Kangweon-Kyungki Math. Jour. **12** (2004), No. 1, pp. 23–32

C_2^3-CONSTRUCTION ON $M_n(k)$

Youngkwon Song

Abstract. Let (B, m_B, k) be a maximal commutative k-subalgebra of a matrix algebra $M_n(k)$. We will construct a maximal commutative k-subalgebra (R, m, k) of $M_{n+3}(k)$ from the algebra B such that the algebra R has dimension greater than the dimension of B by 3. Moreover, we will show a C_i-construction doesn't imply a C_2^3-construction for $i = 1, 2$.

1. Introduction

Let (B, m_B, k) be a maximal commutative k-subalgebra of $M_n(k)$. Then, in [2], W.C. Brown introduced a way to construct a maximal commutative k-subalgebra from the algebra B of smaller dimension by one.

In this paper, we want to construct a maximal commutative subalgebra (R, m, k) of a matrix algebra $M_{n+3}(k)$ from a maximal commutative subalgebra B of $M_n(k)$. This construction is useful to embed a maximal commutative k-subalgebra of matrix algebra in a maximal commutative k-subalgebra of a larger size of matrix algebra. Also we can construct a maximal commutative k-subalgebra from a maximal commutative k-subalgebra of smaller dimension by three. In other words, if there is a maximal commutative k-subalgebra of dimension s, then we can always construct a maximal commutative k-subalgebra of dimension $s + 3$ by using this construction.

Moreover, we will show this construction is neither a C_1-construction nor a C_2-construction defined in [3].

Received December 15, 2003.
2000 Mathematics Subject Classification: 15A27, 15A33.
Key words and phrases: C_2^3-construction.
The present research has been conducted by the Research Grant of Kwangwoon University in 2002.

2. New construction

Let (B, m_B, k) is a finite dimensional commutative k-algebra with identity and N a finitely generated faithful B-module. Then, $R = B \oplus N^{\ell}$ is a maximal commutative k-subalgebra which is called a C_1-construction.

The next theorem present an equivalent condition to be a C_1-construction and the proof can be found in [1].

THEOREM 2.1. [1] *Let (R, m, k) be a commutative k-algebra. Then, R is a C_1-construction if and only if there is an ideal I satisfying the following conditions:*

(1) $Ann_R(I) = I$
(2) $0 \to I \to R \to R/I \to 0$ splits as k-algebras.

The following corollary is obtained directly from Theorem 2.1.

COROLLARY 2.2. [1, 2] *Let (R, m, k) be a commutative k-algebra. If $m^2 = (0)$, then R is a C_1-construction.*

Throughout this paper, the socle of an algebra R will be denoted by $Soc(R)$.

THEOREM 2.3. [2] *Let (B, m_B, k) be a finite dimensional commutative k-algebra with identity and N a finitely generated faithful B-module. Suppose $B \cong Hom_B(N, N)$ via the regular representation. Then there exists an element $w \neq 0 \in Soc(B)$ with $\dim_k(Nw) = 1$.*

Let (B, m_B, k) be a finite dimensional commutative k-algebra with identity. If $R \cong B[X]/(m_B X, X^p - w)$, for some $w \in Soc(B) - \{0\}$ and a positive integer $p > 1$, then we say the algebra R of this form a C_2-construction.

Here is an equivalent condition to be a C_2-construction and can be found in [3].

THEOREM 2.4. [3] *Let (R, m, k) be a commutative k-algebra. Then, R is a C_2-construction if and only if R contains a k-subalgebra (B, m_B, k) and an element $x \in m$ satisfying the following conditions:*

(1) $0 \neq x^p \in Soc(B)$ *for some positive integer* $p > 1$

(2) $m_B x = (0)$

(3) $\dim_k(R) = \dim_k(B) + (p-1)$

For $p = 4$, using C_2-construction, we can construct a maximal commutative k-subalgebra from a maximal commutative subalgebra of smaller dimension by three. But, we can construct a maximal commutative subalgebra from a maximal commutative subalgebra of smaller dimension by three by using the following theorem that is the main result of this paper.

THEOREM 2.5. *Let (B, m_B, k) be a finite dimensional commutative k-algebra with identity and N a finitely generated faithful B-module. Suppose $B \cong Hom_B(N, N)$ via the regular representation. Let $R = B[X, Y, Z]/I$, where the ideal I is as follows:*

$$I = (m_B X, m_B Y, m_B Z, X^2 - w, Y^2 - w, Z^2 - w, XY, YZ, ZX).$$

Here $w \neq 0 \in Soc(B)$ with $dim_k(Nw) = 1$ and $M = N \oplus Nw \oplus Nw \oplus Nw$. Then R is isomorphic to a maximal commutative subalgebra of $M_n(k)$, where $n = dim_k(M)$.

Proof. Let x, y and z be the images of X, Y and Z in R. Then, M is an R-module via

$$(n, n_1 w, n_2 w, n_3 w)x = (n_1 w, nw, 0, 0)$$

$$(n, n_1 w, n_2 w, n_3 w)y = (n_2 w, 0, nw, 0)$$

$$(n, n_1 w, n_2 w, n_3 w)z = (n_3 w, 0, 0, nw)$$

Since N is a finitely generated faithful B-module, M is a finitely generated faithful R-module.

Now, let $f \in Hom_R(M, M)$ and define $\phi_1 : N \to M$ and $\phi_2 : M \to N$ as follows:

$$\phi_1(n) = (n, 0, 0, 0), \quad \phi_2(n, n_1 w, n_2 w, n_3 w) = n.$$

Then, obviously ϕ_1 and ϕ_2 are B-module homomorphisms. Let ϕ be the composition of the homomorphisms ϕ_1, f,and ϕ_2, that is,

$$\phi = \phi_2 f \phi_1.$$

Then ϕ is a B-module homomorphism from N to N. Since B is isomorphic to $Hom_B(N, N)$ via the regular representation, $\phi = \mu_a$ for some $a \in B$. Thus,

$$\phi_2(f(n, 0, 0, 0)) = \phi_2 f \phi_1(n) = \phi(n) = \mu_a(n) = na.$$

By the definition of ϕ_2, there exist three B-module homomorphisms $\psi_i; N \to Nw$, for $i = 1, 2, 3$ such that

$$f(n, 0, 0, 0) = (na, \psi_1(n), \psi_2(n), \psi_3(n)).$$

Since $dim_k(Nw) = 1$, there exists an element $p \in N$ such that $\{pw\}$ is a k-vector space basis of Nw. Thus, there exist $t_1, t_2, t_3 \in k$ such that

$$\psi_i(p) = t_i pw, \quad i = 1, 2, 3.$$

Then, $a + t_1 x + t_2 y + t_3 z \in R$ and eventually we want to show

$$f = \mu_{a+t_1x+t_2y+t_3z}.$$

Since pw generates Nw, we can let

$$n_i w = s_i pw$$

for some $s_i \in k$, $i = 1, 2, 3$. Thus, we want to show the following identity:

$$f(n, s_1pw, s_2pw, s_3pw) = \mu_{a+t_1x+t_2y+t_3z}(n, s_1pw, s_2pw, s_3pw)$$

For the simplicity, let

$$r = a + t_1x + t_2y + t_3z, \;\; A = (n, s_1pw, s_2pw, s_3pw).$$

Then,

$$\begin{aligned} \mu_r(A) &= Ar = (n, s_1pw, s_2pw, s_3pw)(a + t_1x + t_2y + t_3z) \\ &= (na, s_1pwa, s_2pwa, s_3pwa) + (s_1pwt_1, nt_1w, 0, 0) \\ &\quad + (s_2pwt_2, 0, nt_2w, 0) + (s_3pwt_3, 0, 0, nt_3w) \\ &= u + v. \end{aligned}$$

Here,

$$\begin{aligned} u &= (na, nt_1w, nt_2w, nt_3w) \\ v &= (s_1pwt_1 + s_2pwt_2 + s_3pwt_3, s_1pwa, s_2pwa, s_3pwa). \end{aligned}$$

Note that

$$f(A) = f(n, s_1pw, s_2pw, s_3pw) = f(n, 0, 0, 0) + f(0, s_1pw, s_2pw, s_3pw).$$

But, for each i, j,

$$\psi_i(s_j p) = s_j \psi_i(p) = s_j t_i p w.$$

Thus,

$$\begin{aligned} f(0, s_1pw, s_2pw, s_3pw) &= f((s_1p,0,0,0)x + (s_2p,0,0,0)y + (s_3p,0,0,0)z) \\ &= (s_1pa, \psi_1(s_1p), \psi_2(s_1p), \psi_3(s_1p))x \\ &\quad + (s_2pa, \psi_1(s_2p), \psi_2(s_2p), \psi_3(s_2p))y \\ &\quad + (s_3pa, \psi_1(s_3p), \psi_2(s_3p), \psi_3(s_3p))z \\ &= (\psi_1(s_1p), s_1paw, \psi_2(s_1p)w, \psi_3(s_1p)w) \\ &\quad + (\psi_2(s_2p), \psi_1(s_2p)w, s_2paw, \psi_3(s_2p)w) \\ &\quad + (\psi_3(s_3p), \psi_1(s_3p)w, \psi_2(s_3p)w, s_3paw) \\ &= (s_1pt_1w, s_1paw, s_1pt_2w^2, s_1pt_3w^2) \\ &\quad + (s_2pt_2w, s_2pt_1w^2, s_2paw, s_2pt_3w^2) \\ &\quad + (s_3pt_3w, s_3pt_1w^2, s_3pt_2w^2, s_3paw) \\ &= (s_1pt_1w, s_1paw, 0, 0) + (s_2pt_2w, 0, s_2paw, 0) \\ &\quad + (s_3pt_3w, 0, 0, s_3paw) \\ &= (s_1pt_1w + s_2pt_2w + s_3pt_3w, s_1paw, s_2paw, s_3paw). \end{aligned}$$

Note that $nw = spw$ for some $s \in k$. Thus,

$$\begin{aligned} (nwt_1, nwa, 0, 0) &= (spwt_1, spwa, 0, 0) = f(0, spw, 0, 0) \\ &= f(0, nw, 0, 0) = f((n,0,0,0)x) = f(n,0,0,0)x \\ &= (na, \psi_1(n), \psi_2(n), \psi_3(n))x \\ &= (\psi_1(n), naw, \psi_2(n)w, \psi_3(n)w). \end{aligned}$$

This implies that

$$\psi_1(n) = nwt_1.$$

Similarly, we have the followings:

$$\begin{aligned} (nwt_2, 0, nwa, 0) &= (spwt_2, 0, spwa, 0) = f(0, 0, spw, 0) \\ &= f(0, 0, nw, 0) = f((n,0,0,0)y) = f(n,0,0,0)y \\ &= (na, \psi_1(n), \psi_2(n), \psi_3(n))y \\ &= (\psi_2(n), \psi_1(n)w, naw, \psi_3(n)w). \end{aligned}$$

Thus,

$$\psi_2(n) = nwt_2.$$

Finally, we have

$$\begin{aligned}(nwt_3,0,0,nwa) &= (spwt_3,0,0,spwa) = f(0,0,0,spw)\\ &= f(0,0,0,nw) = f((n,0,0,0)z) = f(n,0,0,0)z\\ &= (na,\psi_1(n),\psi_2(n),\psi_3(n))z\\ &= (\psi_3(n),\psi_1(n)w,\psi_2(n)w,naw).\end{aligned}$$

Thus,

$$\psi_3(n) = nwt_3.$$

From the above results, we have the following identity:

$$\begin{aligned}f(n,0,0,0) &= (na,\psi_1(n),\psi_2(n),\psi_3(n))\\ &= (na,nt_1w,nt_2w,nt_3w).\end{aligned}$$

Therefore, we have proved the following two identities:

(1) $f(n,0,0,0) = u$

(2) $f(0,s_1pw,s_2pw,s_3pw) = v$

Thus, we finally obtain

$$f(n,s_1pw,s_2pw,s_3pw) = \mu_{a+t_1x+t_2y+t_3z}(n,s_1pw,s_2pw,s_3pw).$$

Therefore, we have the following result:

$$f = \mu_{a+t_1x+t_2y+t_3z}.$$

Since M is a faithful R-module, R is isomorphic to $Hom_R(M,M)$ via the regular representation. □

Remark. In the above theorem, R thus defined is isomorphic to a maximal commutative subalgebra of $M_n(k)$, where $n = dim_k(M)$. We will call the k-algebra R of the form in Theorem 2.5 a C_2^3-construction.

With a C_2^3-construction, a maximal commutative subalgebra B of $M_n(k)$ with $\dim_k(B) = s$ can be embedded in a maximal commutative subalgebra R of $M_{n+3}(k)$ with $\dim_k(R) = s+3$.

The following theorem provides an equivalent condition for R to be a C_2^3-construction.

THEOREM 2.6. *Let (R,m,k) be a finite dimensional commutative algebra. Then, R is a C_2^3-construction if and only if there exists commutative k-subalgebra (B,m_B,k) and elements $x,y,z \in m$ satisfying the following properties :*

(1) $x^2 = y^2 = z^2 \in Soc(B) - \{0\}$
(2) $xy = yz = zx = 0$
(3) $m_B x = (0) = m_B y = m_B z$
(4) $\dim_k(R) = \dim_k(B) + 3$

Proof. Suppose R is a C_2^3-construction. Then, obviously the four conditions (1),(2),(3), and (4) are satisfied.

Conversely, suppose there exist a k-subalgebra B and elements $x, y, z \in m$ such that the four conditions are satisfied. Let $x^2 = y^2 = z^2 = w \in Soc(B)$ and let I be the following ideal :

$$I = (m_B X, m_B Y, m_B Z, X^2 - w, Y^2 - w, Z^2 - w, XY, YZ, ZX).$$

Define a map

$$\psi : B[X, Y, Z]/I \longrightarrow R$$

by

$$\psi(b + I) = b, \ \ \psi(X + I) = x, \ \ \psi(Y + I) = y, \ \ \psi(Z + I) = z$$

,where $b \in B$. Then, ψ is a k-algebra homomorphism. Suppose $\psi(a + bX + cY + dZ + I) = 0$. Then,

$$a + bx + cy + dz = 0.$$

Here, we may assume $b, c, d \in k$ since $m_B x = m_B y = m_B z = (0)$. Assume $a \neq 0$, then $a \notin m_B$. For, if $a \in m_B$, then we have

$$0 = ax + bx^2 + cxy + dxz = bw.$$

Since $w \neq 0 \in Soc(B)$, we should have $b = 0$. By the similar ways, we can easily have $c = d = 0$. But then $a = 0$ which is impossible. Thus, $a \notin m_B$ and hence $a + bx + cy + dz$ is a unit which is impossible. Thus, we have $a = 0$. If $b \neq 0$, then $x + (b^{-1}c)y + (b^{-1}d)z = 0$ since $a = 0$. By multiplying x each side, we get

$$0 = x^2 + (b^{-1}c)xy + (b^{-1}d)xz = x^2 = w$$

which is impossible and so $b = 0$. Similarly, we can show $c = d = 0$. This implies ψ is monomorphism. Note that

$$dim_k(im(\psi)) = dim_k(B[x, y, z]) = dim_k(B) + 3 = dim_k(R)$$

Therefore, ψ is an isomorphism and we conclude the algebra R is a C_2^3-construction. □

Here we have an example of C_2^3-construction. We will let E_{ij} be the (i,j)-th matrix unit.

EXAMPLE 2.7. Let $R = m \oplus kI_5$ be a maximal k-subalgebra of $M_5(k)$ such that $r \in m$ is of the following form :

$$r = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 \\ d & a & b & c & 0 \end{pmatrix},$$

where $a, b, c, d \in k$.

Let $B = k[E_{51}]$. Then, $Soc(B) = kE_{51} = m_B$. Thus, if we let

$$x = E_{21} + E_{52}, \;\; y = E_{31} + E_{53}, \;\; z = E_{41} + E_{54}$$

then the following conditions can be easily proved :

(1) $x^2 = y^2 = z^2 \in Soc(B) - \{0\}$
(2) $xy = yz = zx = 0$
(3) $m_B x = (0) = m_B y = m_B z$
(4) $\dim_k(R) = \dim_k(B) + 3$

Thus, by Theorem 2.6, R is a C_2^3-construction.

Now, we want to prove that a C_i-construction doesn't imply a C_2^3-construction for $i = 1, 2$.

COROLLARY 2.8. *A C_1-construction doesn't imply a C_2^3-construction.*

Proof. Let $R = m \oplus kI_3$ be a maximal k-subalgebra of $M_3(k)$ such that the element $r \in m$ is of the following form:

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $a, b \in k$. By Corollary 2.2, the algebra R is a C_1-construction since $m^2 = (0)$. But, the algebra R can't be a C_2^3-construction since there are no elements x, y, and z in m whose squares are not zero. Thus, R is a C_1-construction but not a C_2^3-construction by Theorem 2.6. □

COROLLARY 2.9. *A C_2^3-construction doesn't imply a C_1-construction.*

Proof. Let k be the real number field and let $R = m \oplus kI_5$ be a maximal k-subalgebra of $M_5(k)$ as in Example 2.7. Then, R is a C_2^3-construction. Suppose R is a C_1-construction. Then, there exists an ideal I of R such that $Ann_R(I) = I$. If we let $r \in Ann_R(I)$, then for some real numbers a, b, c, d, the element r is of the following form :

$$r = a(E_{21} + E_{52}) + b(E_{31} + E_{53}) + c(E_{41} + E_{54}) + dE_{51}.$$

Since $Ann_R(I) = I$, we have $0 = r^2 = (a^2 + b^2 + c^2)E_{51}$ and hence $a = b = c = 0$. But, then $r = dE_{51}$ and so $Ann_R(I) = kE_{51}$ which is impossible since $E_{21} + E_{52} \in Ann_R(I) = I$. Thus, the algebra R in Example 2.7 is a C_2^3-construction but not a C_1-construction. □

COROLLARY 2.10. *A C_2-construction doesn't imply a C_2^3-construction.*

Proof. Let k be any field and let $R = m \oplus kI_4$ be a maximal k-subalgebra of $M_4(k)$ such that $r \in m$ is of the following form :

$$r = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & 0 & 0 & 0 \end{pmatrix},$$

where $a, b, c \in k$.

If we let $B = k[E_{31}, E_{41}]$ and $r = E_{21} + E_{32}$, then

(1) $E_{31} = r^2 \in Soc(B)$

(2) $rm_B = (0)$

(3) $\dim_k(R) = \dim_k(B) + 1$

This implies R is a C_2-construction by Theorem 2.4.

Now, suppose R is a C_2^3-construction, then R contains a k-subalgebra B such that for some $x, y, z \in m$,

$$x^2, y^2, z^2 \in Soc(B) - \{0\}, \;\; xy = 0 = yz = zx.$$

For some $a_i, b_i, c_i \in k$, the elements $x, y, z \in m$ can be written as follows:

$$x = a_1(E_{21} + E_{32}) + b_1E_{31} + c_1E_{41}$$
$$y = a_2(E_{21} + E_{32}) + b_2E_{31} + c_2E_{41}$$
$$z = a_3(E_{21} + E_{32}) + b_3E_{31} + c_3E_{41}.$$

Thus, we have the following identities :

$$x^2 = a_1^2 E_{31}, \;\; y^2 = a_2^2 E_{31}, \;\; z^2 = a_3^2 E_{31}$$
$$xy = a_1 a_2 E_{31}, \;\; yz = a_2 a_3 E_{31}, \;\; zx = a_3 a_1 E_{31}.$$

But, by the conditions, we have

$$a_1 \neq 0, \;\; a_2 \neq 0, \;\; a_3 \neq 0, \;\; a_1 a_2 = 0, \;\; a_2 a_3 = 0, \;\; a_3 a_1 = 0$$

which is impossible and hence R can't be a C_2^3-construction by Theorem 2.6. Thus, R is a C_2-construction but not a C_2^3-construction. □

References

[1] W.C. Brown and F.W. Call, *Maximal Commutative Subalgebras of* $n \times n$ *Matrices.* Communications in Algebra, **59(12)** (1993), 4439-4460.

[2] W.C. Brown, *Two Constructions of Maximal Commutative Subalgebras of* $n \times n$ *Matrices.* Communications in Algebra, **22(10)** (1994), 4051-4066.

[3] W.C. Brown, *Constructing Maximal Commutative Subalgebras of Matrix Rings in Small Dimensions.* Communications in Algebra, **25(12)** (1997), 3923-3946.

[4] R.C. Courter, *The Dimension of Maximal Commutative Subalgebras of* K_n. Duke Mathematical Journal, **32** (1965), 225-232.

[5] Youngkwon Song, *On the Maximal Commutative Subalgebras of* 14 *by* 14 *Matrices.* Communications in Algebra, **25(12)** (1997), 3823-3840.

[6] Youngkwon Song, *Maximal Commutative Subalgebras of Matrix Algebras.* Communications in Algebra, **27(4)** (1999), 1649-1663.

[7] Youngkwon Song, *Notes on the Constructions of Maximal Commutative Subalgebra of* $M_n(k)$. Communications in Algebra, **29(10)** (2001), 4333-4339.

Department of Mathematics
Research Institute of Basic Science
Kwangwoon University
Seoul 139–701, Korea
E-mail: yksong@daisy.kwangwoon.ac.kr