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Using parametric reasoning to understand solutions to systems of differential equations

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This paper offers an analysis of how students reasoned with the dynamic parameter time to support their mathematical activity and deepen their understandings of mathematical concepts. This mathematical thinking occurred as they participated in a differential equations class before, during, and instruction on solutions to linear systems of differential equations. Students participated in the following identified mathematical practices related to parametric reasoning during this time period: reasoning simultaneously in a qualitative and quantitative manner, reasoning by moving from discrete to continuous imaging of time, and reasoning by imagining the motion. Examples of this reasoning are provided in this report. Implications of this research include the possibility that instructional activities can build on this reasoning to help students learn about the mathematics of change at the middle school, high school, and the university.

Introduction

Mathematics when used as a tool to understand and make predictions in the world today often involves thinking about and using the ideas of mathematical change. Different possible ways this can happen include conceptualizing and using rate, derivative, and different types of parameter. Presently, mathematics education research in the domain of the mathematics of change and variation is a fairly well documented area of literature. Over the past twenty years, several researchers have investigated the concept of rate and how students learn to reason about rate (e.g. Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Confrey & Smith, 1994; Thompson, 1994a). For much of this research, rate (of change) is considered to be a homogeneous ratio of two quantities (Kaput & West, 1994), where the second value in a ratio, b in a:b, is often time. Additionally, students in science and mathematics study rates such as velocity (distance versus time), acceleration (velocity versus time), or rate of growth (cm versus time); these studies may involve conceptualizing rate as a concept in its own right.

In the area of dynamical systems, including those systems that are comprised of differential equations that this study concerns, time is always the independent variable, but often the equations do not explicitly use the variable time, instead it is implicitly used and the changes that occur with time are studied, measured and analyzed to understand and predict real world phenomena. With the increase in our ability to study these systems using technology, and their increase in use in science, economics, and mathematics, it is important to learn how students can reason about these situations. Little of the previous mathematics education research has specifically addressed the idea of parametric time-based reasoning as students use it in dynamical systems and other mathematical areas at all levels of mathematics instruction. The research question that is addressed in this paper is as follows: What general mathematical practices that involve parametric reasoning do students use when they learn about solutions to systems of differential equations?

Defining parametric reasoning

For purposes here, there are two kinds of parameter, each of which are complementary to the other and connect with a kind of parametric reasoning. The two types of parameter are

1) parameter as a higher order variable (sleeping) and
2) parameter with time as variable (dynamic).

Freudenthal (1983) used the idea of a sleeping variable when he described parameter. He suggested that a parameter is a variable that lies quietly and not changing during the course of one problem or situation; then when the situation changes, it wakes up and changes to a different value, then falls back asleep. This concept of variable has been elaborate to propose that a parameter plays at least three functions (Drijvers, 2001): place holder, generalizer, and changing quantity. As a placeholder, a parameter contains specific values, which are used one at a time. As a changing quantity, the parameter is sliding and so has a dynamic notion that its changes causes graphs, functions or equations to change. As a generalizer, the parameter creates a family of functions or equations to generalize over all objects it represents. An example of parameter as a sleeping variable would be the m in

$$y = mx + b$$

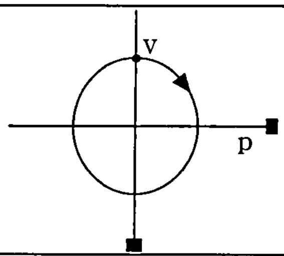
or the α in the differential equation

$$\frac{dP}{dt} = .1P\left(1 - \frac{10}{P}\right)\alpha$$

In these two examples, m and could serve any of the above three functions depending on the activity and the level of students mathematical work.

In contrast, a parameter as dynamic variable is identified as always awake. It is a quantity that changes continuously and its change causes other quantities to change as well. It may not be explicit in the situation, but plays an important role behind the mathematical situation of changing to influence and even define other quantities. One example of this might be the traditional parametric equations that are studied in calculus; two or more functions are defined in terms of time (t) and then graphed on planes (with our without technology) without the appearance of the variable t. Another example of the implicit (time) variable is when students utilize a graph sketched in a phase plane (x-y plane) with arrows to actually create a moving and dynamic function in three dimensions. See Figure 1.

This graph describes the relationship between velocity and position of a mass on a horizontal spring. Qualitatively describe the motion of the spring and make a rough sketch in three dimensions.



<Figure 1> Example of parameter as a dynamic variable

There is some overlap between the two kinds of parameter; the characterization of parameter as changing quantity in the first definition and parameter as a dynamic variable in the second definition appear to be connected. However, in this paper, I focus on the second definition of parameter a dynamically changing quantity and the support and the practices that students participate in with that type of parameter. Therefore, I use this definition: **parametric reasoning is developing and using conceptualizations about time as a dynamic quantity that implicitly or explicitly coordinates with other quantities (possibly functions) to understand and solve problems.** Analysis of the data shows that students used parametric reasoning in several ways as they participate in the differential equations class and these ideas will be delineated.

Connected research

Piaget was one of the earlier researchers to hypothesize how students think about speed and time. According to Piaget (1971), many young children decided their answer to which train is moving faster depending on which train ended up farther along its tracks. Piaget called this an ordinal notion of speed; it comes from a child's intuition of objects passing each other as being faster and claimed that this how fast notion is more primitive than time. He proposed that perception of speed (rate) comes from the physical act of seeing objects pass each other, or objects pass over static locations and our eyes watching the movement. The concept of time then is something that comes after and is constructed by children later in their mental development from conceptions of distance and speed but still quite early in their life.

The idea that children have a perception of speed or rate at an early age led to studies by physics educators about how students may use their physical intuitions to aid in developing mathematical understanding (e.g. Brasell, 1987; Beichner, 1990). These researchers found that when students in physics are studying kinematics (the science of motion), they develop deeper understandings of velocity, distance, and acceleration when they use computer based motion detectors to create graphs by walking or moving their hands and bodies and watching the graphs be created simultaneous to the movement. This indicates that rate is an intensive property, that is, a quality in its own right, as well as an extensive property, a property constructed by combining other properties (Schwartz, 1988), and is an important way to introduce students to rate concepts.

Several researchers have offered ways to think about rate as an extensive property. For example, Thompson (1994b) created a framework for analyzing how students think about rate that involves the conceptualizing of rate as the covariation of distance and time and eventually thinking about rate as a constant ratio of distance and time that does not vary over a set distance. One idea from his framework is that students use the concept of speed-length in their reasoning about rate. Speed-length as he defines it is when students when presented with something like 50 km per hour, actually think of it as a length, in this case, 50 km, and instead of thinking of it as something that can be partitioned, or a proportional value, always think of it as the length gone in one whole time unit. This concept of speed length appears in students reasoning from algebra through differential equations. From another perspective, Confrey and Smith (1994) talk about rate as a constant ratio as they offer rate as a ratio of any two values that are multiplicatively related. Finally, at a more sophisticated mathematical level, Carlson and her colleagues offer a framework for covariational reasoning that provides a way to categorize student mental actions about rate in calculus (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Students reason about covariation between two quantities using different mental actions ranging from knowing that as one functions changes, the other one does as well, to the action of thinking of the instantaneous rate of change as the more advanced conceptualizing of covariation.

Still in the calculus domain, Stroup (2002) in his article on qualitative calculus suggested that the idea of derivative as an intensive quality of a function, and not as a concept that arises from thinking of the limit of a ratio in the formal sense, is as important and valuable an domain of mathematics as the traditional calculus sequences. He proposed that the idea of rate (derivative) can be developed intuitively for students in the early grades and can complement traditional calculus as well as stand on its own. His article defined the notion of intuitive calculus and provides evidence about students reasoning about rate of change and time in a qualitative manner. This paper builds upon Stroups idea to provide analysis of the way students enhance their understanding of solutions to systems of differential equations and other three dimensional functions using time as parameter in intuitive ways.

The idea of time based reasoning also complements recent work from Drijvers (2001) that investigates student understanding of parameter. As mentioned earlier, one role of parameter is that of changing quantity. In Drijverss research, students used computers and real time simulations to understand about changing parameters and the roles they play in creating continuously changing graphs; this may lead to using the parameter as a generalizer to represent a family of functions or graphs that have something in common. Drijvers investigation of parameter provides work on the first type of parameter, while this research is about the second.

Research setting and data collection

Data collection for this report was conducted in a college level differential equations class in the central part of the United States using a classroom teaching experiment methodology (Cobb, 2000). Ten students completed the course, all of whom were engineering or mathematics majors; some were undergraduates and some were graduate students with a wide variety of ages. Data collection was as follows: Each of the nine classes in the analysis was videotaped; whole class discussions were videotaped with two cameras, one in the back focused on the teacher and one in the front focused on the students. Two small groups were videotaped during local discussions. Individual semi-structured task based interviews were conducted two times in the semester with the same six students in the class, before and after formal instruction in systems of differential equations. All the in-class work was collected and copied; all homework and exams were copied as well as to serve as artifacts of student work.

The class was inquiry oriented and so each class involved cycles of learning: whole class discussion, followed by small group discussion, followed by whole class discussion. The learning environment of the classroom required students to discuss frequently the mathematics they were learning. They were asked to express their own ideas, as well as make sense and come to agree or disagree with others ideas. Explanation and justification of mathematics reasoning by students was expected on a regular basis. The teacher facilitated the class by fostering a learning environment that allowed for this exchange of ideas. He also participated in the discussions and created ways to use the discourse to further the mathematical agenda in the class. Ultimately the teacher and the students constituted mathematical meaning together.

Analysis

Analysis of the data used a grounded theory approach (Glaser & Strauss, 1967). The theoretical ideas that develop are called grounded to indicate that they are rooted in the data analysis process and not developed a priori (Cobb & Whitenack, 1996). The analysis has primarily centered around two students (Adam and Brandon) including analysis of their participation in the mathematical practices in the classroom, their work and discourse in their interviews, and their written work both in and out of the class. However, some analysis of the discourse and work of others in the class has also done. Analysis began by transcribing all whole class discussions, all small group discussions and the two individual interviews conducted with six of the students. Then the discourse in class for each of the two men was specifically identified and coded for the kind of thinking being used both in small group and whole class discussions. General themes were identified. Written work was used to triangulate the results and verify the analysis from the discourse. Correlations to previous literature and the new ideas that have emerged were recognized and documented.

Result

For students in this differential equations class, parametric reasoning in systems involved (but was not limited to)

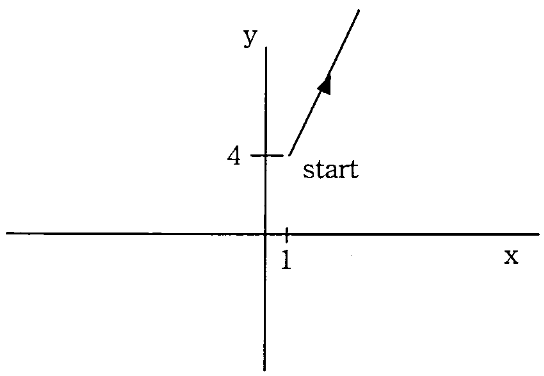
- reasoning qualitatively and quantitatively at the same time
- moving from discrete to continuous imaging of time
- imagining the motion

These practices are not in any order and often occur simultaneously. To make reading easier, each will be discussed separately and examples provided even though the practices cannot be isolated in actuality.

- Reasoning qualitatively and quantitatively at the same time.

In task-based interviews before formal instruction in systems of differential equations, students were asked to consider this situation:

You are up in a hot air balloon looking down at a skateboarder. He has a piece of chalk on the bottom of his board that is drawing a line on the ground as he moves. Below is an example of what you in the hot air balloon see on the ground (dark line). Sketch possible graphs of the x vs. t and y vs. t graphs for the skateboarder.

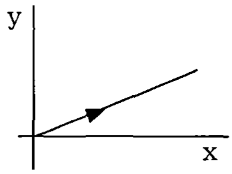


Adam (and others) responded to this task by sketching two graphs. He sketched an x-t graph that started at the origin and was a line segment at about a 30 angle and a y-t graph that started at the origin and was a line segment with a steeper slope. He created the graphs by first actually assigning values to the points and reasoning about the movement discretely. He said words like as it moves 1 this way, it will move maybe 3 this way and it is moving 4 units this way and 1 unit this way. He also marked off two units on the t axis for each and the number 2 on the x-axis and 5 on the y-axis on the respective graphs. This shows he was thinking about the situation from a quantitative standpoint. As he continued to speak about the problem, he then began to talk about the situation and reason using discourse that indicates a change to qualitative thinking, and he did not depend on actual values to complete the task. For example he said, As the skateboarder moves, the graph of x moves like this and sketched it in. He took values of x and y that were originally discrete and extended it to reason that y was moving faster than x in a qualitative way. Then he would discuss with numbers the graphs again. Thus, he would cycle in his work between qualitatively thinking about the situation and quantitative thinking to more generally analyze of the situation.

- Moving from discrete to continuous imaging of time

Another student, Max, discussed the following task in his individual interview before Instruction.

The following is a sketch of x vs. y that is produced over time. Look at each of the following suggested systems of differential equations and decide if any of the choices could describe the graph.



(1) $\frac{dx}{dt}=x+y$, $\frac{dy}{dt}=-x+y$ (2) $\frac{dx}{dt}=x+y$, $\frac{dy}{dt}=x+y$ (3) $\frac{dx}{dt}=2x-2y$, $\frac{dy}{dt}=x-y$

Max: [Draws slash through choice 1.] This is not true. Because x and y [Refers to graph] value is always positive on this graph· So. This dy/dt give us decreasing function. Decreasing rate of change. [Karen: Why?]

Max: Because x and y are positive. So, if we just put those number in. [Refers to choice 1] then x is positive. So. This is not possible· This, any case, on this graph, dy/dt would be negative·Because x value is larger than y value. So this is not true.

Max: [Now points to Choice 2.] This might be true. But, this is saying dx/dt equals dy/dt. Their rate of change are the same. [Karen: Uh, huh.]

Max: [Points to graph.] But, it's not. [Karen: Why not?]

Max: [Places pen along slope line.] It's not, um. The slope way's not one.

Karen: Uh, huh.[Pause.] I'm not sure what the slope one would have to do with this.

Max: After one minute, this same change of x is same as change of y. So. Saying, after one minute, [Refers to slope line.] from here, x is changing from one to three. [Makes notations on x axis.] And, y is changing one, two, three, [Notates on y axis.] too. But. That's, uh, that's not the point on the graph. [Karen: uh huh]

Max: This is saying something, like, after one minute, x has to change two. y has to change, maybe, one. [Karen: Uh, huh.]

Max: And, you can say that, [Refers to graph.] you can tell that slope is not [Swivels pen slightly.] forty-five degree angle·.

Max: So this is not right.[crosses out number 2] And, this. [Points to Choice three.] Might be. [Writes number 4 above 2x.] Six. This is. [Circles choice 3] This could be. Right. Because [Pause.] it's saying it's 2(x-y). So dx/dt is two times larger than dy/dt. Right? [Karen: Uh, huh.]

Max: Because x minus y is always same value. So dx/dt has two times bigger than this number. [2(x-y).]·Which is this. [Refers to x-y.] So let's say dx/dt equals two times dy/dt. [Writes formula on sheet under choice three.] [Karen: Oh. That's nice. Uh, huh.]

Max: And, from this graph, it's not accurate. But, looks like. [Puts right hand over sheet.] At the same time x is moving twice as much as y does.

Max first thinks about the situation in a discrete manner. In the quotation with a single underline, he talks about what happens after one minute. He reasons about the changing graph by thinking in steps. This allows him to develop some frame to think about the situation. As he continues to work on the task, he gradually shifts his time based reasoning to a continuous mode. He thinks of x and y as continuous functions of time and reasons about them as time is moving continuously in a more holistic sense. At this point, in the quotation with the double underline, he does not need to actually think of specific discrete values, but reasons parametrically with a continuous image of time to obtain a correct solution. Three others of the six interviewed before instructional also responded in similar ways.

- Reasoning by imagining the motion

Illustration 1: Three dimensional visualization.

Janvier (1998) suggests that students intuitively understand time and the analysis here supports that and indicates that they build on that intuition to conceptualize solutions of systems of differential equations as based on the continuous parameter time. He names graphs that use time as dependent variable chronicles and posits that students have an inherent facility to create dynamic time-based graphs in their mind. He states, In fact, such changing elements can be represented in the mind as changing. More precisely, the changes of such function-variables are dynamically representable in the mind as the phenomenon is mentally simulated as a temporal event (p. 82). As students learn mathematics that involves the parameter time they have a way to represent the phenomenon in their mind.

For example, in the 7th class day on instructions about systems of differential equations, students were discussing what a-straight line solution (A solution to certain linear systems of homeogenous differential equations that are represented in the phase plane by a straight line through the origin) would look like.

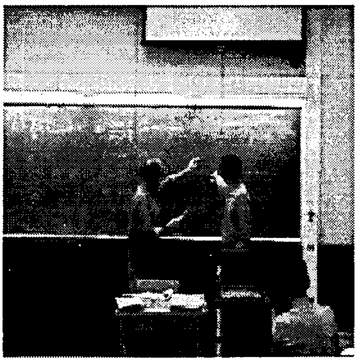


Figure 3. Brandon (on left) demonstrating the motion of the solution with the teacher, as the teacher (on right) uses his arm to be a t-axis

Brandon: Ok, it seems like if you start up here somewhere on this side of this line, it would swing towards that [t-axis] but coming out and then away this way, all the way moving along t, so coming out this way and then for that way, lets say it starts down here on this side of the line, its coming up and also moving along t, and then its swing out this way [John: Just show where it starts at that line] Right here. Then its moving along t moving towards the line. I think you also have to see some vectors along here because if it is off of the line, it's going to move.

In this episode, Brandon went up to the front of the class and showed his idea of the creation of a particular three-dimensional exponential function. To do this, he looked at what he had written on his paper, and then recreated it to show the whole class in a real time situation. He played the motion and watched in his head for himself (and the rest of the class watched as well) as the solution was actually happening.

Illustration 2: Using the fictive motion metaphor

Lakoff and Nunez (2002) suggest that people have different metaphors to understand and reason about functions. One common metaphor is the fictive motion metaphor. Their description of the metaphor suggests that functions are thought of as something in motion. Students use motion (particularly of a graph) to help them develop and deepen their understanding of function. Analysis of the discourse in this differential equation class provides evidence that this mathematical practice is very common as students, particularly Adam and Brandon, develop reasoning ability to understand solutions of differential equations. Discourse in the classroom indicates that students conceptualizations of solutions as being moving objects that create a trace was extremely common. Examples such as:

"its going to go..."

"as time passes, the solution increases exponentially"

"as R goes up, rate of change goes down"

"it never passes the origin, but gets close"

were common to reason about solutions to systems of differential equations. A specific example of the use of the fictive motion metaphor relates to the following problem asked in interview 3:

Without technology, sketch several different graphs of solutions in the x-y plane and support your claims and conclusions for why the graphs look the way they do.

$$\frac{dx}{dt} = 2x$$

$$\frac{dy}{dt} = 2y$$

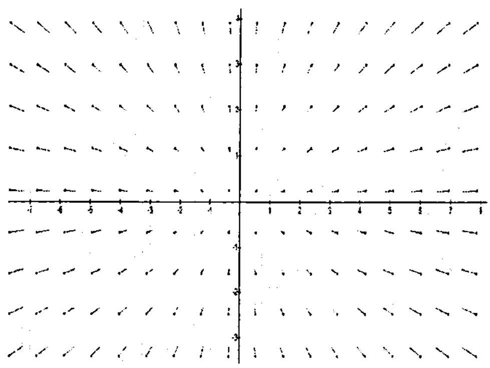


Figure 4. Interview task about straight line solutions

Adam analyzes the situation correctly that all solutions to this system of differential equations can be represented as straight lines in the phase plane. He is then asked to explain why the three dimensional solution curves all be exponential and have a certain shape. His language shows how he is thinking of the motion metaphor as a way to understand the situation:

> Adam: Oh, it is because, no, look, look, when time increases at the same amount of time [Karen, time constantly increases] yeah, but this is going to get bigger and bigger [sketches vectors on the straight line solution] so it is going to start going bigger in this direction [points to x direction] than it is in this direction [t direction] so that is why it is concave down. Otherwise it wouldn't be a straight-line solution because if it slowed down, it would have to slow down to a point and then reverse the other direction so it would run into a different equilibrium point!

This transcription is missing the gestures and inscriptions that he creates as he speaks, but the words show several things. First, he reasons about time being continuous in the first sentence and talks about the vectors growing as time increases. There is no time in the phase plane, but he integrates it into his discourse and talks about the solution curves growth as time passes. In the last sentence, he particularly uses the fictive motion metaphor for function as he talks about the solution slowing down and coming to a stop (slow down to a point) and then reversing direction. For Adam, there is some entity at the end of the function that is actually moving around on a path that changes. He almost gives the function human properties as it controls its own movements to create the straight-line solutions for the problem.

Conclusion

In this paper, three mathematics practices were identified and discussed. At this point, there is no evidence that all students participate in all the different practices, but each of them occur on a regular basis in this particular differential equation class. These three practices form a basis for students time based parametric reasoning. In some ways, parametric reasoning of this type is similar to Freudenthal and Drijvers notion of parameter, but in other ways, it is different.

This analysis suggest three possible practices where students use this facility for understanding temporal events to reason parametrically with time, moving from discrete to continuous imaging of time, reasoning qualitatively and quantitatively, imagining the motion and thinking of function in a way similar to the fictive motion metaphor. These practices allow conceptualizing of solutions to systems of differential equations to occur. Also, conceptualizing solutions may deepen the understanding of time as a parameter, but that is an issue for future work.

The results of this study lead to possibilities that students of mathematics can use this inherent understanding of the dynamic parameter time as a springboard to conceptualizing mathematics of change at the university level, as opposed to Janvier's contention that they are obstacles to learning (1998). As more science and mathematics is based on dynamical systems, which are based on time, understanding how students reason parametrically will be more valuable. Also, this study shows that students are able to reason parametrically before the university level so that using and strengthening parametric reasoning is an area to integrate more explicitly in the 6-12 curriculum. Since students understand time from an early age, these practices may be a part of their thinking as they learn mathematics from middle school on. Future research will hopefully support this. Meanwhile, this research study will continue to investigate parametric reasoning as an individual as well as social practice in the differential equations classroom.

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