sum_{n=0}^\infty \varepsilon_n < \infty$ implies that $\lim_{n \to \infty} y_n = q$, so that $\{x_n\}_{n=0}^\infty$ is almost $T$-stable on $K$;

(iv) $\lim_{n \to \infty} y_n = q$ implies that $\lim_{n \to \infty} \varepsilon_n = 0$.

(v) $\lim_{n \to \infty} s_n = \lim_{n \to \infty} t_n = 0$. 
Proof. Lemma 2.4 ensures that \( F(T) \) is a singleton. That is, \( F(T) = \{ q \} \) for some \( q \in K \). Let \( s_n = \|Tx_{n+1} - Tz_n\|, \ t_n = \|Tp_n - Tw_n\| \) for all \( n \geq 0 \) and

\[
M = 1 + \|x_0 - q\| + \sup\{\|Tx - q\| : x \in K\} \\
+ \sup\{\max\{\|u_n - q\|, \|v_n - q\|, \|y_n - q\| : n \geq 0\}\}.
\]

It is easy to verify that

\[
\max\{\|x_n - q\|, \|z_n - q\|, \|w_n - q\|, \|p_n - q\|\} \leq M, \quad n \geq 0. \tag{3.7}
\]

Note that

\[
\|x_{n+1} - z_n\| \leq b_n\|x_n - Tz_n\| + c_n\|x_n - u_n\| + b'_n\|x_n - Tx_n\| + c'_n\|x_n - v_n\|
\leq 2M(d_n + d'_n) \to 0
\]

and

\[
\|p_n - w_n\| \leq b_n\|y_n - Tw_n\| + c_n\|y_n - u_n\| + b'_n\|y_n - Ty_n\| + c'_n\|y_n - v_n\|
\leq 2M(d_n + d'_n) \to 0
\]
as \( n \to \infty \). Since \( T \) is uniform continuity, it follows that

\[
\lim_{n \to \infty} s_n = \lim_{n \to \infty} t_n = 0. \tag{3.8}
\]

Since \( T \) is strictly hemi-contractive, it follows from Lemma 2.3 that

\[
\Re\langle x - Tx, j(x - q) \rangle \geq k\|x - q\|^2, \quad x \in K,
\]

which implies that

\[
\Re\langle (I - T - kI)x - (I - T - kI)q, j(x - q) \rangle \geq 0, \quad x \in K.
\]

In view of Lemma 2.1, we have

\[
\|x - q\| \leq \|x - q + r((I - T - kI)x - (I - T - kI)q)\|, \quad x \in K, \ r > 0. \tag{3.9}
\]

It follows from (3.1) and (3.5) that for all \( n \geq 0 \),

\[
(1 - d_n)x_n = x_{n+1} - b_nTz_n - c_nu_n \\
= (1 - (1 - k)b_n)x_{n+1} + b_n((1 - k)I - T)x_{n+1} \\
+ b_n(Tx_{n+1} - Tz_n) - c_nu_n, \tag{3.10}
\]

and

\[
(1 - d_n)q = (1 - (1 - k)b_n)q + b_n((1 - k)I - T)q - c_nq. \tag{3.11}
\]
By virtue of (3.9)–(3.11), we infer that for any $n \geq 0$,
\[
(1 - d_n)\|x_n - q\| \geq (1 - (1 - k)b_n)\|x_{n+1} - q + \frac{b_n}{1 - (1 - k)b_n}[(1 - k)I - T]x_{n+1} - (1 - k)I - Tq\| - b_n\|Tx_{n+1} - Tz_n\| - c_n\|u_n - q\|
\geq (1 - (1 - k)b_n)\|x_{n+1} - q\| - b_n\|Tx_{n+1} - Tz_n\| - c_n\|u_n - q\|
\]
which implies that for all $n \geq 0$,
\[
\|x_{n+1} - q\| \leq \frac{1 - d_n}{1 - (1 - k)b_n}\|x_n - q\| + \frac{b_n}{1 - (1 - k)b_n}\|Tx_{n+1} - Tz_n\|
+ \frac{c_n}{1 - (1 - k)b_n}\|u_n - q\|
\leq (1 - \frac{k\beta_n + \gamma_n}{1 - (1 - k)b_n})\|x_n - q\| + k^{-1}b_n\|s_n\| + k^{-1}c_n\|u_n - q\|
\leq (1 - k\beta_n)\|x_n - q\| + k^{-1}b_n\|s_n\| + k^{-1}c_n\|u_n - q\|.
\] (3.12)
Put $\alpha_n = \|x_n - q\|$, $\omega_n = k\beta_n$, $\beta_n = k^{-2}(s_n + r_n\|u_n - q\|)$ and $\gamma_n = 0$ for each $n \geq 0$. Using (3.2) and (3.12), we have
\[
\alpha_{n+1} \leq (1 - \omega_n)\alpha_n + \omega_n\beta_n + \gamma_n, \quad n \geq 0.
\]
Observe that $\sum_{n=0}^\infty \omega_n = \infty$, $\omega_n \in [0, 1]$, $\sum_{n=0}^\infty \gamma_n = 0$ and $\lim_{n \to \infty} \beta_n = 0$. It follows from Lemma 2.2 that $\lim_{n \to \infty} \alpha_n = 0$. That is, $x_n \to q$ as $n \to \infty$.

From (3.1) and (3.6), we get for all $n \geq 0$,
\[
(1 - d_n)y_n = p_n - b_nTw_n - c_nu_n
= \left(1 - (1 - k)b_n\right)p_n + b_n\left(1 - k\right)I - T)p_n
+ b_n(Tp_n - Tw_n) + c_nu_n.
\] (3.13)
It follows from (3.9), (3.11) and (3.13) that
\[
(1 - d_n)\|y_n - q\| \geq (1 - (1 - k)b_n)\|p_n - q
+ \frac{b_n}{1 - (1 - k)b_n}[(1 - k)I - T)p_n - ((1 - k)I - T)q]\|
- b_n\|Tp_n - Tw_n\| - c_n\|u_n - q\|
\geq (1 - (1 - k)b_n)\|p_n - q\| - b_n\|t_n - c_n\|u_n - q\|
\] (3.14)
for all $n \geq 0$. Using (3.1) and (3.14), we immediately conclude that
\[
\|p_n - q\| \leq \frac{1 - d_n}{1 - (1 - k)b_n}\|y_n - q\| + \frac{b_n}{1 - (1 - k)b_n}\|t_n
+ \frac{c_n}{1 - (1 - k)b_n}\|u_n - q\|
\]
\[(1 - kb_n)\|y_n - q\| + k^{-1}b_n t_n + k^{-1}c_n\|u_n - q\| \leq (1 - kb_n)\|y_n - q\| + k^{-1}b_n t_n + k^{-1}c_n\|u_n - q\| + \varepsilon_n \leq (3.15)\]

for any \(n \geq 0\). Thus (3.15) implies that
\[
\|y_{n+1} - q\| \leq \|p_n - q\| + \|y_{n+1} - p_n\|
\leq (1 - kb_n)\|y_n - q\| + k^{-1}b_n t_n + k^{-1}c_n\|u_n - q\| + \varepsilon_n \leq (3.16)
\]

for all \(n \geq 0\).

Suppose that \(\sum_{n=0}^{\infty} \varepsilon_n < \infty\). Set \(\alpha_n = \|y_n - q\|, \omega_n = kb_n, \gamma_n = \varepsilon_n, \beta_n = k^{-2}(t_n + r_n\|u_n - q\|), \ n \geq 0\).

Using Lemma 2.2, (3.3), (3.4) and (3.16), we conclude that \(\alpha_n \to 0\) as \(n \to \infty\). Therefore \(y_n \to q\) as \(n \to \infty\). That is, \(\{x_n\}_{n=0}^{\infty}\) is almost \(T\)-stable on \(K\).

Suppose that \(\lim_{n \to \infty} y_n = q\). It follows from (3.15) that
\[
\varepsilon_n \leq \|y_{n+1} - q\| + \|p_n - q\|
\leq \|y_{n+1} - q\| + (1 - kb_n)\|y_n - q\| + k^{-1}b_n + k^{-1}c_n\|u_n - q\|
\to 0
\]
as \(n \to \infty\). That is, \(\varepsilon_n \to 0\) as \(n \to \infty\). This completes the proof. \(\square\)

Using the technique of proof of Theorem 3.1, we have

**Theorem 3.2.** Let \(X, T, K, \{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}, \{x_n\}_{n=0}^{\infty}, \{z_n\}_{n=0}^{\infty}, \{w_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}, \{p_n\}_{n=0}^{\infty}\) and \(\{\varepsilon_n\}_{n=0}^{\infty}\) be as in Theorem 3.1. Suppose that \(\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}, \{c_n\}_{n=0}^{\infty}, \{a'_n\}_{n=0}^{\infty}, \{b'_n\}_{n=0}^{\infty}\) and \(\{c'_n\}_{n=0}^{\infty}\) are any sequences in \([0, 1]\) satisfying (3.1), (3.4) and
\[
\lim_{n \to \infty} b_n = \lim_{n \to \infty} c_n = \lim_{n \to \infty} b'_n = \lim_{n \to \infty} c'_n = 0; \quad (3.17)
\]
\[
\sum_{n=0}^{\infty} c_n = \infty. \quad (3.18)
\]

Then the conclusions of Theorem 3.1 hold.

**Theorem 3.3.** Let \(K\) be a nonempty convex subset of a uniformly smooth Banach space \(X\) and \(T : K \to K\) be a strictly hemi-contractive operator with \(R(T)\) bounded. Suppose that \(\{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}, \{x_n\}_{n=0}^{\infty}, \{z_n\}_{n=0}^{\infty}, \{w_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}, \{p_n\}_{n=0}^{\infty}, \{\varepsilon_n\}_{n=0}^{\infty}, \{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}, \{c_n\}_{n=0}^{\infty}, \{a'_n\}_{n=0}^{\infty}, \{b'_n\}_{n=0}^{\infty}, \{c'_n\}_{n=0}^{\infty}\) and \(\{r_n\}_{n=0}^{\infty}\) be as in Theorem 3.1. Then there exist nonnegative sequences \(\{s_n\}_{n=0}^{\infty}, \{t_n\}_{n=0}^{\infty}\) and constant \(D > 0\) such that
(i) the sequence \( \{x_n\}_{n=0}^\infty \) converges strongly to the unique fixed point \( q \) of \( T \) and
\[
\|x_{n+1} - q\|^2 \leq (1 - kb_n)\|x_n - q\|^2 + Db_n s_n, \quad n \geq 0,
\]
(ii) \( \|y_{n+1} - q\|^2 \leq (1 - kb_n)\|y_n - q\|^2 + Db_n t_n + D\epsilon_n, \quad n \geq 0,
\]
(iii) \( \sum_{n=0}^\infty \epsilon_n < \infty \) implies that \( \lim_{n \to \infty} y_n = q \), so that \( \{x_n\}_{n=0}^\infty \) is almost \( T \)-stable on \( K \);
(iv) \( \lim_{n \to \infty} y_n = q \) implies that \( \lim_{n \to \infty} \epsilon_n = 0 \).
(v) \( \lim_{n \to \infty} s_n = \lim_{n \to \infty} t_n = 0 \).

Proof. Let \( q \) and \( M \) be as in the proof of Theorem 3.1. Then (3.7) holds. Put
\[
f_n = \|j(z_n - q) - j(x_n - q)\|, \quad g_n = \|j(x_{n+1} - q) - j(x_n - q)\|, \quad h_n = \|j(p_n - q) - j(w_n - q)\|,
\]
\[
k_n = \|j(w_n - q) - j(y_n - q)\| \quad \text{for each } n \geq 0.
\]
Observe that
\[
\|(z_n - q) - (x_n - q)\| \leq b'_n\|x_n - Tx_n\| + c'_n\|x_n - v_n\| \leq 2Md'_n, \tag{3.19}
\]
\[
\|(x_{n+1} - q) - (z_n - q)\| \leq b_n\|x_n - Tx_n\| + c_n\|x_n - u_n\|
+ b'_n\|x_n - Tx_n\| + c'_n\|x_n - v_n\|
\leq 2M(d_n + d'_n), \tag{3.20}
\]
\[
\|(p_n - q) - (w_n - q)\| \leq b_n\|y_n - Tw_n\| + c_n\|y_n - u_n\|
+ b'_n\|y_n - Ty_n\| + c'_n\|y_n - v_n\|
\leq 2M(d_n + d'_n), \tag{3.21}
\]
\[
\|(w_n - q) - (y_n - q)\| \leq b'_n\|y_n - Ty_n\| + c'_n\|y_n - v_n\| \leq 2Md'_n \tag{3.22}
\]
for any \( n \geq 0 \). Using Lemma 2.5, (3.2), (3.3) and (3.19)-(3.22), we infer that
\[
\lim_{n \to \infty} f_n = \lim_{n \to \infty} g_n = \lim_{n \to \infty} h_n = \lim_{n \to \infty} k_n = 0. \tag{3.23}
\]
In view of (3.1), (3.5), (3.7) and Lemma 2.3 and Lemma 2.6, we have
\[
\|z_n - q\|^2 = \|(1 - d'_n)(x_n - q) + b'_n(Tx_n - q) + c'_n(v_n - q)\|^2
\leq (1 - d'_n)^2\|x_n - q\|^2 + 2b'_n \Re(Tx_n - q, j(z_n - q))
+ 2c'_n \Re(v_n - q, j(z_n - q))
\leq (1 - d'_n)^2\|x_n - q\|^2 + 2b'_n \Re(Tx_n - q, j(x_n - q))
+ 2b'_n \Re(Tx_n - q, j(z_n - q) - j(x_n - q))
+ 2c'_n\|v_n - q\|\|z_n - q\|
\leq [(1 - d'_n)^2 + 2b'_n(1 - k)]\|x_n - q\|^2
+ 2b'_n\|Tx_n - q\|\|j(z_n - q) - j(x_n - q)\| + 2M^2c'_n.
\[(1 - d_n')s_n\] for any \(n \geq 0\). Using (3.1), (3.5), (3.7), (3.24) and Lemma 2.3 and Lemma 2.6, we conclude that

\[
\|x_{n+1} - q\|^2 = \|(1 - d_n)(x_n - q) + b_n(Tz_n - q) + c_n(u_n - q)\|^2 \\
\leq (1 - d_n)^2\|x_n - q\|^2 + 2b_n \text{Re}(Tz_n - q, j(x_{n+1} - q)) \\
+ 2c_n \text{Re}(u_n - q, j(x_{n+1} - q)) \\
\leq (1 - d_n)^2\|x_n - q\|^2 + 2b_n \text{Re}(Tz_n - q, j(z_n - q)) \\
+ 2b_n \text{Re}(Tz_n - q, j(x_{n+1} - q) - j(z_n - q)) + 2M^2 c_n \\
\leq (1 - d_n)^2\|x_n - q\|^2 + 2(1 - k)b_n\|z_n - q\|^2 \\
+ 2Mb_n g_n + 2M^2 c_n \\
\leq ((1 - d_n)^2 + 2(1 - k)b_n[(1 - d_n')^2 + 2b_n'(1 - k)])\|x_n - q\|^2 \\
+ 2(1 - k)b_n(2M b_n f_n + 2M^2 c_n') + 2Mb_n g_n + 2M^2 c_n \\
\leq (1 - kb_n)\|x_n - q\|^2 + Db_n s_n
\] (3.25)

for any \(n \geq 0\), where \(s_n = b_n f_n + c_n + g_n + r_n\), \(D = 6M^2\). Let \(\alpha_n = \|x_n - q\|^2\), \(\omega_n = kb_n\), \(\beta_n = k^{-1}Ds_n\) and \(\gamma_n = 0\) for each \(n \geq 0\). Thus Lemma 2.2, (3.2)-(3.4), (3.23) and (3.25) yield that \(\alpha_n \to 0\) as \(n \to \infty\). That is, \(\lim_{n \to \infty} x_n = q\). Similarly, we have

\[
\|w_n - q\|^2 = \|(1 - d_n')(y_n - q) + b_n'(Ty_n - q) + c_n'(v_n - q)\|^2 \\
\leq (1 - d_n')^2\|y_n - q\|^2 + 2b_n' \text{Re}(Ty_n - q, j(w_n - q)) \\
+ 2c_n' \text{Re}(v_n - q, j(w_n - q)) \\
\leq (1 - d_n')^2\|y_n - q\|^2 + 2b_n' \text{Re}(Ty_n - q, j(y_n - q)) \\
+ 2b_n' \text{Re}(Ty_n - q, j(w_n - q) - j(y_n - q)) \\
+ 2c_n'\|v_n - q\|\|w_n - q\| \\
\leq [(1 - d_n')^2 + 2b_n'(1 - k)]\|y_n - q\|^2 \\
+ 2b_n'\|Ty_n - q\|\|j(w_n - q) - j(y_n - q)\| + 2M^2 c_n' \\
\leq [(1 - d_n')^2 + 2b_n'(1 - k)]\|y_n - q\|^2 + 2Mb_n' h_n + 2M^2 c_n
\] (3.26)

for all \(n \geq 0\). Using (3.1), (3.5), (3.7), (3.26) and Lemma 2.3 and Lemma 2.6, we infer that

\[
\|p_n - q\|^2 = \|(1 - d_n)(y_n - q) + b_n(Tw_n - q) + c_n(u_n - q)\|^2
\]
\[
(1 - d_n)^2 \| y_n - q \|^2 + 2b_n \text{Re}(Tw_n - q, j(p_n - q)) + 2c_n \text{Re}(u_n - q, j(p_n - q)) \\
\leq (1 - d_n)^2 \| y_n - q \|^2 + 2b_n \text{Re}(Tw_n - q, j(w_n - q)) + 2b_n \text{Re}(Tw_n - q, j(p_n - q) - j(w_n - q)) + 2M^2 c_n \\
\leq (1 - d_n)^2 \| y_n - q \|^2 + 2(1 - k) b_n \| w_n - q \|^2 + 2Mb_n k_n + 2M^2 c_n \\
\leq ((1 - d_n)^2 + 2(1 - k) b_n [(1 - d'_n)^2 + 2b'_n (1 - k)]) \| y_n - q \|^2 \\
+ 2(1 - k) b_n (2Mb'_n h_n + 2M^2 c'_n) + 2Mb_n k_n + 2M^2 c_n \\
\leq (1 - kb_n) \| y_n - q \|^2 + Db_n t_n \tag{3.27}
\]
for any \( n \geq 0 \), where \( t_n = b'_n h_n + c'_n + k_n + r_n \), It follows from (3.7), (3.27) that
\[
\| y_{n+1} - q \|^2 \leq (\| y_{n+1} - p_n \| + \| p_n - q \|)^2 \\
\leq \| p_n - q \|^2 + \| y_{n+1} - p_n \| (\| y_{n+1} - p_n \| + 2\| p_n - q \|) \\
\leq (1 - kb_n) \| y_n - q \|^2 + Db_n t_n + \epsilon_n (2M + 2M) \\
\leq (1 - kb_n) \| y_n - q \|^2 + Db_n t_n + D\epsilon_n \tag{3.28}
\]
for all \( n \geq 0 \).

Suppose that \( \sum_{n=0}^{\infty} \epsilon_n < \infty \). Let \( \alpha_n = \| y_n - q \|^2 \), \( \omega_n = kb_n \), \( \beta_n = k^{-1} D t_n \) and \( \gamma_n = D \epsilon_n \) for each \( n \geq 0 \). Thus Lemma 2.2, (3.2)–(3.4), (3.23) and (3.28) yield that \( \alpha_n \to 0 \) as \( n \to \infty \). That is, \( \lim_{n=\infty} y_n = q \).

Conversely, suppose that \( \lim_{n=\infty} y_n = q \). By virtue of (3.27) and (3.4), we obtain that
\[
\epsilon_n \leq \| y_{n+1} - q \| + \| p_n - q \| \\
\leq \| y_{n+1} - q \| + [(1 - kb_n) \| y_n - q \|^2 + Db_n t_n]^\frac{1}{2} \\
\to 0
\]
as \( n \to \infty \). This implies that \( \lim_{n=\infty} \epsilon_n = 0 \). This completes the proof. \( \square \)

Similarly, we have

**Theorem 3.4.** Let \( X, T, K, \{ u_n \}_{n=0}^{\infty}, \{ v_n \}_{n=0}^{\infty}, \{ x_n \}_{n=0}^{\infty}, \{ z_n \}_{n=0}^{\infty}, \{ w_n \}_{n=0}^{\infty}, \{ y_n \}_{n=0}^{\infty}, \{ p_n \}_{n=0}^{\infty}, \{ \epsilon_n \}_{n=0}^{\infty} \) be as in Theorem 3.3. Suppose that, \( \{ a_n \}_{n=0}^{\infty}, \{ b_n \}_{n=0}^{\infty}, \{ c_n \}_{n=0}^{\infty}, \{ a'_n \}_{n=0}^{\infty}, \{ b'_n \}_{n=0}^{\infty} \) and \( \{ c'_n \}_{n=0}^{\infty} \) are any sequences in \([0, 1]\) satisfying (3.1), (3.4), (3.17) and (3.18). Then the conclusion of theorem 3.3 hold.
Remark 3.1. The convergence results in Theorem 3.1 and Theorem 3.2 extend, improve and unify Theorems 3.4 and 4.2 of Chang [1], Theorems 3.4 and 4.2 of Chang, Cho, Lee & Kang [2], Theorem of Chidume [3], Theorem 2 of Chidume [4], Theorem 4 of Chidume [5], Theorem 4 of Chidume [6], Theorem 1 of Chidume [7], Theorem 2 of Chidume & Osilike [9], Theorem 4 of Chidume & Osilike [10], Theorem 1 of Chidume & Osilike [11], Theorem 1 of Chidume & Osilike [12], Theorem 1 of Liu [19], the Theorem of Schu [25] and Theorem 4.2 of Tan & Xu [26] in the following ways:

(i) Theorem 3.1 and Theorem 3.2 hold in arbitrary Banach spaces whereas the results in Chang [1], Chang, Cho, Lee & Kang [2], Chidume [3, 4, 5, 6, 7], Chidume & Osilike [9, 10, 11, 12], Schu [25] and Tan & Xu [26] are fulfilled in the restricted $L_p$ (or $l_p$) spaces, $p$-uniformly smooth Banach spaces, real uniformly smooth Banach spaces, real smooth Banach spaces, real Banach spaces, respectively;

(ii) The boundedness of $R(T)$ in Theorem 3.1 and Theorem 3.2 is weaker than the boundedness of the subsets $K$ in Chang [1], Chang, Cho, Lee & Kang [2], Chidume [3, 4, 5, 6, 7], Chidume & Osilike [9, 10, 11, 12], Liu [19], Schu [25] and Tan & Xu [26];

(iii) The Mann iteration methods in Chang [1], Chang, Cho, Lee & Kang [2], Chidume [3], Chidume & Osilike [10], Liu [19], and Schu [25] and the Ishikawa iteration methods in Chang [1], Chang, Cho, Lee & Kang [2], Chidume [4, 5, 6], Chidume & Osilike [9, 11, 12], and Tan & Xu [26] are replaced by the more general Ishikawa iteration method with errors;

(iv) The uniformly continuous strongly pseudocontractive operators in Chang [1], Chang, Cho, Lee & Kang [2], Chidume [6, 7], Chidume & Osilike [12] and Schu [25], the Lipschitz strongly pseudocontractive operators in Chidume [3, 4, 5], Chidume & Osilike [10, 11], Liu [19], and Tan & Xu [26] and the Lipschitz strictly hemi-contractive operators in Chidume & Osilike [9] are replaced by the uniformly continuous strictly hemi-contractive operators;

(v) The iteration parameters $\alpha_n, \beta_n$ in Chidume [4] and Chidume & Osilike [9] deal with the geometry of the underlying Banach space $X$. The iteration parameters \( \{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}, \{c_n\}_{n=0}^{\infty}, \{a'_n\}_{n=0}^{\infty}, \{b'_n\}_{n=0}^{\infty}, \{c'_n\}_{n=0}^{\infty} \) and \( \{r_n\}_{n=0}^{\infty} \) in Theorem 3.1 and Theorem 3.2 are not dependent on the geometric structure of $X$;
(vi) The conditions $0 \leq \alpha_n \leq \beta_n < 1$ in Chidume [4] and Chidume [6] are omitted. The following example reveals that Theorem 3.1 generalizes indeed the corresponding results in Chang [1], Chang, Cho, Lee & Kang [2], Chidume [3, 4, 5, 6, 7], Chidume & Osilike [9, 10, 11, 12], Liu [19], Schu [25] and Tan & Xu [26].

Example 3.1. Let $R$ denote the reals with the usual norm, $K = [0, \infty)$ and define $T : K \to K$ by $Tx = \left(\sin \frac{x}{3}\right)^2$ for all $x \in X$. Set

$$t = \frac{3}{2}, \quad a_n = 1 - \frac{1}{2\sqrt{1 + n}}, \quad b_n = \frac{1}{2\sqrt{1 + n}},$$

$$c_n = \frac{1}{2(1 + n)}, \quad a'_n = 1 - \frac{2}{2 + n}, \quad b'_n = c'_n = \frac{1}{2 + n}$$

for all $n \geq 0$. It is easy to verify that

$$|Tx - Ty| \leq 2\left|\sin \frac{x}{3} - \sin \frac{y}{3}\right| \leq 4\left|\sin \frac{x-y}{6}\right| \leq \frac{2}{3}|x-y|, \quad x, y \in K. \quad (3.29)$$

That is, $T$ is both Lipschitz and uniformly continuous in $K$. Thus (3.29) yields that

$$|(1 + r)(x - y) - rt(Tx - Ty)| \geq (1 + r)|x - y| - rt|Tx - Ty|$$

$$= |x - y| + r(|x - y| - t|Tx - Ty|)$$

$$\geq |x - y|$$

for any $x, y \in K$ and $r > 0$. Hence $T$ is strongly pseudocontractive. Clearly $F(T) = \{0\}$. Thus Lemma 2.4 ensures that $T$ is strictly hemi-contractive. Since $K$ is unbounded and $\sum_{n=0}^{\infty} c_n = \infty$, the results in Chang [1], Chang, Cho, Lee & Kang [2], Chidume [3, 4, 5, 6, 7], Chidume & Osilike [9, 10, 11, 12], Liu [19], Schu [25] and Tan & Xu [26] are not applicable. Let

$$r_n = \frac{1}{\sqrt{1 + n}}$$

for each $n \geq 0$. Then the conditions of Theorem 3.1 are satisfied.

Remark 3.2. Theorem 3.3 and Theorem 3.4 generalize Theorems 3.2 and 4.1 of Chang [1], Theorems 3.3 and 4.1 of Chang, Cho, Lee & Kang [2], Theorem of Chidume [3], Theorem of Chidume [8], Theorem 2 of Chidume & Osilike [9], Theorem 4 of Chidume & Osilike [10], Theorem 4.2 of Tan & Xu [26], Theorem 3.3 of Xu [28], Theorem 2 of Zhou [29] and Theorem 2.1 of Zhou & Jia [30] to the more general class of uniformly smooth Banach spaces and the Ishikawa iteration method with errors.
CONVERGENCE AND ALMOST STABILITY

REFERENCES


(Z. Liu) Department of Mathematics, Liaoning Normal University Dalian, Liaoning 116029, China

(J. S. Ume) Department of Applied Mathematics, Changwon National University, 9 Sarim-dong, Changwon, Gyeongnam 641-773, Korea

Email address: jsume@changwon.ac.kr

(S. M. Kang) Department of Mathematics and Research, Institute of Natural Science, Gyeongsang National University, Chinju 660-701, Korea

Email address: smkang@nongae.gsu.ac.kr